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On submanifolds of submanifolds of a Riemannian manifold

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§ 0. Introduction.

Takahashi [1] proved that (i) in order for a submanifold M^n of an m-dimensional euclidean space E^m to be a minimal submanifold it is necessary and sufficient that the radius vector X satisfies $\Delta X=0$, where Δ denotes the Laplacian in the submanifold M^n, i.e. all the (natural) coordinate functions are harmonic, and (ii) in order that a submanifold of a hypersphere with radius r of a euclidean space is minimal, it is necessary and sufficient that the radius vector X satisfies $\Delta X=(-n/r^2)X$, where n denotes the dimension of the submanifold.

The main purpose of the present paper is to study, a submanifold M^n of a submanifold M^m of a Riemannian manifold M^l being given, the conditions that M^n is minimal in M^m or that M^n is minimal in M^l, and to obtain a theorem which generalizes two results above of Takahashi.

§ 1. Submanifold M^n of a submanifold M^m of a Riemannian manifold M^l.

Let M^l be an l-dimensional Reimannian manifold of class C^∞ covered by a system of coordinate neighborhoods $\{U; x^A\}$ where here and in the sequel the indices $A, B, C, \cdots$ run over the range $\{1, 2, \cdots, l\}$. We denote the components of the metric tensor of M^l by g_{CB}.

Let M^m be an m-dimensional differentiable submanifold of class C^∞ of M^l covered by a system of coordinate neighborhoods $\{V; y^h\}$ where here and in the sequel the indices $h, i, j, \cdots$ run over the range $\{\bar{1}, \bar{2}, \cdots, \bar{m}\}$ $(m \leqq l)$ and the local expression of M^m be

$$x^A = x^A(y^h). \tag{1.1}$$

We put

$$B_i{}^A = \partial_i x^A, \qquad \partial_i = \partial/\partial y^i \tag{1.2}$$

and denote by $C_u{}^A (u, v, w, \cdots = m+1, \cdots, l)$ $l-m$ mutually orthogonal unit normal vectors of M^m in M^l. Then the components of the metric tensor of M^m are given by

$$g_{ji}=g_{CB}B_j{}^C B_i{}^B. \tag{1.3}$$

Since $C_u{}^A$ satisfy

$$g_{CB}B_j{}^C C_u{}^B=0, \qquad g_{CB}C_v{}^C C_u{}^B=\delta_{vu}, \tag{1.4}$$

we have

$$g^{CB}=g^{ji}B_j{}^C B_i{}^B+C_u{}^C C_u{}^B, \tag{1.5}$$

g^{CB} and g^{ji} being contravariant components of the metric tensors of M^l and M^m respectively.

Now the so-called van der Waerden-Bortolotti covariant derivative of $B_i{}^A$ is given by

$$\nabla_j B_i{}^A=\partial_j B_i{}^A+\left\{ {A \atop CB} \right\} B_j{}^C B_i{}^B-\left\{ {h \atop ji} \right\} B_h{}^A \tag{1.6}$$

and is orthogonal to M^m, where $\left\{ {A \atop CB} \right\}$ and $\left\{ {h \atop ji} \right\}$ are Christoffel symbols formed with g_{CB} and g_{ji} respectively. The vector field

$$H^A(M^m, M^l)=\frac{1}{m}\, g^{ji}\nabla_j B_i{}^A \tag{1.7}$$

of M^l defined along M^m and normal to M^m is called the *mean curvature vector* of M^m in M^l. If $H^A(M^m, M^l)$ vanishes, M^m is called a *minimal submanifold* of M^l.

We now consider an n-dimensional differentiable submanifold M^n of class C^∞ of M^m covered by a system of coordinate neighborhoods $\{W, z^a\}$ where here and in the sequel the indices $a, b, c, \cdots$ run over the range $\{\dot{1}, \dot{2}, \cdots, \dot{n}\}$ $(n<m\leqq l)$ and let the local expression of M^n be

$$y^h=y^h(z^a). \tag{1.8}$$

We put

$$B_b{}^h=\partial_b y^h, \qquad \partial_b=\partial/\partial z^b \tag{1.9}$$

and denote by $C_p{}^h$ $(p, q, r, \cdots=n+1, \cdots, m)$ $m-n$ mutually orthogonal unit normal vectors of M^n in M^m. Then the components of the metric tensor of M^n are given by

$$g_{cb}=g_{ji}B_c{}^j B_b{}^i. \tag{1.10}$$

Since $C_p{}^h$ satisfy

$$g_{ji}B_c{}^j C_p{}^i=0, \qquad g_{ji}C_q{}^j C_p{}^i=\delta_{qp}, \tag{1.11}$$

we have

$$g^{ji}=g^{cb}B_c{}^jB_b{}^i+C_p{}^jC_p{}^i, \tag{1.12}$$

g^{cb} being contravariant components of the metric tensor of M^n.

