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Localization of CW-complexes and its applications

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Introduction

In the algebraic topology, in particular in the homotopy theory, abelian groups are often treated by being devided into their "p-primary component" for various primes p.

In the homotopy category of 1-connected CW-complexes, an isomorphism means a homotopy equivalence, which is of course an equivalence relation. As is well known, a homotopy equivalence is such a map that it induces an isomorphism on the integral homology group.

There might be three ways to generalize it in the mod p sense.

First one is to define a p-equivalence so that it induces an isomorphism on the homology group with Z_p-coefficient. A p-equivalence, however, is not in general an equivalence relation even in the category of 1-connected finite CW-complexes. In fact, in [**11**] is shown an example, for which symmetricity does not hold. To make it an equivalence relation, we have to work in the category of p-universal spaces [**12**].

Next one is to define that X and Y are of same p-type, if there exist a space Z and p-equivalences $f: X \to Z$ and $g: Y \to Z$. Then it is easy to see that a relation being of same p-type is an equivalence relation.

The last one is to consider a homotopy equivalence for "localized spaces $X_{(p)}$" of X at p. It is a functor of 1-connected CW-complexes into itself such that if $f: X \to Y$ is a p-equivalence then the localization at p $f_{(p)}: X_{(p)} \to Y_{(p)}$ is a homotopy equivalence. The localization is studied by Adams [**2**], Anderson [**3**], Bousfield-Kan and others. Our construction is a generalization of Adams' telescope [**2**], and has the following advantage:

THEOREM 2.5. *If X is a* 1-*connected CW-complex of finite type, then $H_*(X_{(p)}) \cong H_*(X) \otimes Q_p$ and $\pi_*(X_{(p)}) \cong \pi_*(X) \otimes Q_p$, where Q_p denotes the ring of those fractions, whose denominators, in the lowest form, are prime to p.*

Also we show

COROLLARY 4.3. *X is homotopy equivalent to $\prod_{X_{(0)}} X_{(p)}$ the pull back of $X_{(p)}$ over $X_{(0)}$.*

So we can study the topological properties of X for each prime p

separately.

In this paper, $\mathfrak{C}_1$ denotes the category of 1-connected CW-complexes of finite type, i.e., the i-dim integral homology group is of finite type for each i. Also we denote by $\mathfrak{F}\mathfrak{C}_1$ the category of 1-connected finite CW-complexes.

Let $\boldsymbol{P}$ be a subset of the set of all primes. The notation (0) will be used as the vacant set ϕ. We denote by $Q_{\boldsymbol{P}}$ the ring of those fractions, the denominators of which are, in the lowest form, prime to p for all $p \in \boldsymbol{P}$. If $\boldsymbol{P}$ is the set of all primes, then $Q_{\boldsymbol{P}} = Z$, and if $\boldsymbol{P} = (0)$, then $Q_{\boldsymbol{P}} = Q$ the set of rational numbers. Z_p stands for Z/pZ and Z_0 for Q. $\mathfrak{C}_{\boldsymbol{P}}$ is a class of finite abelian groups without $\boldsymbol{P}$-torsion. $H_*(X)$ means $H_*(X; Z)$. $X = Y$ reads that X is homotopy equivalent to Y.

DEFINITION 0.1. A space X is *$\boldsymbol{P}$-equivalent* to Y, if there exists a map $f: X \to Y$ such that f induces isomorphisms $H_*(X; Z_p) \cong H_*(Y; Z_p)$ for all $p \in \boldsymbol{P}$. Then the map f is called *a $\boldsymbol{P}$-equivalence.*

DEFINITION 0.2. A space $K \in \mathfrak{F}\mathfrak{C}_1$ is called *$\boldsymbol{P}$-universal* if, for any given $\boldsymbol{P}$-equivalence $k: X \to Y$ in the category $\mathfrak{C}_1$, and for an arbitrary map $g: K \to Y$, there is a map $h: K \to X$ and there is a $\boldsymbol{P}$-equivalence $f: K \to K$ such that the following diagram is homotopy commutative:

$$\begin{tikzcd} X \arrow[r, "k"] & Y \\ K \arrow[u, dashed, "h"] \arrow[r, dotted, "f"'] & K \arrow[u, "g"'] \end{tikzcd}$$

or equivalently, for any given $\boldsymbol{P}$-equivalence $k: X \to Y$ in $\mathfrak{F}\mathfrak{C}_1$ and for an arbitrary map $g: X \to K$, there is a map $h: Y \to K$ and there is a $\boldsymbol{P}$-equivalence $f: K \to K$ such that the following diagram is homotopy commutative:

$$\begin{tikzcd} X \arrow[r, "k"] \arrow[d, "g"'] & Y \arrow[d, dashed, "h"] \\ K \arrow[r, dotted, "f"] & K \end{tikzcd}$$

Thus, for a given $\boldsymbol{P}$-equivalence $X \to Y$, if one of X and Y is $\boldsymbol{P}$-universal, there exists a converse $\boldsymbol{P}$-equivalence $Y \to X$, and hence a $\boldsymbol{P}$-equivalence is an equivalence relation in the category of $\boldsymbol{P}$-universal spaces as was stated earlier.

The present paper is organized as follows.

§ 1. A $\boldsymbol{P}$-sequence of a CW-complex.

§ 2. Localization of CW-complexes.

§ 3. Further properties of localization.

§ 4. The pull-back of localized spaces.
§ 5. Localizing $\boldsymbol{P}$-universal spaces.
§ 6. Mod p H-spaces and mod p co-H-spaces.
§ 7. Localization of finite H-complexes.
§ 8. New finite H-complexes.
§ 9. Mod p decomposition of suspended spaces.

In the first three sections we define a localization at $\boldsymbol{P}$ and show the uniqueness as well as the existence of it. We study its properties thoroughly. In § 4, we reconstruct the original space X from its localized spaces $X_{(p)}$. § 5 is used to see how $\boldsymbol{P}$-universal spaces behaves nicely under localization. For example, in the category of $\boldsymbol{P}$-universal spaces, X and Y are $\boldsymbol{P}$-equivalent if and only if $X_{\boldsymbol{P}}$ and $Y_{\boldsymbol{P}}$ are homotopy equivalent. In § 6 various equivalent definitions of a mod p H-space (also of a mod p co-H-space) are given. Examples for them are given, too. § 7 is used to discuss the localization of finite H-complexes, e. g., it is shown that under a certain condition, a finite CW-complex X is an H-space if and only if $X_{(p)}$ is an H-space for all primes p. In § 8, many new finite H-complexes are constructed by mixing homotopy types. The last section, § 9, is devoted to give a mod p decomposition of a suspension of the symmetric product of the Moore space of type (G, n), $G=Z$ or Z_{p^r}, and of a suspension of an H-space with certain conditions. They can give also a mod p decomposition of $SK(Z, n)$ and of $SK(Z_{p^r}, n)$.

§ 1. A $\boldsymbol{P}$-sequence of a CW-complex.

Let X be a CW-complex of finite type and let $\boldsymbol{P}$ be a subset of the set of all primes.

DEFINITION 1.1. $\{X_i, f_i\}$ is *a homology $\boldsymbol{P}$-sequence* of X, if

1) $f_i : X_{i-1} \to X_i$ is a $\boldsymbol{P}$-equivalence with $X_0 = X$,
2) for any n, any i, and any prime q with $(q, p)=1$ for all $p \in \boldsymbol{P}$, there exists $N(>i)$ such that $(f_N \circ \cdots \circ f_i)_* = 0 : H_n(X_{i-1}; Z_q) \to H_n(X_N; Z_q)$.

DEFINITION 1.1′. $\{X_i, f_i\}$ is *a homotopy $\boldsymbol{P}$-sequence* of X, if

1)′ $f_i : X_{i-1} \to X_i$ is a $\boldsymbol{P}$-equivalence with $X_0 = X$,
2)′ for any n, any i, and any prime q with $(q, p)=1$ for all $p \in \boldsymbol{P}$, there exists $N(>i)$ such that $(f_N \circ \cdots \circ f_i)_* \otimes 1 = 0 : \pi_n(X_{i-1}) \otimes Z_q \to \pi_n(X_N) \otimes Z_q$.

THEOREM 1.2. *Let $X, X_i \in \mathfrak{C}_1$. Then $\{X_i, f_i\}$ is a homology $\boldsymbol{P}$-sequence of X if and only if it is a homotopy $\boldsymbol{P}$-sequence of X.*

To prove the theorem, we need to prepare the following. For a given space X, the $(n-1)$-connective space (X, n) is a fibering over X with a fibre map $p : (X, n) \to X$ inducing isomorphisms $p_* : \pi_i((X, n)) \cong \pi_i(X)$ for all $i \geqq n$ and $\pi_i((X, n)) = 0$ for all $i < n$. There exists a fibering

$$(1.1) \qquad K(\pi_n(X), n-1) \longrightarrow (X, n+1) \longrightarrow (X, n).$$

Similarly, the space (n, X) is such a space that there is a fibering $q: X \to (n, X)$ inducing isomorphisms $q_*: \pi_i(X) \cong \pi_i((n, X))$ for all $i \leqq n$ and $\pi_i((n, X)) = 0$ for all $i > n$. Then there exists a fibering

$$(1.2) \qquad K(\pi_n(X), n+1) \longrightarrow (n+1, X) \longrightarrow (n, X).$$

Clearly a $\boldsymbol{P}$-equivalence $f: X \to Y$ induces $\boldsymbol{P}$-equivalences:

$$f_{n,i}: K(\pi_n(X), i) \longrightarrow K(\pi_n(Y), i),$$

$$f_n: (X, n) \longrightarrow (Y, n),$$

$$_nf: (n, X) \longrightarrow (n, Y).$$

By the abuse of the notation, we denote them by the same notation f.

We state easy lemmas without proof.

LEMMA 1.3. *The condition* 2) *of Definition* 1.1 *implies*

3) *For any A, any i, and any prime q with $(q, p) = 1$ for all $p \in \boldsymbol{P}$, there exists $N(> i)$ such that $(f_N \circ \cdots \circ f_i)_* = 0: H_j(X_{i-1}; Z_q) \to H_j(X_N; Z_q)$ for all $0 < j < A$.*

LEMMA 1.3'. *The condition* 2)' *of Definition* 1.1' *implies*

3)' *Foy any A, any i, any prime q with $(q, p) = 1$ for all $p \in \boldsymbol{P}$, there exists $N(> i)$ such that $(f_N \circ \cdots \circ f_i)_* \otimes 1 = 0: \pi_j(X_{i-1}) \otimes Z_q \to \pi_j(X_N) \otimes Z_q$ for all $0 < j < A$.*

Then we show

LEMMA 1.4. *The conditions* 1) *and* 2) *of Definition* 1.1 *imply the following (T_n) for all $n \geqq 2$.*

(T_n): *For any A and any k, there exists $N = N(n, k, A)$ such that $f_{N,k} = f_N \circ \cdots \circ f_k: X_{k-1} \to X_N$ induces $(f_{N,k})_* = 0: H_j((X_{k-1}, n); Z_q) \to H_j((X_N, n); Z_q)$ for all j with $0 < j < A$.*

PROOF. We prove the lemma by induction on n. For $n = 2$, there is nothing to prove, since $(X_k, 2) = X_k$. Suppose (T_n) is true and let us prove (T_{n+1}), $n \geqq 2$. Consider the homology spectral sequence $\{E^r_{p,q}\}$ with Z_q-coefficient associated with a fibering

$$(1.1) \qquad K(\pi_n(X_l), n-1) \longrightarrow (X_l, n+1) \longrightarrow (X_l, n).$$

Then $E^2_{p,q} = H_p((X_l, n); H_q(\pi_n(X_l), n-1; Z_q))$. We may assume that $A \geqq n+2$. Let $N = N(n, l, A)$ and take $f_{N,l+1}: X_l \to X_N$ given in (T_n). Then $(f_{N,l+1})_* = 0$ on $H_n((X_l, n); Z_q)$ by the assumption, and hence $(f_{N,l+1})_* = 0$ on $H_{n-1}(\pi_n(X_l), n-1; Z_q)$ by the suspension isomorphism. So $(f_{N,l+1})^* = 0$ on $H^{n-1}(\pi_n(X_N), n-1; Z_q)$, whence $(f_{N,l+1})^* = 0$ on $H^i(\pi_n(X_N), n-1; Z_q)$ for all $i > 0$, since any element of $H^i(\pi_n(X_l), n-1; Z_q)$ is written as a sum of the cup-products of elements of the form $\mathfrak{p}^I x$, where $x \in H^{n-1}(\pi_n(X_N), n-1; Z_q)$ and $\mathfrak{p}^I$ is a cohomology operation.

Therefore $(f_{N,l+1})_*=0$ on $H_i(\pi_n(X_l), n-1; Z_q)$ for all $i>0$. On the other hand, $(f_{N,l+1})_*=0$ on $H_j((X_l, n); Z_q)$ for all j with $0<j<A$ by the assumption. Thus $(f_{N,l+1})_*=0$ on $E^2_{i,j}$ and hence it is trivial on $E^\infty_{i,j}=D_{i,j}/D_{i-1,j+1}$ for any (i, j) with $j>0$ and for any $(i, 0)$ with $0<i<N$, where $H_{i+j}((X_l, n+1); Z_q)=D_{i+j,0} \supset D_{i+j-1,1} \supset \cdots \supset D_{-1,i+j+1}=0$. So the triviality of $(f_{N,l+1})_*$ on $E^\infty_{i,j}$ implies $(f_{N,l+1})_*(D_{i,j}) \subset D_{i-1,j+1}$. We put $N_{i+1}=N(n, N_i, A)$ and $f_{N_{i+1},N_i}=f_{N_{i+1}} \circ \cdots \circ f_{N_i}$: $X_{N_{i-1}} \to X_{N_{i+1}}$ inductively starting with $N_0=k$. Then $f_{N_i,k}=f_{N_i} \circ \cdots \circ f_k$: $X_{k-1} \to X_{N_i}$ induces the trivial homomorphism on $H_i((X_{k-1}, n+1); Z_q)$. Take $N(n+1, k, A)=N_A$ and $f_{N_A,k}=f_{N_A} \circ \cdots \circ f_k$. Then $(f_{N_A,k})_*=0$ on $H_i((X_{k-1}, n+1); Z_q)$ for all $0<i<A$, so (T_{n+1}) holds. Q. E. D.

