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On skew product transformations with quasi-discrete spectrum

By Nobuo AOKI

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§ 1. Introduction.

Let X and Y be unit intervals with Borel measurability and Lebesgue measure. Let $\Omega = X \otimes Y$ be the unit square with the usual direct product measurability and measure. We consider the following skew product (measure preserving) transformation defined on Ω; let T be the measure preserving transformation with the α-function defined by $T:(x, y) \to (x+\gamma, y+\alpha(x))$ (additions modulo 1) where γ is an irrational number and $\alpha(\cdot)$ a real-valued measurable function defined on X.

The purpose of this paper is to give a criterion in order that the transformation T has quasi-discrete spectrum.

I am greatly indebted to the referee for many improvements on this paper.

§ 2. Definitions.

Let (Z, Σ, m) be a finite measure space and T an invertible measure preserving transformation on Z. We recall the following definition of quasi-proper functions [1]. Let $G(T)_0$ be the set

$$\{\beta \in K : V_T f = \beta f, \ \|f\|_2 = 1 \text{ for } f \in L^2(Z)\},$$

where V_T is the unitary operator induced by the transformation T and K the unit circle in the complex plane. For each positive integer i, let $G(T)_i \subset L^2(Z)$ be the set of all normalized functions f such that $V_T f = gf$ where $g \in G(T)_{i-1}$. The set $G(T)_i$ is the set of quasi-proper functions of order at most i. We put $G(T) = \bigcup_{i \geq 0} G(T)_i$. The transformation T is said to have *quasi-discrete spectrum* if the set $G(T)$ spans $L^2(Z)$. If the set $G(T)_1$ of order 1 spans $L^2(Z)$, then it is well-known that T has discrete spectrum. If the transformation T is ergodic, then $|f(x)| = 1$ for arbitrary $f \in G(T)$. This implies that $G(T)$ is a

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Throughout this paper, any equality between functions are taken as the equality for almost all values of the variables.

multiplicative abelian group. The group K is a subgroup of the group $G(T)$, and since K is a complete group, K is a direct factor in $G(T)$. From this, there is a subgroup $O(T)$ such that $G(T)=K\otimes O(T)$. If the transformation T is totally ergodic, then the group $O(T)$ is an orthonormal base of $L^2(Z)$.

From now on, we consider the following skew product transformation

$$T:\ (x, y) \longrightarrow (x+\gamma,\ y+\alpha(x)) \qquad \text{(additions modulo 1)},$$

where γ is an irrational number and $\alpha(\cdot)$ a real-valued measurable function on X. Let Γ be the set of all real-valued measurable functions on X. We define by Θ the submodule of Γ, whose elements $\xi(x)\in\Theta$ are of the form

$$\xi(x)=\theta(x)-\theta(x+\gamma)$$

for some $\theta(x)\in\Gamma$. Since Ω is the two-dimensional torus, the set of functions $G=\{\psi_{p,q}(x, y)\}$:

$$\psi_{p,q}(x, y)=\exp\{2\pi i(px+qy)\}, \quad \text{where} \quad p, q=0, \pm 1, \pm 2, \cdots,$$

forms an orthonormal base of $L^2(\Omega)$. Let H_q be the closed linear subspace of $L^2(\Omega)$ which is spanned by $\{\psi_{p,q}(x, y)\}$ for fixed q and $p=0, \pm 1, \pm 2, \cdots$. It is clear that $L^2(\Omega)$ is decomposed into the direct sum of H_q, $q=0, \pm 1, \pm 2, \cdots$, which are mutually orthogonal and that each H_q is invariant under the unitary operator V_T induced by the skew product transformation T as above. The subspace H_q is the set of all functions of the form $f(x)\exp\{2\pi iqy\}$ where $f\in L^2(\Omega)$. Especially the subspace H_0 is the set of functions depending only on the value of x-coordinate. We denote by $H_0^{\perp}$ the orthocomplement of H_0; $H_0^{\perp}=\sum_{q\neq 0}\oplus H_q$.

§ 3. Anzai's results.

Let T and S be skew product transformations with α-functions $\alpha(x)$ and $\beta(x)$ respectively. For α-functions $\alpha(x)$ and $\beta(x)$, if

$$\alpha(x)-\beta(x+u) \quad \text{or} \quad \alpha(x)+\beta(x+u)$$

belongs to Θ for some $u\in X$, then $\alpha(x)$ and $\beta(x)$ are called to be *equivalent*.

The following three theorems appear in Anzai [3].