The so-called van der Waerden-Bortolotti covariant derivative of $B_b{}^h$ along M^n is given by

$$\nabla_cB_b{}^h=\partial_cB_b{}^h+\left\{{h \atop ji}\right\}B_c{}^jB_b{}^i-\left\{{a \atop cb}\right\}B_a{}^h \tag{1.13}$$

and is orthogonal to M^n, where $\left\{{a \atop cb}\right\}$ are Christoffel symbols formed with g_{cb}. The vector field

$$H^h(M^n, M^m)=\frac{1}{n}g^{cb}\nabla_cB_b{}^h \tag{1.14}$$

of M^m defined along M^n and normal to M^n is the mean curvature vector of M^n in M^m. If $H^h(M^n, M^m)$ vanishes along M^n, M^n is a minimal submanifold of M^m.

Now the submanifold M^n of M^m can be regarded as a submanifold of M^l and its local expression is

$$x^A=x^A(y^h(z^a)) \tag{1.15}$$

and consequently

$$B_b{}^A=B_b{}^hB_h{}^A, \tag{1.16}$$

the fundamental tensors being related by

$$g_{cb}=g_{ji}B_c{}^jB_b{}^i=g_{CB}B_j{}^CB_i{}^BB_c{}^jB_b{}^i=g_{CB}B_c{}^CB_b{}^B. \tag{1.17}$$

The mutually orthogonal unit normals of M^n in M^l are

$$C_p{}^A=C_p{}^iB_i{}^A \quad \text{and} \quad C_u{}^A, \tag{1.18}$$

$C_p{}^A$ being tangent to M^m and $C_u{}^A$ being normal to M^m.

The van der Waerden-Bortolotti covariant derivative of $B_b{}^A$ is given by

$$\nabla_cB_b{}^A=\partial_cB_b{}^A+\left\{{A \atop CB}\right\}B_c{}^CB_b{}^B-\left\{{a \atop cb}\right\}B_a{}^A \tag{1.19}$$

and is orthogonal to M^n. The mean curvature vector of M^n in M^l is given by

$$H^A(M^n, M^l)=\frac{1}{n}g^{cb}\nabla_cB_b{}^A \tag{1.20}$$

and normal to M^n. If $H^A(M^n, M^l)$ vanishes, then M^n is minimal in M^l.

§ 2. Relations between mean curvature vectors.

Now taking the van der Waerden-Bortolotti covariant derivative of (1.16), we find [2]

$$\nabla_c B_b{}^A = (\nabla_c B_b{}^h) B_h{}^A + B_c{}^j B_b{}^i \nabla_j B_i{}^A , \tag{2.1}$$

from which

$$\frac{1}{n} g^{cb} \nabla_c B_b{}^A = \left(\frac{1}{n} g^{cb} \nabla_c B_b{}^h\right) B_h{}^A + \left(\frac{1}{n} g^{cb} B_c{}^j B_b{}^i\right) \nabla_j B_i{}^A , \tag{2.2}$$

or

$$H^A(M^n, M^l) = H^h(M^n, M^m) B_h{}^A + H^A(M^n, M^m, M^l) , \tag{2.3}$$

where we have put

$$H^A(M^n, M^m, M^l) = \left(\frac{1}{n} g^{cb} B_c{}^j B_b{}^i\right) \nabla_j B_i{}^A . \tag{2.4}$$

We call $H^A(M^n, M^m, M^l)$ the *relative mean curvature vector* of M^n with respect to M^m and M^l. $H^A(M^n, M^m, M^l)$ is a vector field normal to M^m.

The relative mean curvature vector $H^A(M^n, M^m, M^l)$ can be written as

$$H^A(M^n, M^m, M^l) = \frac{m}{n} H^A(M^m, M^l) - \frac{1}{n} C_p{}^j C_p{}^i \nabla_j B_i{}^A , \tag{2.5}$$

since

$$g^{cb} B_c{}^j B_b{}^i = g^{ji} - C_p{}^j C_p{}^i .$$

From (2.3) we have

THEOREM 2.1. *The mean curvature vector of M^n in M^l is the sum of the mean curvature vector of M^n in M^m and the relative mean curvature vector of M^n with respect to M^m and M^l.*

COROLLARY 2.2. *In order that M^n be minimal in M^m, it is necessary and sufficient that the mean curvature vector of M^n in M^l be normal to M^m.*