LEMMA 1.4′. *The conditions* 1)′ *and* 2)′ *of Definition* 1.1′ *imply the following* (I_n) *for all* $n \geqq 2$.

(I_n); *For any B, and any k, there exists $M=M(n, k, B)$ such that $(f_{M,k})_*=0$: $H_j((n, X_{k-1}); Z_q) \to H_j((n, X_M); Z_q)$ for all j with $0<j<B$.*

PROOF. Clearly $n=2$ is true. For $(2, X_k)=K(\pi_2(X_k), 2)$, since X_k is 1-connected. Then $(f_{M,k+1})^*=0$ on $H^*(\pi_2(X_M), 2; Z_q)$ for some M and hence $(f_{M,k+1})_*=0$ on $H_j(\pi_2(X_k), 2; Z_q)$ for all $j>0$ as before. The statement (I_n) for $n>2$ is then established similarly by induction using the homology spectral sequence with Z_q-coefficient associated with a fibering

$$K(\pi_n(X_l), n+1) \longrightarrow (n+1, X_l) \longrightarrow (n, X_l)\,. \tag{1.2}$$

Q. E. D.

(PROOF OF THEOREM 1.2.)

Let (X_i, f_i) satisfy 1) and 2) of Definition 1.1. Then by Lemma 1.4, for any n, any i, and any prime q with $(p, q)=1$ for all $p \in \boldsymbol{P}$, there exists N such that $f_{N,k*}=0: H_n((X_{k-1}, n); Z_q) \to H_n((X_N, n); Z_q)$, where $H_n((X_j, n); Z_q) \cong \pi_n(X_j) \otimes Z_q$ for any j. So it follows that $f_{N,k*} \otimes 1: \pi_n(X_{k-1}) \otimes Z_q \to \pi_n(X_N) \otimes Z_q$ is trivial.

Conversely, for any n, take sufficiently large m, then $H_n((m, X_i); Z_q) \cong H_n(X_i; Z_q)$. So the condition 2) in Definition 1.1 follows from 1)′ and 2)′ of Definition 1.1′ by Lemma 1.4′. Q. E. D.

REMARK 1.1″. In the Definitions 1.1 and 1.1′, the condition that q is a prime with $(q, p)=1$ for all $p \in \boldsymbol{P}$ can be replaced by that q is an integer with $(q, p)=1$ for all $p \in \boldsymbol{P}$.

From now on we call the homology $\boldsymbol{P}$-sequence (equivalently the homotopy $\boldsymbol{P}$-sequence) merely *the $\boldsymbol{P}$-sequence* by virtue of Theorem 1.2.

DEFINITION 1.5. Let $\{X_i, q_i\}$ and $\{Y_i, h_i\}$ be $\boldsymbol{P}$-sequences of X and Y respectively, and let $f: X \to Y$ be a map. *A morphism* $\{f_i\}$ between two sequences: $\{X_i, g_i\} \to \{Y_i, h_i\}$ covering f is defined as follows: For any i, there exist $\rho(i)$ $(\geqq \rho(i-1))$ and maps $f_i: X_i \to Y_{\rho(i)}$ such that $f_0=f: X \to Y$ and the following diagram is homotopy commutative.

$$\begin{CD} X_{i-1} @>{g_i}>> X_i \\ @V{f_{i-1}}VV @VV{f_i}V \\ Y_{\rho(i-1)} @>>{h_{\rho(i,i-1)}}> Y_{\rho(i)} \end{CD}$$

where $h_{\rho(i,i-1)} = h_{\rho(i)} \circ \cdots \circ h_{\rho(i-1)+1}$.

DEFINITION 1.6. Let $\{f_i\}$ and $\{f'_i\}$ be two morphisms between $\boldsymbol{P}$-sequences: $\{X_i, g_i\} \to \{Y_i, h_i\}$. Then $\{f_i\}$ and $\{f'_i\}$ are homotopic, if there exists a morphism $\{H_i\} : \{X_i \times I, g_i \times 1\} \to \{Y_{\varphi(i)}, h_{\varphi(i)}\}$ covering the homotopy $f \sim f'$ with $\varphi(i) \geqq \operatorname{Max}(\varphi(i-1), \rho'(i), \rho(i))$ such that

1) $H_i(\ , 0) = f_i$ and $H_i(\ , 1) = f'_i$ in $Y_{\varphi(i)}$,

2) $H_{i+1} \circ (g_i \times 1) \cong h_{\varphi(i)} \circ H_i$ rel. $X_i \times \partial I$.

PROPOSITION 1.7. *Let $\{X_i, g_i\}$ and $\{Y_i, h_i\}$ be $\boldsymbol{P}$-sequences of X and Y respectively. Let $X_i \in \mathfrak{F}\mathfrak{C}_1$. Let $f: X \to Y$ be arbitrary. Then there exists a morphism $\{f_i\} : \{X_i\} \to \{Y_{\rho(i)}\}$ covering f. Further, it is unique up to homotopy.*

PROOF. We prove it by induction starting with $f_0 = f$. Assume that $f_k : X_k \to Y_{\rho(k)}$ is constructed;

$$\begin{CD} X_k @>{g_{k+1}}>> X_{k+1} \\ @V{f_k}VV @VVV \\ Y_{\rho(k)} @>>{h_{\rho(k+1,k)}}> Y_{\rho(k+1)} \end{CD}$$

We may consider that g_{k+1} is an inclusion of a subcomplex by taking a mapping cylinder, if necessary. The obstruction to extending f_k over X_{k+1} lies in $H^{i+1}(X_{k+1}, X_k ; \pi_i(Y_{\rho(k)}))$. Remark that $H^*(X_{k+1}, X_k) \in \mathfrak{C}_{\boldsymbol{P}}$, since g_{k+1} is a $\boldsymbol{P}$-equivalence. We assume that f_k is already extended over $(X_{k+1}, X_k)^{(r)}$ in Y_{N_r} for some $N_r \geqq \rho(k)$. Then the obstruction to extending over $(X_{k+1}, X_k)^{(r+1)}$ lies in $H^{r+1}(X_{k+1}, X_k ; \pi_r(Y_{N_r}))$. Then by the condition 2)′ in Definition 1.1′, the obstruction is zero in $Y_{N_{r+1}}$ for some $N_{r+1} \geqq N_r$. Since X_{k+1} is finite dimensional, we obtain a map $f_{k+1}: X_{k+1} \to X_{\rho(k+1)}$ extending f_k. The uniqueness up to homotopy can be proved quite similarly. Q. E. D.

DEFINITION 1.8. *$\{X_i, g_i\}$ is homotopy equivalent to $\{Y_i, h_i\}$*, if there exist morphisms $f_i : \{X_i, g_i\} \to \{Y_i, h_i\}$ and $f'_i : \{Y_i, h_i\} \to \{X_i, g_i\}$ such that morphisms $\{f'_{\rho(i)} \circ f_i\}$ and $\{f_{\varphi(i)} \circ f'_i\}$ cover 1_X and 1_Y respectively.

THEOREM 1.9. (1) *For any subset $\boldsymbol{P}$ of the set of all primes and for any X, there exists a $\boldsymbol{P}$-sequence $\{X_i\}$ of X, where $X_i \in \mathfrak{F}\mathfrak{C}_1$, if $X \in \mathfrak{F}\mathfrak{C}_1$.*

(2) *It is unique up to homotopy type, if $X_i \in \mathfrak{F}\mathfrak{C}_1$.*

Before proving, let us recall the notion of the fibred sum (or the pushout) of CW-complexes. Given a diagram of CW-complexes

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Y \\ Z & \end{tikzcd}$$

construct a CW-complex $Y \bigvee_X Z = Y \bigcup_f (X \times I) \bigcup_g Z$ by identifying $(x, 0) \sim f(x)$ and $(x, 1) \sim g(x)$. Let $j_1 : Y \to Y \bigvee_X Z$ and $j_2 : Z \to Y \bigvee_X Z$ be the natural inclusions. Clearly $j_1 \circ f \cong j_2 \circ g$. Let W be another CW-complex, and let $a : Y \to W$ and $b : Z \to W$ be maps such that $a \circ f \cong b \circ g$. Then there exists a map $w : Y \bigvee_X Z \to W$ such that the following is homotopy commutative:

$$\begin{tikzcd} & Y \arrow[dr, "j_1"] \arrow[drr, "a"] & & \\ X \arrow[ur, "f"] \arrow[dr, "g"'] & & Y \bigvee_X Z \arrow[r, "w"] & W \\ & Z \arrow[ur, "j_2"'] \arrow[urr, "b"'] & & \end{tikzcd}$$

LEMMA 1.10. *f is a $\boldsymbol{P}$-equivalence if and only if j_2 is a $\boldsymbol{P}$-equivalence. Similarly for g and j_1.*

PROOF. Clearly the cofibre of g and j_1 are naturally homotopy equivalent. So it follows from the five lemma. Q. E. D.

(PROOF OF THEOREM 1.9.)

1) It suffices to construct a homotopy $\boldsymbol{P}$-sequence. Let $i \geqq 2$ and q be a given prime with $(q, p) = 1$ for all $p \in \boldsymbol{P}$. Consider a $\boldsymbol{P}$-equivalence $f : X \to X'$, which induces $f_* \otimes 1 : \pi_i(X) \otimes Z_q \to \pi_i(X') \otimes Z_q$. Let $g_j : S^i \to X'$, $j \in J$, be representatives of a basis for the image of $f_* \otimes 1$. Let $\bigvee_J S^i$ be a bouquet of spheres and put $g = \bigvee_J g_j : \bigvee_J S^i \to X'$. Let $q : \bigvee_J S^i \to \bigvee_J S^i$ be a map such that it is of degree q on each S^i. Take $X_{q,i} = \bigvee S^i \bigvee_{\vee S^i} X'$ the fibred sum of g and q. Then the map $\bar{f} = j_{X'} \circ f : X \to X_{q,i}$ is a $\boldsymbol{P}$-equivalence by Lemma 1.10 and it induces $\bar{f}_* \otimes 1 = 0 : \pi_i(X) \otimes Z_q \to \pi_i(X_{q,i}) \otimes Z_q$. Now consider the set I of triples (i, q, r) for all $i \geqq 2$, all $r \geqq 1$ and all primes q with $(q, p) = 1$ for any $p \in \boldsymbol{P}$. We then give I a linear order. Starting with the identity map $1_X : X \to X$, we iterate the above construction for every pair (i, q) of I in that order. Then we can obtain a $\boldsymbol{P}$-sequence of X. Remark that $X_i \in \mathfrak{F}\mathfrak{C}_1$, if $X \in \mathfrak{F}\mathfrak{C}_1$.

2) Let $\{X_i\}$ be a $\boldsymbol{P}$-sequence of X with $X_i \in \mathfrak{F}\mathfrak{C}_1$. By the construction of the "telescope" of Adams [2], we may assume that X_i is a subcomplex of X_{i+1}. Then let $\bigcup_i X_i$ be the union of X_i and let $j_X : X = X_0 \to \cup X_i$ be the

natural inclusion. Let Y be another space and $\{Y_i\}$ a $\boldsymbol{P}$-sequence of Y. Let $f: X \to Y$ be a given map. Then by Proposition 1.7, there exists a morphism $\{f_i\}: \{X_i\} \to \{Y_{\rho(i)}\}$. So it induces a map $\bar{f}: \cup X_i \to \cup Y_i$ compatible with f. Furthermore such $\bar{f}$ is unique up to homotopy. In particular, taking $X = Y$ and $f = 1_X$, (so $Y_i \in \mathfrak{F}\mathfrak{C}_1$), we obtain a homotopy equivalence $1_X: \cup X_i \cong \cup Y_i$. Namely the complex $\cup X_i$ is unique up to homotopy type. Q.E.D.

§2. Localization of CW-complexes.

Let $\boldsymbol{P}$ be a given subset of the set of all primes. Let $X \in \mathfrak{F}\mathfrak{C}_1$ and let $\{X_i\}$ be a $\boldsymbol{P}$-sequence of X. We may assume that X_i is a subcomplex of X_{i+1} and $X_i \in \mathfrak{F}\mathfrak{C}_1$.

DEFINITION 2.1. *The localization of X at $\boldsymbol{P}$*, denoted by $X_{\boldsymbol{P}}$, is defined to be $X_{\boldsymbol{P}} = \cup X_i$. For a map $f: X \to Y$, where $X \in \mathfrak{F}\mathfrak{C}_1$, the induced map is denoted by $l_{\boldsymbol{P}}(f): X_{\boldsymbol{P}} \to Y_{\boldsymbol{P}}$, or sometimes by $f_{\boldsymbol{P}}$, if there is no misunderstanding.

By Theorem 1.9, $X_{\boldsymbol{P}}$ is determined up to homotopy type. Also by Proposition 1.7 $l_{\boldsymbol{P}}(f): X_{\boldsymbol{P}} \to Y_{\boldsymbol{P}}$ is unique up to homotopy.

Let $X \in \mathfrak{C}_1$. Denote the n-skeleton of X by $X^{(n)}$, which is a finite complex for all n. Then $X_{\boldsymbol{P}}^{(n)}$ is uniquely determined up to homotopy type. There is a natural map $X_{\boldsymbol{P}}^{(n)} \to X_{\boldsymbol{P}}^{(n+1)}$ induced from the inclusion $X^{(n)} \to X^{(n+1)}$.

DEFINITION 2.2.

$$X_{\boldsymbol{P}} = \varinjlim_{n} X_{\boldsymbol{P}}^{(n)}.$$

Let $f: X \to Y$ be a given map. Then we may assume that f is cellular, i.e. $f^{(n)}: X^{(n)} \to Y^{(n)}$. Hence it induces $l_{\boldsymbol{P}}(f^{(n)}): X_{\boldsymbol{P}}^{(n)} \to Y_{\boldsymbol{P}}^{(n)}$, which is unique up to homotopy by Proposition 1.7. Thus we obtain a map $l_{\boldsymbol{P}}(f): X_{\boldsymbol{P}} \to Y_{\boldsymbol{P}}$.

NOTATION. When $\boldsymbol{P}$ consists of one prime p, we denote $X_{\boldsymbol{P}} = X_{(p)}$.

When $\boldsymbol{P} = \phi$, the vacant set, we denote $X_{\boldsymbol{P}} = X_{(0)}$.