THEOREM A. *A skew product transformation T with an α-function $\alpha(x)$ is ergodic when $\alpha(x)=mx+c$ for a non-zero integer m and a real number c.*

THEOREM B. *An ergodic skew product transformation T with an α-function $\alpha(x)$ has discrete spectrum if and only if $\alpha(x)$ is equivalent with a constant function λ, where λ is an irrational number linearly independent of γ.*

THEOREM C. *Let T and S be ergodic skew product transformations with α-functions $\alpha(x)$ and $\beta(x)$ respectively. If T and S are isomorphic, that is, if* *there exists a measure preserving transformation V of Ω onto itself such that $VTV^{-1}=S$, then between $\alpha(x)$ and $\beta(x)$ there exists the following relation,*

$$\alpha(x)-\beta(x+u)\in\Theta \quad or \quad \alpha(x)+\beta(x+u)\in\Theta\,,$$

where u is an element of X. And accordingly V is of the following form:

$$V(x, y)=(x+u,\ \theta(x)+y) \quad or \quad V(x, y)=(x+u,\ \theta(x)-y)\,.$$

Conversely, if

$$\alpha(x)-\beta(x+u)=\theta(x)-\theta(x+\gamma)$$

holds for some $u\in X$ and $\theta(x)\in\Gamma$, then $VTV^{-1}=S$ holds, where $V(x, y)=(x+u,\ \theta(x)+y)$;

and if

$$\alpha(x)+\beta(x+u)=\theta(x+\gamma)-\theta(x)$$

holds for some $u\in X$ and $\theta(x)\in\Gamma$, then $VTV^{-1}=S$ holds, where $V(x, y)=(x+u,\ \theta(x)-y)$.

§ 4. The theorem.

As before, Ω is the two-dimensional torus $X\otimes Y$ and T is a totally ergodic skew product transformation defined by $T:(x, y)\rightarrow(x+\gamma,\ y+\alpha(x))$ (additions modulo 1), where γ is an irrational number and $\alpha(\cdot)$ is a real valued measurable function on X.

THEOREM. *With the notations as above, the following statements are equivalent:*

(i) *The transformation T has quasi-discrete spectrum.*

(ii) *The α-function $\alpha(x)$ is equivalent with either a function $mx+c$ where m is a non-zero integer and c a real number, or a constant function λ where λ is an irrational number linearly independent of γ.*

PROOF. Proof of (ii)$\Rightarrow$(i). If the α-function $\alpha(x)$ is equivalent with a function $mx+c$ where m is some non-zero integer and c some real number, then, by Theorem C, T is isomorphic to the transformation S defined by

$$S(x, y)=(x+\gamma,\ y+mx+c)\,.$$

Therefore, it is enough to show that the transformation S has quasi-discrete spectrum. From the facts that for an arbitrary integer p

$$V_S \exp\{2\pi ipx\}=\exp\{2\pi ip\gamma\}\exp\{2\pi ipx\}$$

and for arbitrary integers p and q

$$\begin{aligned}&V_S \exp\{2\pi i(px+qy)\}\\ &\qquad=\exp\{2\pi i(p+qc)\}\exp\{2\pi iqx\}\exp\{2\pi i(px+qy)\}\end{aligned}$$

hold, each $\exp\{2\pi ipx\}$ is a proper function of order 1, each $\exp\{2\pi i(px+qy)\}$ is a proper function of order 2 and since G spans $L^2(\Omega)$, the transformation S has quasi-discrete spectrum. If the α-function $\alpha(x)$ is equivalent with a constant function λ where λ is an irrational number linearly independent of the irrational number γ, then by Theorem B the transformation T has quasi-discrete spectrum and $G(T)=G(T)_1$.

Proof of (i)$\Rightarrow$(ii). If the transformation $T:(x, y)\to(x+\gamma, y+\alpha(x))$ has quasi-discrete spectrum, then the group $G(T)$ spans $L^2(\Omega)$. It is clear that $\{\psi_p(x): p=0, \pm1, \pm2, \cdots\}\subset G(T)_1$ where $\psi_p(x)=\exp\{2\pi ipx\}$. We consider the following cases: either

$$G(T)_1=G(T) \quad \text{or} \quad G(T)_1\neq G(T).$$

Step I. The case of $G(T)_1=G(T)$. It is clear that the transformation T with an α-function $\alpha(x)$ has discrete spectrum. By Theorem B, the α-function $\alpha(x)$ is equivalent with a constant function λ where λ is an irrational number linearly independent of γ.