THEOREM 2.3. *In order for M^n to be minimal in M^l, it is necessary and sufficient that M^n is minimal in M^m and the relative mean curvature of M^n with respect to M^m and M^l vanishes.*

§ 3. Concurrent vector field [3]

We consider a vector field v^A of M^l defined along M^n and assume that v^A is concurrent along M^n, that is,

$$B_b{}^A + \nabla_b v^A = 0 . \tag{3.1}$$

From this equation, we have

$$\nabla_c B_b{}^A + \nabla_c \nabla_b v^A = 0 , \tag{3.2}$$

and consequently

$$H^A(M^n, M^l)+\frac{1}{n}g^{cb}\nabla_c\nabla_b v^A=0 \tag{3.3}$$

or, by (2.3),

$$H^h(M^n, M^m)B_h{}^A+H^A(M^n, M^m, M^l)+\frac{1}{n}g^{cb}\nabla_c\nabla_b v^A=0\,. \tag{3.4}$$

Thus, $H^A(M^n, M^m, M^l)$ being normal to M^m, if $\frac{1}{n}g^{cb}\nabla_c\nabla_b v^A$ is normal to M^m, we have $H^h(M^n, M^m)=0$. Conversely, if $H^h(M^n, M^m)=0$, then $\frac{1}{n}g^{cb}\nabla_c\nabla_b v^A$ is normal to M^m. Hence we have

THEOREM 3.1. *Suppose that there exists a vector field v^A of M^l defined along M^n and concurrent along M^n. In order for M^n to be minimal in M^m, it is necessary and sufficient that $\Delta v^A=g^{cb}\nabla_c\nabla_b v^A$ is normal to M^m.*

In particular, if $M^m=M^l$, then we have

THEOREM 3.2. *Suppose that there exists a vector field v^h of M^m defined along M^n and concurrent along M^n. In order for M^n to be minimal in M^m, it is necessary and sufficient that $g^{cb}\nabla_c\nabla_b v^h=0$,* i.e. *$\Delta v^h=0$, where Δ denotes the Laplacian operator in M^n.*

COROLLARY 3.3. (Takahashi) *Suppose that M^n is a submanifold of a hypersphere S^m in a euclidean space E^{m+1}. Then the radius vector X of S^m considered along M^n is concurrent. Thus in order for M^n to be minimal, it is necessary and sufficient that $\Delta X=g^{cb}\nabla_c\nabla_b X$ be normal to S^m, that is, proportional to X.*

COROLLARY 3.4. (Takahashi) *Suppose that M^n is a submanifold of a euclidean space E^m. Then M^n is minimal in E^m if and only if each (natural) coordinate function of M^n in E^m is harmonic.*

§ 4. Submanifolds umbilical with respect to a normal.

We consider a unit vector field e^A of M^l defined along M^n and normal to M^m and assume that M^n is umbilical with mean curvature β with respect to e^A. We choose e^A as the first normal $C_{m+1}{}^A$ to M^m, then we have equations of Gauss of M^n in M^l:

$$\nabla_c B_b{}^A=h_{cbp}C_p{}^A+\beta g_{cb}C_{m+1}{}^A+h_{cbm+2}C_{m+2}{}^A+\cdots+h_{cbl}C_l{}^A\,, \tag{4.1}$$

h_{cbp}, βg_{cb}, h_{cbm+2}, $\cdots$, h_{cbl} being second fundamental forms with respect to $C_p{}^A$, $C_{m+1}{}^A$, $\cdots$, $C_l{}^A$ respectively, from which

$$\begin{aligned} H^A(M^n, M^l)=\ &\frac{1}{n}g^{cb}h_{cbp}C_p{}^A+\beta C_{m+1}{}^A \\ &+\frac{1}{n}g^{cb}h_{cbm+2}C_{m+2}{}^A+\cdots+\frac{1}{n}g^{cb}h_{cbl}C_l{}^A\,. \end{aligned} \tag{4.2}$$

We denote by α the mean curvature of M^n in M^l, then from equation (4.2), we see that if $\alpha^2 \leqq \beta^2$, then we have $\alpha^2=\beta^2$ and

$$g^{cb}h_{cbp}=0\,, \qquad g^{cb}h_{cbm+2}=\cdots=g^{cb}h_{cbl}=0\,,$$

which show that M^n is minimal in M^m and M^n is minimal in M^l if and only if $\beta=0$. Thus we have

THEOREM 4.1. *Let e^A be a unit vector field of M^l defined along M^n and normal to M^m and assume that M^n is umbilical with mean curvature β with respect to e^A. If the mean curvature α of M^n in M^l satisfies $\alpha^2 \leqq \beta^2$, then M^n is minimal in M^m and is minimal in M^l if and only if $\beta=0$.*

We now assume that M^m is totally umbilical in M^l. Then we have

$$\nabla_j B_i{}^A = g_{ji}\alpha_u C_u{}^A\,, \tag{4.3}$$

where α_u is a vector field in the normal bundle of M^m in M^l and

$$\nabla_j C_v{}^A = -\alpha_v B_j{}^A + l_{jvu} C_u{}^A\,, \tag{4.4}$$

where l_{jvu} is the third fundamental tensor of M^m in M^l and skew symmetric in v and u.