PROPOSITION 2.3. *Let $X, Y \in \mathfrak{C}_1$.*

(1) *$X_{\boldsymbol{P}}$ is determined uniquely up to homotopy type.*

(2) *$f: X \to Y$ induces a map $l_{\boldsymbol{P}}(f): X_{\boldsymbol{P}} \to Y_{\boldsymbol{P}}$, which is unique up to homotopy.*

The proof is obvious.

THEOREM 2.4. *The localization at $\boldsymbol{P}$ has the following properties:*

(1) *The correspondence $X \to X_{\boldsymbol{P}}$ is a functor from the homotopy category of 1-connected CW-complexes of finite type to the homotopy category of 1-connected countable CW-complexes.*

(2) *There exists a natural inclusion $j_X: X \to X_{\boldsymbol{P}}$.*

(3) *If $f: X \to Y$ is a $\boldsymbol{P}$-equivalence, then $f_{\boldsymbol{P}}: X_{\boldsymbol{P}} \to Y_{\boldsymbol{P}}$ is a homotopy equivalence.*

PROOF. (1) is Proposition 2.3. (2) is clear from the construction. (3) It suffices to prove it for X and Y of $\mathfrak{F}\mathfrak{C}_1$. Let $f: X \to Y$ be a $\boldsymbol{P}$-equivalence and $\{Y_i\}$ a $\boldsymbol{P}$-sequence of Y. Then $X \to Y = Y_0 \to Y_1 \to Y_2 \to \cdots$ is also a $\boldsymbol{P}$-sequence of X. Then by the uniqueness of localization of X, we have that $X_{\boldsymbol{P}} = Y_{\boldsymbol{P}}$, i.e., $l_{\boldsymbol{P}}(f): X_{\boldsymbol{P}} \to Y_{\boldsymbol{P}}$ is a homotopy equivalence. Q. E. D.

THEOREM 2.5. $X \in \mathfrak{C}_1$. *Let* $j_X: X \to X_{\boldsymbol{P}}$ *be the inclusion.*

(1) $H_*(X_{\boldsymbol{P}}) \cong H_*(X) \otimes Q_{\boldsymbol{P}}$. *Moreover* $j_{X*}: H_*(X) \to H_*(X_{\boldsymbol{P}})$ *is equivalent to* $1 \otimes j: H_*(X) \otimes Z \to H_*(X) \otimes Q_{\boldsymbol{P}}$, *where* $j: Z \to Q_{\boldsymbol{P}}$ *is the canonical inclusion.*

(2) $\pi_*(X_{\boldsymbol{P}}) \cong \pi_*(X) \otimes Q_{\boldsymbol{P}}$. *Moreover* $j_{X*}: \pi_*(X) \to \pi_*(X_{\boldsymbol{P}})$ *is equivalent to* $1 \otimes j: \pi_*(X) \otimes Z \to \pi_*(X) \otimes Q_{\boldsymbol{P}}$.

PROOF. It suffices to prove (1), since the argument is quite same for the homotopy functor.

Consider the homomorphism

$$j_{X*} \otimes 1: H_*(X) \otimes Q_{\boldsymbol{P}} \longrightarrow H_*(X_{\boldsymbol{P}}) \otimes Q_{\boldsymbol{P}}.$$

We note that $H_*(X_{\boldsymbol{P}}) \cong \varinjlim_i H_*(X_i)$ and that $j_{X*}: H_*(X) \to H_*(X_{\boldsymbol{P}})$ is equivalent to the canonical inclusion: $H_*(X_0) \to \varinjlim_i H_*(X_i)$. Since $\varinjlim_i$ and $Q_{\boldsymbol{P}}$ commute, we have that

$$\begin{aligned} H_*(X_{\boldsymbol{P}}) \otimes Q_{\boldsymbol{P}} &= (\varinjlim_i H_*(X_i)) \otimes Q_{\boldsymbol{P}} \\ &= \varinjlim_i (H_*(X_i) \otimes Q_{\boldsymbol{P}}). \end{aligned}$$

Obviously $f_{i*}: H_*(X_{i-1}) \to H_*(X_i)$ is $\mathfrak{C}_{\boldsymbol{P}}$-isomorphic, since $f_{i*}: H_*(X_{i-1}; Z_p) \to H_*(X_i; Z_p)$ is isomorphic, and since $H_*(X_j)$ is of finite type for all j. Therefore $f_{i*} \otimes 1: H_*(X_{i-1}) \otimes Q_{\boldsymbol{P}} \to H_*(X_i) \otimes Q_{\boldsymbol{P}}$ is isomorphic, and hence $\varinjlim_i (H_*(X_i) \otimes Q_{\boldsymbol{P}}) \cong H_*(X) \otimes Q_{\boldsymbol{P}}$. Now we will prove that $1 \otimes j: H_*(X_{\boldsymbol{P}}) \otimes Z \to H_*(X_{\boldsymbol{P}}) \otimes Q_{\boldsymbol{P}}$ is isomorphic. Take an arbitrary element α from $H_*(X_{\boldsymbol{P}}) \otimes Q_{\boldsymbol{P}} \cong \varinjlim_i (H_*(X_i) \otimes Q_{\boldsymbol{P}})$ and let $x \otimes \frac{n}{m} \in H_*(X_i) \otimes Q_{\boldsymbol{P}}$ be a representative of α, where m is an integer with $(m, p) = 1$ for all $p \in \boldsymbol{P}$. By the condition 2) of Definition 1.1, there exists an integer N such that $(f_{N,i+1})_* x = my$ for some $y \in H_*(X_N)$, where $(f_{N,i+1})_*: H_*(X_i) \to H_*(X_N)$. Then $(1 \otimes j)(y \otimes n) = x \otimes \frac{n}{m}$. Thus $1 \otimes j$ is epimorphic. Suppose that $(1 \otimes j)(x \otimes 1) = 0$ in $H_*(X_{\boldsymbol{P}}) \otimes Q_{\boldsymbol{P}}$. Clearly x is of order d, where $(d, p) = 1$ for any prime p of $\boldsymbol{P}$. Let $x_m \in H_i(X_m)$ be a representative of x. Then there exists an element $x'_m \in H_{i+1}(X_m; Z_d)$ such that $\partial x'_m = x_m$ where $\partial: H_{i+1}(X_m; Z_d) \to H_i(X_m)$, since x_m is of order d. By the definition of the $\boldsymbol{P}$-sequence, there exist N and a $\boldsymbol{P}$-equivalence $f_{m+N,m+1}: X_m \to X_{m+N}$ such that $(f_{m+N,m+1})_* = 0: H_{i+1}(X_m; Z_d) \to H_{i+1}(X_{m+N}; Z_d)$. By naturality we obtain that $(f_{m+N,m+1})_*(x_m)=0$, and hence $x=\{x_m\}=\{(f_{m+N,m+1})_*(x_m)\}=0$. Thus $1\otimes j$ is monomorphic. Then we have the following commutative diagram:

$$\begin{array}{ccc} H_*(X)\cong H_*(X)\otimes Z & \xrightarrow{1\otimes j} & H_*(X)\otimes Q_P \\ \downarrow j_{X*}\otimes 1 & & \cong \downarrow j_{X*}\otimes 1 \\ H_*(X_P)\cong H_*(X_P)\otimes Z & \xrightarrow[\cong]{1\otimes j} & H_*(X_P)\otimes Q_P \end{array}$$

Thus $j_{X*}: H_*(X)\to H_*(X_P)$ is equivalent to $1\otimes j: H_*(X)\otimes Z\to H_*(X)\otimes Q_P$.

Q. E. D.

REMARK 2.6. For $X\in\mathfrak{C}_1$, we can construct a $\boldsymbol{P}$-sequence $\{X_i, f_i\}$ of X in such a way that $X_i\in\mathfrak{C}_1$ for all i (cf. Theorem 1.9). This fact is used in the above proof.

THEOREM 2.7. *Let $\boldsymbol{P}\subset\boldsymbol{Q}$ be given subsets of the set of all primes. Then there exists a map $j_{P,Q}: X_Q\to X_P$ satisfying the following properties:*

(1) *$j_{P,Q}$ is a $\boldsymbol{P}$-equivalence.*

(2) *If $\boldsymbol{Q}$ is the set of all primes (and hence $X_Q=X$), then $j_{P,Q}: X_Q\to X_P$ coincides with the canonical inclusion.*

(3) *For $\boldsymbol{P}\subset\boldsymbol{Q}\subset\boldsymbol{R}$, $j_{P,Q}\circ j_{Q,R}\cong j_{P,R}$.*

(4) *Let $X\in\mathfrak{F}\mathfrak{C}_1$. Then an arbitrary map $f: X_Q\to Y_Q$ induces $f_P: X_P\to Y_P$ such that the following diagram commutes up to homotopy:*

$$\begin{array}{ccc} X_Q & \xrightarrow{f} & Y_Q \\ j_{P,Q}\downarrow & & \downarrow j_{P,Q} \\ X_P & \xrightarrow{f_P} & Y_P \end{array}$$

The proof is quite easy and left to the reader.

DEFINITION 2.8. Let $X, Y\in\mathfrak{C}_1$. We define that X and Y *have the same $\boldsymbol{P}$-type*, if there exist $Z\in\mathfrak{C}_1$ and two $\boldsymbol{P}$-equivalences $f: X\to Z$ and $g: Y\to Z$.

PROPOSITION 2.9. *If X and Y have the same $\boldsymbol{P}$-type, then X_P is homotopy equivalent to Y_P.*

Further if either X or $Y\in\mathfrak{F}\mathfrak{C}_1$, then the converse is true.

If we denote by g_P^{-1} the homotopy inverse of the homotopy equivalence g_P, then a homotopy equivalence from X_P to Y_P is given by $g_P^{-1}\circ f_P$.

§ 3. Further properties of localization.

Let X, Y and $Z\in\mathfrak{C}_1$.

THEOREM 3.1. (1) *If $X\xrightarrow{f} Y\xrightarrow{g} Z$ is a cofibering, then $X_P\xrightarrow{f_P} Y_P\xrightarrow{g_P} Z_P$ is homotopy equivalent to a cofibering.*

(2) *If* $X \xrightarrow{f} Y \xrightarrow{g} Z$ *is a fibering, then* $X_P \xrightarrow{f_P} Y_P \xrightarrow{g_P} Z_P$ *is homotopy equivalent to a fibering.*

PROOF. (1) Let $C(f_P)$ be the cofiber of f_P and let $j: Y_P \to C(f_P)$ be the projection. Then there exists a map $h: C(f_P) \to Z_P$ such that g_P is homotopic to $h \circ j: Y_P \to C(f_P) \to Z_P$. Let $Z \xrightarrow{\pi} SX$ be the canonical boundary map. Then π induces $\pi_P: Z_P \to S(X_P)$, since clearly $(SX)_P = S(X_P)$ holds. (More general formula will be proved below.) Consider the homology exact sequence:

$$\cdots \longrightarrow H_i(X) \longrightarrow H_i(Y) \longrightarrow H_i(Z) \xrightarrow{\partial} H_{i-1}(X) \longrightarrow \cdots,$$

and hence we have an exact sequence by Theorem 2.5

$$\cdots \longrightarrow H_i(X_P) \longrightarrow H_i(Y_P) \longrightarrow H_i(Z_P) \longrightarrow H_{i-1}(X_P) \longrightarrow \cdots,$$

since tensoring Q_P is an exact functor. So by the five lemma we obtain that $h_*: H_i(C(f_P)) \to H_i(Z_P)$ is an isomorphism for all i. Thus $C(f_P)$ is homotopy equivalent to Z_P. (2) can be proved quite similarly. Q. E. D.

COROLLARY 3.2.

(1) $(X \times Y)_P = X_P \times Y_P$.

(2) $(X \wedge Y)_P = X_P \wedge Y_P$.

(3) $(X \vee Y)_P = X_P \vee Y_P$.

PROPOSITION 3.3.

(1) $X_P \wedge Y = (X \wedge Y)_P$.

(2) $(\Omega X)_P = \Omega(X_P)$, *if* X *is* 2-*connected.*

PROOF. (1) will be obtained by making use of the Künneth formula. (2) Let $\{X_i, f_i\}$ be a $\boldsymbol{P}$-sequence of X. Then $\{\Omega X_i, \Omega f_i\}$ can be a $\boldsymbol{P}$-sequence of ΩX. Q. E. D.

Let $K, X \in \mathfrak{C}_1$. We denote by $[K, X]$ the set of homotopy classes of maps: $K \to X$. Recall that $[K, X]$ is an abelian group, if K is a double suspended space. The canonical map $j_p: X \to X_{(p)}$ induces then a homomorphism $j_{p*}: [K, X] \to [K, X_{(p)}]$. Then we have

THEOREM 3.4. *Let* $K, X \in \mathfrak{F}\mathfrak{C}_1$. *Assume that* K *is a double suspended space. Then an element* α *of* $[K, X]$ *is trivial if and only if* $j_{p*}(\alpha) = 0$ *in* $[K, X_{(p)}]$ *for every prime* p.

The proof is an application of the theory of finitely generated abelian groups. (cf. Theorem 4.7.)

§ 4. The pull-back of localized spaces

The purpose of this section is to reconstruct the original space X from its localized spaces X_P.

Let $\boldsymbol{P}_i$, $i \in I$, be subsets of the set of all primes. Put $\boldsymbol{P}=\bigcap_I \boldsymbol{P}_i$ and $\bar{\boldsymbol{P}}=\bigcup_I \boldsymbol{P}_i$. Then by Theorem 2.6 there are canonical maps $\bar{\varphi}_i: X_{\bar{P}} \rightarrow X_{P_i}$, $\varphi_i: X_{P_i} \rightarrow X_P$ and $\varphi: X_{\bar{P}} \rightarrow X_P$ according to the inclusions $\boldsymbol{P} \rightarrow \boldsymbol{P}_i \rightarrow \bar{\boldsymbol{P}}$. In particular, for any set $\boldsymbol{Q}$, there is a canonical map $\varphi_Q: X_Q \rightarrow X_{(0)}$, where $X_{(0)}$ is the localization at ϕ, the vacant set ($\boldsymbol{Q} \supset \phi$). Let us denote by $\prod_{X_P} X_{P_i}$ the pull-back (or the fibred product) of φ_i over X_P. In the below, let $X \in \mathfrak{C}_1$.