Step II. In the case of $G(T)_1\neq G(T)$, it follows that there exist a function $f(x, y)\in G(T)_2-G(T)_1$ and a function $g(x, y)\in G(T)_1-K$ such that

$$f(T(x, y))=g(x, y)f(x, y)\,. \tag{1}$$

For the above function $g(x, y)$, we have

$$g(T(x, y))=e^{2\pi i\lambda}g(x, y)\,. \tag{2}$$

From (2), we have

$$\int g(x+\gamma, y+\alpha(x))\exp\{-2\pi iqy\}dy=e^{2\pi i\lambda}\int g(x, y)\exp\{-2\pi iqy\}dy\,. \tag{3}$$

Put

$$g_q(x)=\int g(x, y)\exp\{-2\pi iqy\}dy\,. \tag{4}$$

From (3) and (4), we have

$$g_q(x+\gamma)=\exp\{2\pi i(\lambda-q\alpha(x))\}g_q(x)\,. \tag{5}$$

Taking the absolute value of both sides of (5),

$$|g_q(x+\gamma)|=|g_q(x)|\,.$$

Since the number γ is irrational, the function $|g_q(x)|$ is a non-negative constant C_q. If $C_q\neq0$, then there exists a function $\theta_q(x)\in\Gamma$ such that

$$g_q(x)=C_q\exp\{2\pi i\theta_q(x)\}\,. \tag{6}$$

Since the function $g(x, y)$ is not identically zero, there exists an integer q such that $C_q\neq0$. Let q be such an integer. Replacing $g_q(x)$ in (5) by (6), we have

(7) $$q\alpha(x)-\lambda=\theta_q(x)-\theta_q(x+\gamma).$$

If $q\neq 0$, then the equation (7) shows that $\alpha(x)$ is equivalent with the constant function λ/q. But, on account of Theorem B together with the result 8° in [1], this is impossible since $G(T)_2\neq G(T)_1$. Thus we obtain

$$g(x, y)=C_0\exp\{2\pi i\theta_0(x)\}.$$

The latter equation implies that the function $g(x, y)$ must be some non-constant function $C_0\psi_m(x)$ in $G(T)_1$. From this fact, we have

$$f(T(x, y))=C_0 e^{2\pi imx}f(x, y)$$

where C_0 is some constant with $|C_0|=1$. We define

$$f_q(x)=\int f(x, y)\exp\{-2\pi iqy\}\,dy$$

as the equation (4). We have

$$f_q(x+\gamma)=\exp\{2\pi i(mx-q\alpha(x)+\lambda')\}f_q(x),$$

where $C_0=\exp\{2\pi i\lambda'\}$, and

$$|f_q(x+\gamma)|=|f_q(x)|.$$

The latter equality implies that there exist a non-zero constant k_p and $\theta_q(x)\in\Theta$ such that

$$f_q(x)=k_q\exp\{2\pi i\theta_q(x)\}.$$

The function $f_q(x)\psi_q(y)$ is the proper function belonging to the generalized proper value $e^{2\pi i(mx+\lambda')}$. Since the group $O(T)$ is an orthonormal base of $L^2(\Omega)$, we see that

$$f(x, y)=k_q f_q(x)\psi_q(y)$$

holds for some integer q. Since the function $f(x, y)$ is an arbitrary member in $G(T)_2-G(T)_1$, each member of $G(T)_2-G(T)_1$ is of the form $k_m f_m(x)\exp\{2\pi imy\}$, where $f_m(x)\in L^2(X)$. It is easy to verify that each member of $G(T)_1$ is also of the form $f_n(x)\exp\{2\pi iny\}$ where $f_n(x)\in L^2(X)$.

From the same arguments as above, it follows that $G(T)_2=G(T)_3$. Since

$$L^2(\Omega)=\sum_{-\infty}^{\infty}\oplus H_q=\overline{\text{span}\, G(T)},$$

we obtain

(8) $$H_1\cap(G(T)_2-G(T)_1)\neq\phi.$$

The relation (8) guarantees that there exists a function $f_1(x)\psi_1(y)$ in $H_1\cap(G(T)_2-G(T)_1)$ belonging to some generalized proper value $e^{2\pi i(mx+c)}$ such that

$$f_1(x+\gamma)=\exp\{2\pi i(mx-\alpha(x)+c)\}f_1(x).$$

Taking the absolute value of both sides of the above equation, we obtain $|f_1(x+\gamma)|=|f_1(x)|$. This implies that there exist a non-zero constant k_1 and $\theta(x)\in\Theta$ such that $f_1(x)=k_1\exp\{2\pi i\theta(x)\}$. Thus we obtain

$$\alpha(x)-(mx+c)=\theta(x)-\theta(x+\gamma).$$

From the fact mentioned above, we see that the α-function $\alpha(x)$ is equivalent with a function $mx+c$ where m is some integer and c some real number.

Josai University
Sakado, Saitama

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