From (4.3), we have

$$H^A(M^m, M^l)=\frac{1}{m}g^{ji}\nabla_j B_i{}^A=\alpha_u C_u{}^A \tag{4.5}$$

and

$$H^A(M^n, M^m, M^l)=\Big(\frac{1}{n}g^{cb}B_c{}^j B_b{}^i\Big)\nabla_j B_i{}^A=\alpha_u C_u{}^A\,. \tag{4.6}$$

Consequently we see that

$$H^A(M^m, M^l)=H^A(M^n, M^m, M^l)=\alpha_u C_u{}^A \tag{4.7}$$

and

$$H^A(M^n, M^l)=H^h(M^n, M^m)B_h{}^A+\alpha_u C_u{}^A\,. \tag{4.8}$$

Thus, for the covariant derivative of $H^A(M^n, M^l)$, we have

$$\begin{aligned}\nabla_c(H^A(M^n, M^l))=&\{\nabla_c(H^h(M^n, M^m))-\alpha^2 B_c{}^h\}B_h{}^A\\&+\{\partial_c\alpha_u+l_{jvu}B_c{}^j\alpha_v\}C_u{}^A\,,\end{aligned} \tag{4.9}$$

where α is the length of $H^A(M^m, M^l)$, that is,

$$\alpha=\sqrt{\alpha_u\alpha_u}\,. \tag{4.10}$$

We also assume that $H^A(M^n, M^l)$ is parallel in the normal bundle, and then we have, from (4.9),

$$\partial_c\alpha_u+l_{jvu}B_c{}^j\alpha_v=0\,, \tag{4.11}$$

from which

$$\alpha_u \partial_c \alpha_u = 0$$

and consequently the length α of $H^A(M^m, M^l)$ is constant on M^n.

Then, from (4.7) and (4.11), we have

$$\nabla_b(H^A(M^m, M^l)) = -\alpha^2 B_b{}^h B_h{}^A, \tag{4.12}$$

from which

$$\nabla_c \nabla_b(H^A(M^m, M^l)) = -\alpha^2(\nabla_c B_b{}^h)B_h{}^A - \alpha^2 B_c{}^j B_b{}^i \nabla_j B_i{}^h, \tag{4.13}$$

and consequently

$$\begin{aligned} \Delta(H^A(M^m, M^l)) &= g^{cb}\nabla_c \nabla_b(H^A(M^m, M^l)) \\ &= -n\alpha^2 H^h(M^n, M^m)B_h{}^A - n\alpha^2 H^A(M^n, M^m, M^l). \end{aligned} \tag{4.14}$$

Thus we have

THEOREM 4.2. *Assume that M^m is totally umbilical in M^l, then the mean curvature vector $H^A(M^m, M^l)$ of M^m in M^l coincides with the relative mean curvature vector of M^n with respect to M^m and M^l. Moreover assume that the mean curvature vector $H^A(M^n, M^l)$ of M^n in M^l is parallel in the normal bundle, then the length α of the mean curvature vector $H^A(M^m, M^l)$ of M^m in M^l is constant and in the case in which α is different from zero, in order for M^n to be minimal in M^m, it is necessary and sufficient that*

$$\Delta(H^A(M^m, M^l)) = -n\alpha^2 H^A(M^n, M^m, M^l)$$

on M^n.

COROLLARY 4.3. *Let M^m be totally umbilical in M^{m+1} with non-zero mean curvature or be totally umbilical in M^{m+1} of constant sectional curvature, then in order that M^n is minimal in M^m, it is necessary and sufficient that the normal C to M^m in M^{m+1} satisfies $\Delta C = g^{cb}\nabla_c \nabla_b C = fC$, f being a constant.*

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Bibliography

[1] T. Takahashi, Minimal immersions of Riemannian manifolds, J. Math. Soc. Japan, **18** (1966), 380–385.

[2] K. Yano, Sur quelques propriétés de V_l dans V_m dans V_n, Proc. Imp. Acad. Tokyo, **16** (1940), 173–177.

[3] K. Yano, Sur le parallélisme et la concourance dans l'espace de Riemann, Proc. Imp. Acad. Japan, **19** (1943), 189–197.