THEOREM 4.1. *$\prod_{X_P} X_{P_i}$ is homotopy equivalent to $X_{\bar{P}}$.*

PROOF. It suffices to prove the theorem for $I=\{1,2\}$. We use the above notations. By the property of the fibred product, there exists a map $f: X_{\bar{P}} \rightarrow \prod_{X_P} X_{P_i}$ such that the following diagram is homotopy commutative:

$$\begin{tikzcd}
X_{\bar{P}} \arrow[rr, "\bar{\varphi}_1"] \arrow[dr, "f"] \arrow[dd, "\bar{\varphi}_2"'] & & X_{P_1} \arrow[dd, "\varphi_1"] \\
& \prod_{X_P} X_{P_i} \arrow[ur, "q_1"'] \arrow[dl, "q_2"] & \\
X_{P_2} \arrow[rr, "\varphi_2"'] & & X_P
\end{tikzcd}$$

where $q_i: \prod_{X_P} X_{P_i} \rightarrow X_{P_i}$ is the projection to the ingredient. We will show that the map f induces an isomorphism $f_*: \pi_j(X_{\bar{P}}) \rightarrow \pi_j(\prod_{X_P} X_{P_i})$ for all j. Let $\alpha \in \pi_j(X_{\bar{P}}) \cong \pi_j(X) \otimes Q_{\bar{P}}$ be an element such that $f_*(\alpha)=0$. Then $\bar{\varphi}_{1*}(\alpha)=q_{1*}f_*(\alpha)=0$, so α is a torsion element of order prime to $\boldsymbol{P}_1$. Similarly it is shown that α is of order prime to $\boldsymbol{P}_2$. Hence α is of order prime to $\bar{\boldsymbol{P}}$. Namely, $\alpha=0$ in $\pi_j(X) \otimes Q_{\bar{P}} \cong \pi_j(X_{\bar{P}})$. Next we show that f_* is epimorphic. To that end, we decompose $\pi_j(X)$ in the following way:

$$\pi_j(X) \cong F+T_P+T_{P_1-P}+T_{P_2-P}+T',$$

where F is a free subgroup, T_P, T_{P_1-P}, T_{P_2-P} are $\boldsymbol{P}$-torsion, $(\boldsymbol{P}_1-\boldsymbol{P})$-torsion, $(\boldsymbol{P}_2-\boldsymbol{P})$-torsion subgroups respectively, and T' is the other torsion subgroup. Let α be an arbitrary element of $\pi_j(\prod_{X_P} X_{P_i})$. Then $\varphi_{1*}q_{1*}(\alpha)=\varphi_{2*}q_{2*}(\alpha)$, since $\varphi_1 \circ q_1 = \varphi_2 \circ q_2$. So we can write down as

$$q_{1*}(\alpha)=\frac{n}{m}\alpha_1+\alpha_2+x, \qquad x \in T_{P_1-P},$$

$$q_{2*}(\alpha)=\frac{n}{m}\alpha_1+\alpha_2+y\,, \qquad y\in T_{P_2-P}\,,$$

where $\frac{n}{m}\in Q_{\bar{P}}$, $\alpha_1\in F$, $\alpha_2\in T_P$.

Put $\beta=\frac{n}{m}\alpha_1+\alpha_2+x+y\in\pi_j(X)\otimes Q_{\bar{P}}=\pi_j(X_{\bar{P}})$. Then it is obvious that $f_*(\beta)=\alpha$. In fact, $\pi_i(\prod_{X_P} X_{P_j})$ has only $\bar{P}$-torsion. Q. E. D.

COROLLARY 4.2. *Let P be a subset of the set of all primes. Let $\bar{P}$ be the complement of P in the set. Then $X_P\underset{X_{(0)}}{\times}X_{\bar{P}}$ is homotopy equivalent to X.*

More generally,

COROLLARY 4.3. *Let $\bigcup_i P_i$ be a disjoint decomposition of the set of all primes. Then $\prod^{i}_{X_{(0)}} X_{P_i}$ is homotopy equivalent to X. In particular, X is homotopy equivalent to $\prod^{p}_{X_{(0)}} X_{(p)}$, the pull-back of $\varphi_p: X_{(p)}\to X_{(0)}$ over $X_{(0)}$ for all primes.*

COROLLARY 4.4. *X is homotopy equivalent to Y if and only if there exists a map $f: X\to Y$ inducing a homotopy equivalence $l_{(p)}(f): X_{(p)}\to Y_{(p)}$ for all primes p.*

THEOREM 4.5. *Let $X, Y\in\mathfrak{F}\mathfrak{C}_1$, and let P and Q be two subsets of the set of all primes. Assume that we are given a $P\cap Q$-equivalence $f: X\to Y$. Then there exist a space Z and a Q-equivalence $g: X_{P\cup Q}\to Z$ and a P-equivalence $h: Z\to Y_{P\cup Q}$ such that $f_{P\cup Q}=h\circ g$. Further, $Z\in\mathfrak{F}\mathfrak{C}_1$, if $P\cup Q$ is the set of all primes.*

PROOF. It follows from Theorem 2.4 that $f_{P\cap Q}: X_{P\cap Q}\to Y_{P\cap Q}$ is a homotopy equivalence. Let $w_P: X_P\to X_{P\cap Q}$ and $w_Q: Y_Q\to Y_{P\cap Q}$ be the canonical maps obtained by Theorem 2.7. Denote by $Z=X_P\underset{Y_{P\cap Q}}{\times}Y_Q$ the pull-back of $f_{P\cap Q}\circ w_P$ and w_Q over $Y_{P\cap Q}$. Then the rest of the proof is clear from the construction of Z. Q. E. D.

Similarly one can prove

THEOREM 4.6. (Mixing homotopy type.) (cf. [23].) *Let $\bigcup_i P_i$, $i\in I$, be a disjoint decomposition of the set of all primes. Let $X_i\in\mathfrak{C}_1$, $i\in I$, satisfy that $(X_i)_{(0)}$ is of same homotopy type for all $i\in I$. Then there exists $X\in\mathfrak{C}_1$ with a P_i-equivalence $X\to X_i$ for all $i\in I$. Furthermore $X\in\mathfrak{F}\mathfrak{C}_1$, if $X_i\in\mathfrak{F}\mathfrak{C}_1$ for all $i\in I$.*

In the above theorem, the finiteness of X, when $X_i\in\mathfrak{F}\mathfrak{C}_1$ for all $i\in I$, can be proved as follows. $H_*(X;Q)$ is finite dimensional, since $H_*(X_i;Q)$ is finite dimensional for all $i\in I$. Since $H_*(X_i;Z_p)$ is finite dimensional, so is $H_*(X;Z_p)$. Besides, the finite dimension has a common maximum number for Q and all primes p simultaneously. Hence $X\in\mathfrak{F}\mathfrak{C}_1$.

THEOREM 4.7. *Let $X, Y\in\mathfrak{F}\mathfrak{C}_1$. Then an element α of $[SX, Y]$ is trivial if and only if $j_{p*}(\alpha)=0$ in $[SX, Y_{(p)}]$ for every prime p, where $j_p: Y\to Y_{(p)}$ is the canonical map of localization at p.*

PROOF. The necessity is clear. We prove the sufficiency. Let p and q be primes with $(p, q)=1$. Consider the following diagram:

$$\begin{tikzcd}
\Omega Y_{(p,q)} \arrow[rrr] \arrow[d] & & & \Omega Y_{(p)} \arrow[d, "\Omega(w_p)"] \\
\Omega Y_{(q)} \arrow[rrr, "\Omega(w_q)"] \arrow[d, "b'"] & & & \Omega Y_{(0)} \arrow[d, "b"] \\
F_p \arrow[rrr, "1"] \arrow[d, "a'"] & & & F_p \arrow[d, "a"] \\
Y_{(p,q)} \arrow[rrr, "w'_q"] \arrow[ddd, "w'_p"'] & & & Y_{(p)} \arrow[ddd, "w_p"] \\
 & Y \arrow[ul, "j_{p,q}"'] \arrow[urr, "j_p"] \arrow[ddl, "j_q"] & & \\
 & & SX \arrow[ul, "\alpha"'] & \\
Y_{(q)} \arrow[rrr, "w_q"'] & & & Y_{(0)}
\end{tikzcd}$$

where the two vertical sequences are fiberings associated with the fibred product in the bottom square and j_p, j_q, $j_{p,q}$ are canonical inclusions. Similarly for w_p, w'_p, w_q, w'_q. ($Y_{(p,q)}$ denotes the localization at $\{p, q\}$). First we assume $j_{p*}(\alpha)=j_{q*}(\alpha)=0$. Then there exists a map $f\colon SX\to F_p$ such that $a'\circ f\cong j_{p,q}\circ\alpha$, since $w'_p\circ j_{p,q}\circ\alpha\cong j_q\circ\alpha\cong 0$. Also there exists a map $g\colon SX\to\Omega Y_{(0)}$ such that $b\circ g\cong f$, since $a\circ f\cong w'_q\circ a'\circ f\cong w'_q\circ j_{p,q}\circ\alpha\cong j_p\circ\alpha\cong 0$. It satisfies that $j_{p,q}\circ\alpha\cong a'\circ f\cong a'\circ b\circ g$. Next consider the commutative diagram of abelian groups:

$$\begin{array}{ccc}
[SX, \Omega Y_{(p,q)}] & \xrightarrow{(\Omega w'_q)_*} & [SX, \Omega Y_{(p)}] \\
\big\downarrow (\Omega w'_p)_* & & \big\downarrow (\Omega w_p)_* \\
[SX, \Omega Y_{(q)}] & \xrightarrow{(\Omega w_q)_*} & [SX, \Omega Y_{(0)}]
\end{array}$$

As is well known, it is equivalent to the following commutative one:

$$\begin{array}{ccc}
[S^2X, Y_{(p,q)}] & \xrightarrow{w'_{q*}} & [S^2X, Y_{(p)}] \\
\big\downarrow w'_{p*} & & \big\downarrow w_{p*} \\
[S^2X, Y_{(q)}] & \xrightarrow{w_{q*}} & [S^2X, Y_{(0)}] .
\end{array}$$

Then a simple computation shows that the cokernel of w'_{p*} is isomorphic to that of w_{p*}. So the relation $j_{p,q}\circ\alpha\cong a'\circ b\circ g$ implies that $j_{p,q}\circ\alpha\cong 0$. These arguments show that, if $j_p\circ\alpha\cong 0$ for every prime p of $\boldsymbol{P}$, then $j_{\boldsymbol{P}}\circ\alpha=0$ in $[SX, Y_{\boldsymbol{P}}]$. However, when $\boldsymbol{P}$ is the set of all primes, $Y\cong Y_{\boldsymbol{P}}$ and $\alpha\cong j_{\boldsymbol{P}}\circ\alpha\cong 0$. Q. E. D.

We end this section with the following

CONJECTURE 4.8. Let $X, Y\in\mathfrak{F}\mathfrak{C}_1$. Then an element α of $[X, Y]$ is trivial if and only if $j_{p*}(\alpha)=0$ in $[X, Y_{(p)}]$ for all primes p.

§ 5. Localizing $\boldsymbol{P}$-universal spaces.

Throughout this section we work in $\mathfrak{F}\mathfrak{C}_1$.

Let $\boldsymbol{P}$ be a subset of the set of all primes. Let us recall the following theorem which is essentially proved in [**12**].

THEOREM 5.1. *$K\in\mathfrak{F}\mathfrak{C}_1$ is $\boldsymbol{P}$-universal if and only if one of the following conditions is satisfied:*

(1) *For any prime q, $q\notin\boldsymbol{P}$, and for any $i>0$, there exists a $\boldsymbol{P}$-equivalence $f: K\to K$ such that the induced homomorphism $f_*: H_i(K; Z_q)\to H_i(K; Z_q)$ is trivial.*

(2) *For any prime q, $q\notin\boldsymbol{P}$, and for any $i>0$, there exists a $\boldsymbol{P}$-equivalence $f: K\to K$ such that the induced homomorphism $f_*\otimes 1: \pi_i(K)\otimes Z_q\to\pi_i(K)\otimes Z_q$ is trivial.*

DEFINITION 5.2. $K\in\mathfrak{F}\mathfrak{C}_1$ is called *$\boldsymbol{P}$-convertible*, if for any $L\in\mathfrak{F}\mathfrak{C}_1$ and for any $\boldsymbol{P}$-equivalence $h: K\to L$, there exists a converse $\boldsymbol{P}$-equivalence $k: L\to K$.

THEOREM 5.3. *Let $X\in\mathfrak{F}\mathfrak{C}_1$.*

(A) *Then the following four conditions are equivalent:*

(1) *X is $\boldsymbol{P}$-universal.*

(2) *There exists a $\boldsymbol{P}$-sequence $\{X_i\}$ of X such that $X_i=X$.*

(3) *$l_{\boldsymbol{P}}: [Y, X]\to[Y_{\boldsymbol{P}}, X_{\boldsymbol{P}}]$ is quasi-epic for any $Y\in\mathfrak{F}\mathfrak{C}_1$, that is, for any element $\alpha\in[Y_{\boldsymbol{P}}, X_{\boldsymbol{P}}]$, there exist a homotopy equivalence $h: X_{\boldsymbol{P}}\to X_{\boldsymbol{P}}$ and a map $g: Y\to X$ such that $l_{\boldsymbol{P}}(g)=h\circ\alpha$.*

(4) *X is $\boldsymbol{P}$-convertible.*

(B) *One of the above conditions implies the following*

(5) *$l_{\boldsymbol{P}}: [X, Y]\to[X_{\boldsymbol{P}}, Y_{\boldsymbol{P}}]$ is quasi-epic in the above sense.*

PROOF. (A). [(1) *implies* (2)]. Let $\{X_i, f_i\}$ be an arbitrary $\boldsymbol{P}$-sequence of X. By induction we will show that X_i can be replaced by X for all $i\geqq 0$. The case $i=0$ is trivial, since $X_0=X$. We should note here that $X_i\in\mathfrak{F}\mathfrak{C}_1$. Suppose $X_i=X$. Since $X=X_i$ is $\boldsymbol{P}$-universal, there exists a converse $\boldsymbol{P}$-equivalence $g_{i+1}: X_{i+1}\to X=X_i$ for a $\boldsymbol{P}$-equivalence $f_{i+1}: X=X_i\to X_{i+1}$. Then we can replace X_{i+1} by X via g_{i+1}.

[(2) *implies* (1)]. Suppose we are given a $\boldsymbol{P}$-sequence $\{X_i, f_i\}$ of X with $X_i = X$. Then by definition, for any $n>0$ and for any prime q, $q \notin \boldsymbol{P}$, there exists $i>0$ such that $(f_i \circ \cdots \circ f_1)_* = 0$: $H_n(X_0; Z_q) \to H_n(X_i; Z_q)$. So the $\boldsymbol{P}$-equivalence $f_i \circ \cdots \circ f_1$ satisfies (1) of Theorem 5.1.

[(2) *implies* (3)]. Let $\alpha \in [Y_{\boldsymbol{P}}, X_{\boldsymbol{P}}]$ be arbitrary and $f: Y \to X$ a representative of α. Let $j_Y: Y \to Y_{\boldsymbol{P}}$ be the canonical inclusion. Then there exists $i \geqq 0$ such that the composite map $f \circ j_Y: Y \to Y_{\boldsymbol{P}} \to X_{\boldsymbol{P}}$ is factored through X_i, since Y is a finite complex. Namely, there exists a map $g: Y \to X_i$ such that $f \circ j_Y \simeq j_i \circ g$, where $j_i: X_i \to X_{\boldsymbol{P}}$ is the obvious inclusion. Therefore $l_{\boldsymbol{P}}(g) = h \circ \alpha$ with some homotopy equivalence $h: X_{\boldsymbol{P}} \to X_{\boldsymbol{P}}$.

[(3) *implies* (4)]. Let $Y \in \mathfrak{F}\mathfrak{C}_1$ be given. Let $f: X \to Y$ be an arbitrary $\boldsymbol{P}$-sequence. Then by Theorem 2.4 $l_{\boldsymbol{P}}(f): X_{\boldsymbol{P}} \to Y_{\boldsymbol{P}}$ is a homotopy equivalence. Let $k: Y_{\boldsymbol{P}} \to X_{\boldsymbol{P}}$ be its homotopy inverse. Then by (3) there exists a map g: $Y \to X$ such that $l_{\boldsymbol{P}}(g)$ is a homotopy equivalence. Hence g is a $\boldsymbol{P}$-equivalence.

[(4) *implies* (2)]. This is just the same as in [(1) *implies* (2)].

(B). [(1) *implies* (5)]. Let $f: X_{\boldsymbol{P}} \to Y_{\boldsymbol{P}}$ be an arbitrary map. Let $\{Y_i, h_i\}$ be a $\boldsymbol{P}$-equivalence of Y. Since X is a finite complex, the composite $f \circ j_X: X \to Y_{\boldsymbol{P}}$ is factored through Y_i for some i, that is, there exists a map $g: X \to Y_i$ such that $f \circ j_X \simeq j_i \circ g$, where $j_i: Y_i \to Y_{\boldsymbol{P}}$ is the obvious inclusion. Now $h_i \circ \cdots \circ h_1$: $Y = Y_0 \to Y_i$ is a $\boldsymbol{P}$-equivalence. Since X is $\boldsymbol{P}$-universal, there exist a $\boldsymbol{P}$-equivalence $k: X \to X$ and a map $d: X \to Y$ such that the following diagram is homotopy commutative:

$$\begin{tikzcd} X \arrow[r, "k"] \arrow[d, "d"'] & X \arrow[r, "j_X"] \arrow[d, "g"'] & X_{\boldsymbol{P}} \arrow[d, "f"] \\ Y \arrow[r, "h_i \circ \cdots \circ h_1"'] & Y_i \arrow[r, "j_i"'] & Y_{\boldsymbol{P}} \end{tikzcd}$$

Thus there exists a homotopy equivalence $a: Y_{\boldsymbol{P}} \to Y_{\boldsymbol{P}}$ such that $l_{\boldsymbol{P}}(d) = a \circ f$.

Q. E. D.

COROLLARY 5.4. *In the category of* $\boldsymbol{P}$*-universal spaces,* X *and* Y *are* $\boldsymbol{P}$*-equivalent if and only if* $X_{\boldsymbol{P}}$ *and* $Y_{\boldsymbol{P}}$ *are homotopy equivalent.*

REMARK 5.5. Let X be $\boldsymbol{P}$-universal. Then $X_{\boldsymbol{P}}$ is a finite dimensional 1-connected CW-complex. Actually, the dimension of the telescope $\cup X_i =$ $\dim X+1$, since $X = X_i$.

THEOREM 5.6. *Let* X *be* $\boldsymbol{P}$*-universal. Then*

$$[S^n_{\boldsymbol{P}}, X_{\boldsymbol{P}}] \cong \pi_n(X) \otimes Q_{\boldsymbol{P}} \qquad \textit{for } n \geqq 2\,.$$

Before proving we state an easy lemma without proof:

LEMMA 5.7. *Let* A *be a* $\boldsymbol{Q_P}$*-module and let* B *be a finitely generated (as a* $\boldsymbol{Z}$*-module) abelian subgroup of* A*. Assume that, for each element* $x \in A$*, there* *exists m such that* $mx \in B$ *and* $(m, p) = 1$ *for any* $p \in \boldsymbol{P}$. *Then* $A \cong B \otimes Q_{\boldsymbol{P}}$.

(Proof of Theorem 5.6)

Consider the morphism $l_{\boldsymbol{P}}: [S^n, X] \to [S^n_{\boldsymbol{P}}, X_{\boldsymbol{P}}]$. Since $[S^n_{\boldsymbol{P}}, X_{\boldsymbol{P}}]$ is a $Q_{\boldsymbol{P}}$-module, the kernel of $l_{\boldsymbol{P}}$ contains a $\bar{\boldsymbol{P}}$-torsion subgroup of $[S^n, X]$, where $\bar{\boldsymbol{P}}$ denotes the complement of $\boldsymbol{P}$ in the set of all primes. Let $\{X_i, f_i\}$ be a $\boldsymbol{P}$-sequence of X. Take $\alpha \in [S^n, X] \cong \pi_n(X)$ such that $l_{\boldsymbol{P}}(\alpha) = 0$. Then there exists i such that the composite $f_i \circ \alpha: S^n \to X \to X_i$ is null homotopic. (Note that $X_i = X$, since X is $\boldsymbol{P}$-universal.) Thus α is a torsion element of order prime to $\boldsymbol{P}$. Therefore the kernel of $l_{\boldsymbol{P}}$ is isomorphic to a $\bar{\boldsymbol{P}}$-torsion subgroup of $\pi_n(X)$, and hence we obtain a monomorphism $l'_{\boldsymbol{P}}: \pi_n(X: \boldsymbol{P}) \to [S^n_{\boldsymbol{P}}, X_{\boldsymbol{P}}]$, where $\pi_n(X: \boldsymbol{P})$ denotes a $\boldsymbol{P}$-primary component of $\pi_n(X)$. The image of $l'_{\boldsymbol{P}}$ then satisfies the condition of Lemma 5.7, since X is $\boldsymbol{P}$-universal. Thus we get the theorem. Q. E. D.

§ 6. Mod $\boldsymbol{P}$ H-spaces and mod $\boldsymbol{P}$ co-H-spaces.

In this section we work in $\mathfrak{F}\mathfrak{C}_1$.

Definition 6.1. A pointed complex (X, e) is called *an H-space*, if there exists a map $\mu: X \times X \to X$ preserving a base point such that $\mu \circ i_1 \cong \mu \circ i_2 \cong 1_X$, where $i_j: X \to X \times X$ is the obvious inclusion. The map μ is called *a multiplication* or *an H-structure* on X. Let $\boldsymbol{P}$ be a subset of the set of all primes. X is called *a* mod $\boldsymbol{P}$ *H-space*, if $\mu \circ i_1 \cong \mu \circ i_2 \cong l$, which is a $\boldsymbol{P}$-equivalence. Similarly as above, μ is called *a* mod $\boldsymbol{P}$ *multiplication* or *a* mod $\boldsymbol{P}$ *H-structure* on X.

Dually we define

Definition 6.1'. A pointed complex (X, e) is called *a co-H-space*, if there exists a map $\varphi: X \to X \vee X$ preserving a base point such that $p_1 \circ \varphi \cong p_2 \circ \varphi \cong 1_X$, where $p_j: X \vee X \to X$ is the obvious projection. The map φ is called *a co-H-structure* on X. X is called *a* mod $\boldsymbol{P}$ *co-H-space*, if $p_1 \circ \varphi \cong p_2 \circ \varphi \cong l$, which is a $\boldsymbol{P}$-equivalence. The map φ is called *a* mod $\boldsymbol{P}$ *co-H-structure.*

Suppose we are given spaces X and Y and maps $k: X \to Y$ and $h: Y \to X$.

Definition 6.2. X is *dominated* (or $\boldsymbol{P}$*-dominated*) by Y, if the composite $h \circ k: X \to Y \to X$ is a homotopy equivalence (a mod $\boldsymbol{P}$ equivalence).

First we consider the localization at 0 of H-spaces. The following theorem is essentially due to Arkowitz-Curjel [**5**].

Theorem 6.3. *The following statements are equivalent.*

(1) X *is a* mod 0 *H-space.*

(2) $X_{(0)}$ *is an H-space.*

(3) $X_{(0)} = \prod_{i \in I} K(Q, n_i)$, *where I is a finite set and* n_i *is an odd integer.*

(4) *All k-invariants are of finite order in the Postnikov decomposition of* X.

Proof. The equivalence between (1) and (4) is just Theorem of [**5**].

Further according to Theorem of [5], X is a mod 0 H-space if and only if there exists a 0-equivalence $\prod_i S^{n_i} \to X$ with n_i odd, that is equivalent to that $X_{(0)} = \prod_i K(Q, n_i)$ by Theorem 2.4, since $S^{n_i}_{(0)} = K(Q, n_i)$. Now we show the equivalence between (1) and (2).

[(1) *implies* (2)]. By the assumption there exists a map $\mu: X \times X \to X$ such that $i_1 \circ \mu \cong i_2 \circ \mu \cong l$, which is a 0-equivalence. So by localizing we get that $i_{1(0)} \circ \mu_{(0)} \cong i_{2(0)} \circ \mu_{(0)} \cong l_{(0)}$, which is a homotopy equivalence of $X_{(0)}$. Since $X_{(0)}$ is a CW-complex, $X_{(0)}$ is an H-space by the Dold's theorem.

[(2) *implies* (1)]. Note that $X_{(0)}$ is rationally finite dimensional, since $H_*(X; Q) \cong H_*(X_{(0)})$ by Theorem 2.5. Hence $H^*(X; Q) \cong \Lambda(x_1, \cdots, x_r)$ with $\deg x_i$ odd. So by Theorem 2.5 of [11], X is 0-universal. Now by the assumption we have a multiplication $\mu: X_{(0)} \times X_{(0)} = (X \times X)_{(0)} \to X_{(0)}$. Since $l_0: [X \times X, X] \to [(X \times X)_{(0)}, X_{(0)}]$ is quasi-epic by Theorem 5.3, there exists a map $\bar{\mu}: X \times X \to X$ such that $\bar{\mu} \circ i_1 \cong \bar{\mu} \circ i_2$ is a 0-equivalence of X. Q. E. D.

Dually we have ([4]):

THEOREM 6.3$'$. *The following statements are equivalent.*

(1)$'$ *X is a* mod 0 *co-H-space.*

(2)$'$ *$X_{(0)}$ is a co-H-space.*

(3)$'$ $X_{(0)} = \bigvee_{i \in I} S^{n_i}_{(0)}$, *where I is a finite set.*

(4)$'$ *All k'-invariants are of finite order in the homology decomposition.*

Next we will discuss a mod $\boldsymbol{P}$ version of the above theorems.

THEOREM 6.4. *Let $X \in \mathfrak{F}\mathfrak{C}_1$. Then the following conditions are equivalent.*

(1) *X is a* mod $\boldsymbol{P}$ *H-space.*

(2) *$X_{\boldsymbol{P}}$ is an H-space.*

(3) *X is $\boldsymbol{P}$-dominated by a* mod $\boldsymbol{P}$ *H-space.*

PROOF. [(1) *implies* (2)]. We localize $\mu \circ i_1$ and $\mu \circ i_2$ at $\boldsymbol{P}$. Then they give a homotopy equivalence: $X_{\boldsymbol{P}} \to (X \times X)_{\boldsymbol{P}} = X_{\boldsymbol{P}} \times X_{\boldsymbol{P}} \to X_{\boldsymbol{P}}$. So it is easy to see that $X_{\boldsymbol{P}}$ is an H-space.

[(2) *implies* (1)]. By the assumption we have a multiplication $\mu': X_{\boldsymbol{P}} \times X_{\boldsymbol{P}} \to X_{\boldsymbol{P}}$. Now X is $\boldsymbol{P}$-universal by Theorem 2.5 of [11], since $H^*(X; Q) \cong H^*(X_{\boldsymbol{P}}; Q) \cong \Lambda(x_1, \cdots, x_l)$ with $\deg x_i$ odd. From Theorem 5.3 follows the existence of such a map $\mu: X \times X \to X$ that $\mu \circ i_1 \cong \mu \circ i_2 \cong h$, which is a $\boldsymbol{P}$-equivalence. Hence X is a mod $\boldsymbol{P}$ H-space.

[(1) *implies* (3)]. The proof is clear.

[(3) *implies* (1)]. Let Y be a mod $\boldsymbol{P}$ H-space dominating X with maps $k: X \to Y$ and $h: Y \to X$ such that $h \circ k$ is a $\boldsymbol{P}$-equivalence. Let $\mu: Y \times Y \to Y$ be a mod $\boldsymbol{P}$ H-structure such that $\mu \circ i_1 \cong \mu \circ i_2 \cong l$ is a $\boldsymbol{P}$-equivalence. By Lemma 3.3 of [12], there exists a positive integer r such that l^r is the identity of $H_*(Y; Z_p)$ for all $p \in \boldsymbol{P}$, where $l^r = l \circ \cdots \circ l$ the r-fold iteration. Then the composite of maps

$$X\times X \xrightarrow{k\times k} Y\times Y \xrightarrow{l^{r-1}\times l^{r-1}} Y\times Y \xrightarrow{\mu} Y \xrightarrow{h} X$$

gives X a mod $\boldsymbol{P}$ H-structure. Q. E. D.

Dually we have

THEOREM 6.4′. *Let $X\in\mathfrak{F}\mathfrak{C}_1$. Then the following conditions are equivalent.*

(1)′ *X is a* mod $\boldsymbol{P}$ *co-H-space.*

(2)′ *$X_{\boldsymbol{P}}$ is a co-H-space.*

(3)′ *X is $\boldsymbol{P}$-dominated by a* mod $\boldsymbol{P}$ *co-H-space.*

Let $\mu: X\times X\to X$ be a mod $\boldsymbol{P}$ H-structure on X such that $\mu\circ i_1\cong\mu\circ i_2\cong h$, which is a $\boldsymbol{P}$-equivalence.

DEFINITION 6.5. X is mod $\boldsymbol{P}$ *homotopy associative,* if $\mu\circ(\mu\times h)\cong\mu\circ(h\times\mu)$. Dually we define *a* mod $\boldsymbol{P}$ *homotopy coassociativity* on a mod $\boldsymbol{P}$ co-H-space.

THEOREM 6.6. *Let $X\in\mathfrak{F}\mathfrak{C}_1$.*

(A) *The following statements are equivalent.*

(1) *X is a* mod $\boldsymbol{P}$ *homotopy associative H-space.*

(2) *$X_{\boldsymbol{P}}$ is a homotopy associative H-space.*

(B) *Moreover if $\boldsymbol{P}\ni 2$ and 3, or if $\boldsymbol{P}\not\ni 2$ nor 3, then one of* (1) *and* (2) *is equivalent to the following:*

(3) *X is $\boldsymbol{P}$-dominated by a homotopy associative H-space.*

PROOF. (A) The equivalence between (1) and (2) can be proved as before.

(B) The proof for [(3) *implies* (1)] is quite analogous to that for [(3) *implies* (1)] of Theorem 6.4. However, the proof for [(2) *implies* (3)] needs further results on the localization of H-complexes. So it will be at the end of the next section.

Elementary but non-trivial examples for a mod $\boldsymbol{P}$ H-space, $\boldsymbol{P}\not\ni 2$, are odd dimensional spheres. Let p be a prime. Then, as is expected, the mod p structure on S^n, n: odd, is unique for sufficiently large p. More precisely,

THEOREM 6.7. *Let p be an odd prime. Then the number of* mod p *H-structures, up to homotopy, of S^n (n: odd) is equal to the order of $\pi_{2n}(S^n:p)$.*

PROOF. The number of mod p H-structures on S^n is equal to that of H-structures on $S^n_{(p)}$. It is equal to the number of elements of $[S^n_{(p)}\times S^n_{(p)}, S^n_{(p)}\vee S^n_{(p)}; S^n_{(p)}, *]$ by [**15**]. Then the theorem follows from the fact that $[S^n_{(p)}\times S^n_{(p)}, S^n_{(p)}\vee S^n_{(p)}; S^n_{(p)}, *]=[S^n_{(p)}\wedge S^n_{(p)}, S^n_{(p)}]=[S^{2n}_{(p)}, S^n_{(p)}]=\pi_{2n}(S^n:p)$ by Theorem 5.6. Q. E. D.

Now let us recall the notion of A_n-form (or A_n-space) due to Stasheff [**20**]. For example, an A_2-space, an A_3-space, an A_∞-space are an H-space, a homotopy associative H-space and an H-space equivalent to a loop space, respectively.

As is well known [**20**], $S^{2n-1}_{(p)}$ admits an A_{p-1}-form.

PROPOSITION 6.8. *If $S_{(p)}^{2n-1}$ admits an A_p-form, then $n|p-1$.*

PROOF. If $S_{(p)}^{2n-1}$ admits an A_p-form, then there exists a "projective p-space" X over $S_{(p)}^{2n-1}$, [20], such that $H^*(X; Z_p) \cong Z_p[x_{2n}]/(x_{2n}^{p+1})$. To prove the proposition it suffices to show that $\mathfrak{p}^1$ is non-trivial in $H^*(X; Z_p)$. For, if $\mathfrak{p}^1 x_{2n}^r \neq 0$, by taking the degree, $2p-2+2nr = 2nk$, and hence $n|p-1$. Let r be such a number that $\mathfrak{p}^{p^r}$ is non-trivial but $\mathfrak{p}^{p^i} = 0$ for $i < r$ in $H^*(X; Z_p)$. Clearly such r exists, since $\mathfrak{p}^n x_{2n} = x_{2n}^p \neq 0$. Then from the structure of $H^*(X; Z_p)$ and from the factorization of $\mathfrak{p}^{p^r}$ by secondary operations ([18]), we get $r=0$. This completes the proof. Q. E. D.

THEOREM 6.9 (Adams). (1) *Let $\boldsymbol{P} \not\ni 2$. Then S^{2n-1} is a* mod $\boldsymbol{P}$ *H-space for all n.*

(2) *Let $\boldsymbol{P} \ni 2$. Then S^{2n-1} is a* mod $\boldsymbol{P}$ *H-space if and only if $n = 1, 2, 4$.*

(3) *Let $\boldsymbol{P} \not\ni 2$ nor 3. Then S^{2n-1} is a* mod $\boldsymbol{P}$ *homotopy associative H-space for all n.*

(4) *Let $\boldsymbol{P} \not\ni 2$ and $\boldsymbol{P} \ni 3$. Then S^{2n-1} is a* mod $\boldsymbol{P}$ *homotopy associative H-space if and only if $n = 1, 2$.*

PROOF. First recall the classical result that the obstruction to extend the map $\iota_{2n-1} \vee \iota_{2n-1}: S^{2n-1} \vee S^{2n-1} \to S^{2n-1}$ over $S^{2n-1} \times S^{2n-1}$ is the Whitehead product $[\iota_{2n-1}, \iota_{2n-1}]$, which is trivial for $n = 1, 2, 4$ and is of order 2 otherwise.

(1) In any case there exists a map $S^{2n-1} \times S^{2n-1} \to S^{2n-1}$ of type (2, 2) for any n. So, if $\boldsymbol{P} \not\ni 2$, S^{2n-1} is a mod $\boldsymbol{P}$ H-space.

(2) Let $\boldsymbol{P} \ni 2$. If $n = 1, 2, 4$, then S^{2n-1} is an H-space, and hence it is a mod $\boldsymbol{P}$ H-space. If $n \neq 1, 2, 4$, then the obstruction $[\iota_{2n-1}, \iota_{2n-1}]_{\boldsymbol{P}} \neq 0$, and hence $S_{\boldsymbol{P}}^{2n-1}$ is not an H-space.

(3) If $\boldsymbol{P} \not\ni 2, 3$, then clearly $S_{\boldsymbol{P}}^{2n-1}$ is a homotopy associative H-space, and hence S^{2n-1} is a mod $\boldsymbol{P}$ homotopy associative H-space.

(4) Let $\boldsymbol{P} \not\ni 2$ and $\boldsymbol{P} \ni 3$. If $n = 1, 2$, then S^{2n-1} is an associative H-space, and hence S^{2n-1} is a mod $\boldsymbol{P}$ homotopy associative H-space. Conversely, suppose that $S_{\boldsymbol{P}}^{2n-1}$ is a homotopy associative H-space. Then $S_{(3)}^{2n-1}$ is also a homotopy associative H-space. Then by Proposition 6.8 we have that $n|2$, and hence $n = 1, 2$. Q. E. D.

§ 7. Localization of finite H-complexes.

In this section we work in $\mathfrak{F}\mathfrak{C}_1$ again. First we show

THEOREM 7.1. (cf. [23].) *Let $2 \leqq n \leqq \infty$.*

(1) *If X is an A_n-space, then $X_{(p)}$ is an A_n-space for every prime p and for $p = 0$.*

(2) *If $X_{(p)}$ is an A_n-space and if the canonical map $\varphi_p: X_{(p)} \to X_{(0)}$ is an*

A_n-map for all primes p, then X is itself an A_n-space.

PROOF. (1) is clear. (2) follows from Corollary 4.3. Q. E. D.

When applying the above theorem, we have to check that the map $\varphi_p : X_{(p)} \to X_{(0)}$ is an A_n-map. For $n=2, 3$ and ∞, the following proposition gives a sufficient condition for that. In the below $\beta_i(X)$ and $\gamma_i(X)$ denote the i-th Betti number of X and the rank of $\pi_i(X)$ respectively.

PROPOSITION 7.2. *Let $n=2, 3$ or ∞. Suppose that $\beta_i(X\wedge X)\gamma_i(X)=0$ for all i. Then X is an A_n-space if and only if $X_{(p)}$ is an A_n-space for all primes p and for $p=0$.*

PROOF. If $\beta_i(X\wedge X)\gamma_i(X)=0$ for all i, then it is clear that the multiplication on $X_{(0)}=\prod K(Q, 2n_i-1)$ is unique up to homotopy. Then $\varphi_p : X_{(p)}\to X_{(0)}$ is an A_n-map. The rest is clear. Q. E. D.

More generally we will prove the following

THEOREM 7.3. *Let $\bigcup_{i=1}^{r} \boldsymbol{P}_i$ be a disjoint decomposition of the set of all primes. Let $X_i \in \mathfrak{F}\mathfrak{C}_1$, $1\leqq i\leqq r$, be a* mod $\boldsymbol{P}_i$ *H-space such that there exists a homotopy equivalence $h_i : (X_i)_{(0)}\to(X_1)_{(0)}$, which is an H-map, for all i. Then there exists a finite H-complex X such that $X_{\boldsymbol{P}_i}=(X_i)_{\boldsymbol{P}_i}$. Further, if each X_i is a* mod $\boldsymbol{P}_i$ *homotopy associative H-space, X is a homotopy associative H-space.*

PROOF. By the assumption, $(X_i)_{\boldsymbol{P}_i}$ is an H-space, and hence it induces an H-structure on $(X_i)_{(0)}$. Denote the canonical map by $\varphi_i : (X_i)_{\boldsymbol{P}}\to(X_i)_{(0)}$. Then the composite map $h_i\circ\varphi_i : (X_i)_{\boldsymbol{P}}\to(X_i)_{(0)}\to(X_1)_{(0)}$ is an H-map. Put $X=\prod_{(X_1)_{(0)}}(X_i)_{\boldsymbol{P}_i}$, the pull back of $h_i\circ\varphi_i$ over $(X_1)_{(0)}$. Then by Theorem 4.6 X is a finite complex. Obviously X is an H-space. The rest of the theorem is clear. Q. E. D.

LEMMA 7.4. *Let X be a space such that $H^*(X; Q)\cong\Lambda(x_1, \cdots, x_r)$ with* $\deg x_i=n_i$ *odd. Further suppose that a given H-structure on $X_{(0)}$ induces an associative Hopf algebra structure on $H^*(X_{(0)}; Q)$. Then there exists a homotopy equivalence $X_{(0)}\to\prod_{i=1}^{r} K(Q, n_i)$, which is an H-map.*

PROOF. By the Hopf-Samelson theorem [**16**], we can choose primitive generators y_i $(1\leqq i\leqq r)$ such that $H^*(X; Q)\cong\Lambda(y_1, \cdots, y_r)$ with $\deg y_i=n_i$. We may consider that y_i is represented by a map $f_i : X\to K(Q, n_i)$. Then the required map is obtained by

$$X \xrightarrow[\Delta]{} \prod_i X \xrightarrow[f_1\times\cdots\times f_r]{} \prod_{i=1}^{r} K(Q, n_i),$$

where Δ is the diagonal map. Q. E. D.

COROLLARY 7.5. *Let $\bigcup_{i=1}^{r} \boldsymbol{P}_i$ be a disjoint decomposition of the set of all primes. Let $X_i\in\mathfrak{F}\mathfrak{C}_1$, $1\leqq i\leqq r$, be a* mod $\boldsymbol{P}_i$ *H-space such that $H^*(X_i; Q)\cong\Lambda(x(i)_1, \cdots, x(i)_l)$ is an associative Hopf algebra for all $1\leqq i\leqq r$, with* $\deg x(i)_j$

$=n_j$ *odd for* $1 \leqq i \leqq r$. *Then there exists a finite H-complex X such that* $X_{P_i} = (X_i)_{P_i}$.

PROOF. It suffices to show that X_i satisfies the condition of Theorem 7.3. Actually, we have an H-equivalence: $(X_i)_{(0)} \to \prod_{j=1}^{l} K(Q, n_j)$ for all $1 \leqq i \leqq r$.

Q. E. D.

REMARK 7.6. If each of X_i is of the same rational type and if each of X_i is one of the following, then the conditions of the theorem are satisfied.

(1) X_i is mod $\boldsymbol{P}_i$ homotopy associative.

(2) $\beta_j(X_i \wedge X_i)\gamma_j(X_i) = 0$.

(3) X_i is $\boldsymbol{P}_i$-equivalent to a product of spaces satisfying (1) or (2).

(PROOF OF THEOREM 6.6: continued) [(2) *implies* (3)].

Let $\mu: X_P \times X_P \to X_P$ be a homotopy associative multiplication. Then μ induces a homotopy associative multiplication $\mu_{(0)}: X_{(0)} \times X_{(0)} \to X_{(0)}$ by Theorem 2.7. Then by the Hopf-Samelson Theorem, we have that $H^*(X_{(0)}; Q) \cong \Lambda(y_1, \cdots, y_r)$, where $\deg y_i = n_i$ is odd and y_i is primitive for every i. By Lemma 7.4, there is an H-equivalence $a: X_{(0)} \to \prod_{i=1}^{r} K(Q, n_i)$. Let $\boldsymbol{Q}$ be the complement of $\boldsymbol{P}$ in the set of all primes.

(Case: $\boldsymbol{P} \ni 2, 3$)

Put $Y = \prod_{i=1}^{r} S^{n_i}$. Then by Theorem 6.9, $Y_{\boldsymbol{Q}}$ is a homotopy associative H-space. Again by Lemma 7.4 there is an H-equivalence $b: Y_{(0)} \to \prod_{i=1}^{r} K(Q, n_i)$. Denoting by $j_P: X_P \to X_{(0)}$, $(j_Q: Y_Q \to Y_{(0)})$ the canonical map, we consider the pull back $Z = X_P \times_{\Pi K(Q, n_i)} Y_Q$ of $a \circ j_P$ and $b \circ j_Q$ over $\prod_{i=1}^{r} K(Q, n_i)$. Then by Theorem 7.3, Z is a homotopy associative finite H-complex. Further, there exists a $\boldsymbol{P}$-equivalence $X \to Z$ (and hence a $\boldsymbol{P}$-equivalence $Z \to X$, too). So X is $\boldsymbol{P}$-dominated by a homotopy associative H-space.

(Case: $\boldsymbol{P} \not\ni 2$ nor 3)

Clearly, there exist sets of integers $(m_1, \cdots, m_r)$ and $(k_1, \cdots, k_s)$ such that $X \times \prod_{i=1}^{r} S^{m_i}$ has the same 0-type of $\prod_{i=1}^{s} SU(k_i)$. For simplicity put $Y = \prod_{i=1}^{r} S^{m_i}$. Then $(X \times Y)_P = X_P \times Y_P$ is homotopy associative, since $\boldsymbol{P} \not\ni 2$ nor 3. Similarly as above, we denote by Z the pull back over $\prod K(Q, n_i) \times \prod K(Q, m_i)$ of H-maps $(X \times Y)_P \to \prod K(Q, n_i) \times \prod K(Q, m_i)$ and $(\prod SU(k_i))_Q \to \prod K(Q, n_i) \times \prod K(Q, m_i)$. Then Z is a homotopy associative finite H-complex. Here the map $Z \to (X \times Y)_P$ is factored as: $Z \xrightarrow{j_Z} Z_P \xrightarrow{h} (X \times Y)_P$, where j_Z is a natural inclusion and h is a homotopy equivalence. Since X and Z are $\boldsymbol{P}$-universal spaces, there exist maps $f: X \to Z$ and $g: Z \to X$ such that the following is homotopy commutative:

$$
\begin{tikzcd}
 & X_P \times Y_P \arrow[dr, "\pi"] \arrow[d, shift left, "h^{-1}"] & \\
X_P \arrow[ur, "i"] \arrow[r, "h^{-1}\circ i"'] & Z_P \arrow[u, shift left, "h"] \arrow[r, "\pi\circ h"'] & X_P \\
X \arrow[u, "j_X"] \arrow[r, "f"] & Z \arrow[u, "j_Z"] \arrow[r, "g"] & X \arrow[u, "j_X"']
\end{tikzcd}
$$

where i and π are the obvious inclusion and projection. Thus $g\circ f$ is a $\boldsymbol{P}$-equivalence, and hence X is $\boldsymbol{P}$-dominated by Z. Q. E. D.

§ 8. New finite H-complexes.

For a simply connected finite H-complex X, the classical Hopf theorem states that $H^*(X; Q) \cong \Lambda(x_1, \cdots, x_l)$ with $\deg x_i = n_i$ odd. Then $\sum_{i=1}^{l} n_i = \dim X$. l is called *the rank* of X and the sequence $(n_1, \cdots, n_l)$ is called *the (rational) type* of X. Recently, Hilton-Roitberg [8] have discovered a finite H-complex of type (3, 7), which is a principal S^3-bundle over S^7 and not of the same homotopy type of $Sp(2)$. Similar examples are also discovered by Stasheff [**21**]. In this section we will construct more finite H-complexes by making use of the theorems in the previous sections.

Let G be a compact, connected, simply connected topological group and let H be a closed subgroup such that $G/H = S^{2n+1}$, $(n \geqq 1)$. We consider a principal H-bundle: $H \to G \to S^{2n+1}$ with a characteristic class $\alpha \in \pi_{2n}(H)$ of finite order d. Let $k: S^{2n+1} \to S^{2n+1}$ be a map of degree k. We denote by E_k the bundle induced by k from the above principal bundle. Then k induces a bundle map $\tilde{k}: E_k \to G$. In the below, $\nu_p(k)$ denotes the exponent of p in the factorization of an integer k into prime powers.

THEOREM 8.1. *Suppose that $\nu_p(k) = 0$ or $\nu_p(k) \geqq \nu_p(d)$ for any prime p. Then E_k is an H-space if and only if $\nu_2(k) = 0$ or $n = 1, 3$. Further, E_k is a homotopy associative H-space if $\nu_2(k) = \nu_3(k) = 0$.*

PROOF. Let l be minimal positive integer such that $d \mid lk$. Consider the following commutative diagram:

$$\begin{tikzcd}
H \arrow[r, "1_H"] \arrow[d] & H \arrow[r, "1_H"] \arrow[d] & H \arrow[d] \\
E_{kl} \arrow[r, "\tilde{l}"] \arrow[d] & E_k \arrow[r, "\tilde{k}"] \arrow[d] & G \arrow[d] \\
S^{2n+1} \arrow[r, "l"] & S^{2n+1} \arrow[r, "k"] & S^{2n+1}
\end{tikzcd}$$

Note that $E_{kl}=H\times S^{2n+1}$, since $d\,|\,kl$. Clearly we have: $\tilde{k}: E_k\rightarrow G$ is a p-equivalence, if $\nu_p(k)=0$. $\tilde{l}: H\times S^{2n+1}\rightarrow E_k$ is a p-equivalence, if $\nu_p(k)\geqq\nu_p(d)$. Assume that $\nu_2(k)\neq 0$ (and hence $\nu_2(k)\geqq\nu_2(d)$). Then $\tilde{l}: H\times S^{2n+1}\rightarrow E_k$ is a 2-equivalence. So, if E_k is an H-space, S^{2n+1} is a mod 2 H-space, and hence $n=1$ or 3 by Theorem 6.9.

Now suppose that $\nu_2(k)=0$ or $n=1, 3$. Put $\boldsymbol{P}_1=\{p \text{ a prime} \mid \nu_p(k)=0\}$. Denote by $\boldsymbol{P}_2$ the complement of $\boldsymbol{P}_1$ in the set of all primes. Let φ be the multiplication on G and φ' the restriction of φ on H. Denote by s the map $S^{2n+1}\times S^{2n+1}\rightarrow S^{2n+1}$ of type (2, 2). Let $a: S^{2n+1}{}_{\boldsymbol{P}_2}\rightarrow S^{2n+1}{}_{\boldsymbol{P}_2}$ be a map dividing by 2, if $\boldsymbol{P}_2\not\ni 2$. Let $\mu=a\circ s_{\boldsymbol{P}_2}$, if $\boldsymbol{P}_2\not\ni 2$, and let μ be the ordinary multiplication localized at $\boldsymbol{P}_2$, if $\boldsymbol{P}_2\ni 2$. By introducing a multiplication $\varphi'_{\boldsymbol{P}_2}$ and μ on $H_{\boldsymbol{P}_2}$ and $(S^{2n+1})_{\boldsymbol{P}_2}$ separately, we obtain a multiplication $\psi: (H\times S^{2n+1})_{\boldsymbol{P}_2}\times(H\times S^{2n+1})_{\boldsymbol{P}_2}\rightarrow(H\times S^{2n+1})_{\boldsymbol{P}_2}$. Since E_k is $\boldsymbol{P}_1$- and $\boldsymbol{P}_2$-dominated by G and $H\times S^{2n+1}$ respectively, E_k is a mod $\boldsymbol{P}_i$ H-spaces. So we define a multiplication μ_i on $(E_k)_{\boldsymbol{P}_i}$ as follows:

$$\mu_1=(\tilde{k}_{\boldsymbol{P}_1})^{-1}\circ\varphi_{\boldsymbol{P}_1}\circ(\tilde{k}_{\boldsymbol{P}_1}\times\tilde{k}_{\boldsymbol{P}_1}): (E_k)_{\boldsymbol{P}_1}\times(E_k)_{\boldsymbol{P}_1}\rightarrow G_{\boldsymbol{P}_1}\times G_{\boldsymbol{P}_1}\rightarrow G_{\boldsymbol{P}_1}\rightarrow(E_k)_{\boldsymbol{P}_1},$$

$$\begin{aligned}\mu_2=\tilde{l}_{\boldsymbol{P}_2}\circ\psi\circ((\tilde{l}_{\boldsymbol{P}_2})^{-1}\times(\tilde{l}_{\boldsymbol{P}_2})^{-1}): (E_k)_{\boldsymbol{P}_2}\times(E_k)_{\boldsymbol{P}_2}&\rightarrow(H\times S^{2n+1})_{\boldsymbol{P}_2}\times(H\times S^{2n+1})_{\boldsymbol{P}_2}\\ &\rightarrow(H\times S^{2n+1})_{\boldsymbol{P}_2}\rightarrow(E_k)_{\boldsymbol{P}_2},\end{aligned}$$

where $(\tilde{k}_{\boldsymbol{P}_1})^{-1}$ and $(\tilde{l}_{\boldsymbol{P}_2})^{-1}$ are homotopy inverses of $\tilde{k}_{\boldsymbol{P}_2}$ and $\tilde{l}_{\boldsymbol{P}_2}$ respectively. Then by the fact that $\widetilde{k\circ l}=\tilde{k}\circ\tilde{l}$ and by Theorem 2.7 we obtain a homotopy commutative diagram

$$\begin{tikzcd}
(H\times S^{2n+1})_{(0)}\times(H\times S^{2n+1})_{(0)} \arrow[rr, "(\widetilde{kl})_{(0)}\times(\widetilde{kl})_{(0)}"] \arrow[dr, "\tilde{l}_{(0)}\times\tilde{l}_{(0)}"'] \arrow[dd, "\psi_{(0)}"'] & & G_{(0)}\times G_{(0)} \arrow[dd, "\varphi_{(0)}"] \\
& (E_k)_{(0)}\times(E_k)_{(0)} \arrow[ur, "\tilde{k}_{(0)}\times\tilde{k}_{(0)}"'] & \\
(H\times S^{2n+1})_{(0)} \arrow[rr, "(\widetilde{kl})_{(0)}"] \arrow[dr, "\tilde{l}_{(0)}"'] & & G_{(0)} \\
& (E_k)_{(0)} \arrow[ur, "\tilde{k}_{(0)}"'] &
\end{tikzcd}$$

By (4) of Theorem 2.7 μ_1 and μ_2 induce two multiplications $(\mu_1)_{(0)}$ and $(\mu_2)_{(0)}$ on $(E_k)_{(0)}$ induced by $\psi_{(0)}$ and $\varphi_{(0)}$ respectively. But by chasing the above diagram one can see that $(\mu_1)_{(0)}$ is homotopic to $(\mu_2)_{(0)}$. Hence by Theorem 7.1, E_k is an H-space. The assertion for homotopy associativity of E_k, when $\nu_2(k)=\nu_3(k)=0$, is easily checked. Q. E. D.

REMARK. This theorem is proved by Harrison by the following form: *Write $\alpha=\alpha_2+\alpha_3+\cdots+\alpha_q$, where α_p is of p-power order. Write $k\alpha=\sum\varepsilon_p\alpha_p$.*

Let $\varepsilon_p=0$ or ±1 for any p. Then E_k is an H-space if and only if

1) $\varepsilon_2\neq 0$ *or,*

2) $n=1, 3$.

But the above expression of the theorem is easily checked to be equivalent to ours.

EXAMPLE 8.2 (Hilton-Roitberg-Stasheff [8], [21]). *Let* $(G, H)=(Sp(2), Sp(1))$. *Then E_k is an H-space if $k\neq 2$ (4).*

EXAMPLE 8.3 (Curtis-Mislin [7]). *Let* $(G, H)=(SU(4), SU(3))$.

(1) *Any E_k is an H-space.*

(2) *There are exactly four homotopy types of such spaces.*

PROOF. Recall $\pi_6(SU(3))\cong Z_6$. (1) is clear. To prove (2) we need

LEMMA 8.4. $E_k=E_{-k}$.

So, $E_1=E_5$ and $E_2=E_4$. Of course $E_0=S^7\times SU(3)$ and $E_1=SU(4)$ are different. Then $E_2\neq E_0$, $E_2\neq E_1$. For $(E_2)_{(2)}\neq(E_1)_{(2)}$ and $(E_2)_{(3)}\neq(E_0)_{(3)}$. Similarly $E_3\neq E_i$ for $i=0, 1, 2$, since $(E_3)_{(2)}\neq(E_0)_{(2)}$, and since $(E_3)_{(3)}\neq(E_i)_{(3)}$ for $i=1, 2$. Q. E. D.

Let p be a prime. Recall [17] that X is called *p-regular*, if there exists a p-equivalence $\prod_{i=1}^{l} S^{n_i} \to X$, and that X is called *quasi p-regular*, if there exists a p-equivalence $\prod S^{n_i} \times \prod B_{n_j}(p) \to X$, where $B_{n_j}(p)$ is such a space that $H^*(B_{n_j}(p)\,;\,Z_p) \cong \Lambda(x_j, \mathfrak{p}^1 x_j)$ with $\deg x_j = 2n_j+1$.

Let G be a compact, connected, simply connected, simple Lie group. Then by the Hopf theorem

$$H^*(G\,;\,Q) \cong \Lambda(x_1, \cdots, x_l) \qquad \text{with } \deg x_i = 2n_i+1\,,$$

where l is the rank of G, and $\sum(2n_i+1) = \dim G$. Then

THEOREM 8.5 (Kumpel, Serre, Mimura-Toda). (1) *G is p-regular if and only if $p > n_l$.*

(2) *G is quasi p-regular if and only if $p > N(G)$, where*

$N(G)$	G
n	$Sp(n)$
$\frac{n}{2}$	$SU(n)$
$\frac{n-1}{2}$	$\mathrm{Spin}(n)$
3	$G_2,\ F_4,\ E_6$
7	$E_7,\ E_8$

For a proof see [**10**].

REMARK 8.5′. It follows from Theorem 6.4 and Theorem 8.5 that $B_{n_i}(p)$ is a $\bmod p$ H-space, if $n_i \leqq p-1$.

THEOREM 8.6. *Let p be an odd prime.*

(1) *There exist infinitely many finite H-complexes which are p-regular for a given p.*

(2) *There exist infinitely many finite H-complexes, which are quasi p-regular for a given p.*

PROOF. (1) Put $S(G) = \prod_{i=1}^{l} S^{2n_i+1}$. Apparently $S(G)$ is a $\bmod p$ H-space. Let $\boldsymbol{Q}$ be the complement of $\{p\}$ in the set of all primes. Denote by $S_p(G)$ the pull back of the maps $(S(G))_{(p)} \to G_{(0)}$ and $G_{\boldsymbol{Q}} \to G_{(0)}$ over $G_{(0)}$. Then by Corollary 7.5 and Remark 7.6, $S_p(G)$ is a finite H-complex. Clearly $S_p(G)$ is always p-regular.

(2) We put, for $1 \leqq k \leqq a-1$,

$$B(G) = \prod_{i=1}^{k} B_{n_i}(p) \times \prod_{i=k+1}^{a-1} S^{2n_i+1} \times \prod_{i=b+1}^{l} S^{2n_i+1}\,,$$

where a and b are such numbers that $n_a = p$ and $n_b = n_k + p$ respectively. Similarly as above we mix the homotopy type of $B(G)$ and G. We denote by $B_p(G)$ the pull back of the maps $B(G)_{(p)} \to G_{(0)}$ and $G_Q \to G_{(0)}$ over $G_{(0)}$. Then $B_p(G)$ is a finite H-complex, which is always quasi p-regular. Q. E. D.

REMARK 8.7. $S_p(G)$ is not p-equivalent to any product of Lie groups, if $n_l \geqq p$. Similarly $B_p(G)$ is not p-equivalent to any product of Lie groups and spheres, if $N(G) \geqq p$.

Next we give some examples of a finite H-complex which is of (rational) type (3, 11).

THEOREM 8.8. *There exist at least four different finite H-complexes of type* (3, 11).

PROOF. We choose a map $f: S^{11} \to V_{7,2}$ such that $f^*: H^*(V_{7,2}; Z_3) \cong H^*(S^{11}; Z_3)$. We consider the bundle $B_1'(3)$ induced by f from the bundle $G_2/S^3 = V_{7,2}$. Then as is easily seen, $B_1'(3)$ is a S^3-bundle over S^{11} with the characteristic class $\alpha_2(3)$, which is a generator of $\pi_{10}(S^3 : 3) \cong Z_3$. It is also clear that $B_1'(3)$ is a mod 3 H-space. Let $\boldsymbol{Q}$ be the complement of {3, 5} in the set of all primes. Now we mix the homotopy types using the ingredients given in the following table.

	{3}	{5}	$\boldsymbol{Q}$
X_1	$S^3 \times S^{11}$	$S^3 \times S^{11}$	G_2
X_2	$B_1'(3)$	$S^3 \times S^{11}$	G_2
X_3	$S^3 \times S^{11}$	$B_1(5)$	G_2
X_4	$B_1'(3)$	$B_1(5)$	G_2

The pull backs X_i are all finite H-complexes and all have different homotopy types. Note that $X_4 = G_2$. Q. E. D.

REMARK 8.9. According to Hubbuck, if a finite H-complex X of rank 2 has 2-torsion, then

$$H^*(X; Z_2) \cong H^*(G_2; Z_2).$$

So X_i's are such H-complexes.

THEOREM 8.10. (1) *There exist several finite H-complexes which have only 3-torsion.*

(2) *There exists a homotopy associative finite H-complex which has only 5-torsion.*

PROOF. Denote by $\bar{p}$ the complement of p in the set of all primes.

(1) The pull back given by the following diagram gives an example for (1), since F_4 has just 2 and 3 torsions.

$$\begin{tikzcd}[column sep=small]
 & (F_4\times S^7\times S^{19})_{(3)} \arrow[dr] & \\
X \arrow[ur] \arrow[dr] & & (Sp(6))_{(0)} \\
 & (Sp(6))_{(\bar{3})} \arrow[ur] &
\end{tikzcd}$$

Similar examples can be obtained by using E_6, E_7 and E_8.

(2) An example for (2) is obtained by

$$\begin{tikzcd}[column sep=small]
 & (E_8\times S^7\times S^{11}\times S^{19}\times S^{31}\times S^{43}\times S^{51}\times S^{55})_{(5)} \arrow[dr] & \\
Y \arrow[ur] \arrow[dr] & & (Sp(15))_{(0)} \\
 & (Sp(15))_{(\bar{5})} \arrow[ur] &
\end{tikzcd}$$

Q. E. D.

§ 9. Mod p decomposition of suspended spaces.

Throughout this section let p denote an odd prime.

DEFINITION 9.1. A co-H-space X is mod p *decomposable into* r *spaces*, if there exist r spaces X_i with $\tilde{H}^*(X_i; Z_p)\neq 0$, $1\leqq i\leqq r$, and there exists a p-equivalence $f: X\to \bigvee_{i=1}^{r} X_i$, where $\vee$ is the wedge sum.

For simplicity we denote $X\cong_p \bigvee_{i=1}^{r} X_i$. If $X\in\mathfrak{F}\mathfrak{C}_1$, then the direction of a p-equivalence between X and $\vee X_i$ is not important, since there is always a converse p-equivalence.

CONDITION 9.2. For a connected finite CW-complex X,

D_p: (1) There exist homogeneous elements $x_i\in\tilde{H}^*(X; Z_p)$, $1\leqq i\leqq s$, such that $\tilde{H}^*(X; Z_p)$ has a basis consisting of monomials in x_i's.

(2) There exists a map $\phi^k: X\to X$ such that $(\phi^k)^* x_i = kx_i$ for $1\leqq i\leqq s$, where k is a primitive root modulo p.

Now suppose that X satisfies the condition D_p. Then each element of a basis of $\tilde{H}^*(X; Z_p)$ has not only the cohomological degree but also the rank, which is defined to be the degree of monomial. Then according to the rank, we obtain a direct sum decomposition:

$\tilde{H}^*(X; Z_p)\cong\sum_n A_n^*$, where A_n^* consists of elements of rank n.

Then we also have

$\tilde{H}^*(SX; Z_p)\cong\sum_n SA_n^*$, where SA_n^* denotes the module spanned by the suspension of the elements of A_n^*.

Put $B_m = \sum_{n=m+k(p-1)} SA_n^*$; i.e., $\tilde{H}^*(SX; Z_p) \cong \sum_{m=1}^{p-1} B_m$. Let r be the number such that $B_m \neq 0$; i.e., $B_{m_1} \neq 0, \cdots, B_{m_r} \neq 0$.

THEOREM 9.3. *Let X be a connected finite CW-complex satisfying the condition D_p. Then SX is* mod p *decomposable into r spaces. Namely there exist r spaces X_{m_i}, $i=1, \cdots, r$, and a p-equivalence $f: SX \to \bigvee_{i=1}^{r} X_{m_i}$ such that $H^*(X_{m_i}: Z_p) \cong B_{m_i}$.*

PROOF. Let k be a primitive root modulo p. Let $\psi^k: X \to X$ be the map given by (2) of D_p. Let $-k^j: S^1 \to S^1$ be a map of degree $-k^j$. The map $(-k^j) \wedge 1_X: SX \to SX$ will also be denoted by $-k^j$. We consider the map

$$g_j = (S\psi^k - k^j): SX \xrightarrow{\varphi} SX \vee SX \xrightarrow{S\psi^k \vee (-k^j)} SX \vee SX \xrightarrow{\pi} SX,$$

where φ is the canonical map shrinking the equator of SX and π is the obvious projection. Then for any x of SA_n^*, $g_j^*(x) = (k^n - k^j)x$. Recall that $k^n - k^j \equiv 0$ (p) if and only if $n - j \equiv 0$ $(p-1)$, since k is a primitive root modulo p. Note that ψ^k, and hence $S\psi^k$, is a p-equivalence, and hence it is a 0-equivalence. Then there exists a sufficiently large number N such that for every $j \geqq N$, g_j is a 0-equivalence, since SX is a finite CW-complex. SX is p-universal for any p by Theorem 4.2 of [12], since it is a co-H-space. So, by Theorem 5.3, there is a p-sequence $\{A_i, f_i\}$ of SX such that $A_i = SX$ for all $i \geqq 0$. We put $\tilde{g}_j = g_{pN+j}: SX \to SX$ for $1 \leqq j \leqq p-1$. Let m be an integer with $1 \leqq m \leqq p-1$. Let $S_m = \{A_i, \tilde{f}_i\}$ be a sequence obtained by inserting 0-equivalence $\tilde{g}_j$, $j \neq m$, infinitely many times in the p-sequence $\{A_i, f_i\}$. Although S_m is not a p-sequence any longer, it is a "subsequence" of a 0-sequence of SX. By constructing a telescope, we obtain a space, which is denoted by $(SX)_{(p,m)}$, and also inclusions

$$(SX)_{(p)} \xrightarrow{j_1} (SX)_{(p,m)} \xrightarrow{j_2} (SX)_{(0)}$$

such that the composite of them is the canonical map $j_{0,p}: (SX)_{(p)} \to (SX)_{(0)}$. Let $\boldsymbol{Q}$ denote the complement of $\{p\}$ in the set of all primes. Put $X_m = (SX)_{(p,m)} \underset{(SX)_{(0)}}{\times} (SX)_{\boldsymbol{Q}}$ the pull back of j_2 and the map $j_{0,\boldsymbol{Q}}: (SX)_{\boldsymbol{Q}} \to (SX)_{(0)}$ over $(SX)_{(0)}$. Then X_m has the homotopy type of a finite CW-complex, since j_2 is a 0-equivalence. Also we have that $(X_m)_{(p)} = (SX)_{(p,m)}$ (cf. the following diagram).

$$\begin{tikzcd}
(SX)_{(p)} \arrow[dd, "j_1"'] & & SX \arrow[ll] \arrow[dl, dashed, "f_m"] \arrow[dd] \\
& X_m \arrow[dl] \arrow[dr] & \\
(SX)_{(p,m)} \arrow[dr, "j_2"'] & & (SX)_Q \arrow[dl] \\
& (SX)_{(0)} &
\end{tikzcd}$$

Furthermore, by the property of the pull back, we obtain a map $f_m: SX \to X_m$ such that the following diagram is homotopy commutative:

$$\begin{tikzcd}
(SX)_{(p)} \arrow[r, "j_1"] & (SX)_{(p,m)} = (X_m)_{(p)} \\
SX \arrow[u, "j_p"] \arrow[r, "f_m"] & X_m \arrow[u, "j_p"]
\end{tikzcd}$$

where j_p is the canonical inclusion. So the induced homomorphism $(j_1)_*$: $H_*((SX)_{(p)}; Z_p) \to H_*((SX)_{(p,m)}; Z_p)$ is an epimorphism, the kernel of which is isomorphic to $\sum SA_i^*$, where $\sum$ is over all i with $i \not\equiv m$ $(p-1)$. The required p-equivalence $f: SX \to \bigvee_{m=1}^{p-1} X_m$ is obtained as the composite of the maps

$$SX \xrightarrow{\bar{\varphi}} \bigvee^{p-1} SX \xrightarrow{\vee f_m} \bigvee_{m=1}^{p-1} X_m,$$

where $\bar{\varphi}$ is the $(p-2)$-iterations of φ. Q. E. D.

PROPOSITION 9.4. *Each of the following satisfies the condition D_p.*

(1) *A connected finite H-complex X such that $H^*(X; Z_p)$ is primitively generated.*

(2) *The m-th symmetric product $SP^m(M(G, n))$ of the Moore space $M(G, n)$ of type (G, n), where $G = Z$ or Z_{p^r}.*

PROOF. (1) The map $\psi^k: X \to X$ is obtained as the composite of the maps: $X \xrightarrow{\Delta} \underbrace{X \times \cdots \times X}_{k} \xrightarrow{\mu} X$, where Δ is the diagonal map and μ is the $(k-1)$

iterations of the product. If $H^*(X; Z_p)$ is primitively generated, by the Borel's theorem [6], we obtain an additive basis of $H^*(X; Z_p)$ consisting of monomials of primitive elements. (2) is also easily checked. (For the structure of $H^*(SP^m(M(G, n)); Z_p)$ see [13], [14].) Q. E. D.

COROLLARY 9.5. (1) *If X is a connected finite H-complex such that $H^*(X; Z_p)$ is primitively generated, then SX is* mod p *decomposable into $(p-1)$ spaces.*

(2) *$S(SP^m(M(G, n)))$ is* mod p *decomposable into $(p-1)$ spaces for $G=Z$ or Z_{p^r}. In particular $S(CP^n) \cong_p \bigvee_{i=1}^{p-1} X_i$.*

For there is a homeomorphism $SP^m(M(Z, 2)) = CP^m$.

We denote by L_p^{2n+1} the lens space. Then

PROPOSITION 9.6. *$S(L_p^{2n+1})$ is* mod p *decomposable.*

PROOF. It suffices to show that L_p^{2n+1} satisfies the condition D_p. We consider S^{2n+1} as the unit sphere in C^{n+1}. We define a map $\bar{\phi}^k: S^{2n+1} \to S^{2n+1}$ as $\bar{\phi}^k(z_1, \cdots, z_{n+1}) = (z_1^k/\rho, \cdots, z_{n+1}^k/\rho)$ with $\rho = \sqrt{\sum_{i=1}^{n+1} |z_i^k|^2}$. Then $\bar{\phi}^k$ induces a map $\phi^k: L_p^{2n+1} \to L_p^{2n+1}$, since L_p^{2n+1} is the orbit space of Z_p-action on S^{2n+1}. Then it is not difficult to see that L_p^{2n+1} with ϕ^k satisfies D_p. Q. E. D.

We denote by QP^n the quaternionic projective space. Then

PROPOSITION 9.7. $S(QP^n) \cong_p \bigvee_{i=1}^{\frac{p-1}{2}} X_{2i}$.

PROOF. By Corollary 9.5 there is a p-equivalence $f: S(CP^{2n}) \to \bigvee_{i=1}^{p-1} X_i$. Since $S(CP^{2n})$ is p-universal, there is a converse p-equivalence $g: \bigvee_{i=1}^{p-1} X_i \to S(CP^{2n})$. Let $j: \bigvee_{i=1}^{\frac{p-1}{2}} X_{2i} \to \bigvee_{i=1}^{p-1} X_i$ be the obvious inclusion. Let $h_n: CP^{2n} \xrightarrow[i]{} CP^{2n+1} \xrightarrow[\eta]{} QP^n$ be the composite of the inclusion i and the natural map η. Then $Sh_n \circ j$ gives the required p-equivalence. Q. E. D.

REMARK 9.8. Since the infinite symmetric product $SP^\infty(M(G, n))$ is the Eilenberg-MacLane space $K(G, n)$, Corollary 9.5 gives a mod p decomposition of $S(K(G, n))$ for $G=Z$ or Z_{p^r}.

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