Notes on minimal immersions

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§ 0. Introduction.

In this note we shall study minimal immersions of Riemannian manifolds in some Riemannian manifolds with certain properties. In particular, positions of compact minimal submanifolds (with oriented boundary or without boundary) in complete Riemannian manifolds with some curvature conditions will be our main concern.

One of the essential tools in our study here is an integral-geometric inequality, to the effect that if the Laplacian Δf of a smooth function f defined over a compact Riemannian manifold (without boundary) has definite sign, then f is constant everywhere, where the Laplace-Beltrami operator is taken with respect to the induced Riemannian metric on the minimal submanifold under consideration. Hermann [7] used this to prove a uniqueness theorem for minimal submanifolds in a complete Riemannian manifold of non-positive curvature, which is a clue for us to prove that a compact minimal submanifold in a product Riemannian manifold $V^k \times R^m$, V^k being compact, is contained in $V^k \times \{p\}$, $p \in R^m$.

Another tool is a well-known concavity property, used first by Tompkins [19], by which various results on isometric immersions are obtained, for example, when an ambient manifold is of non-positive curvature [10, 14] or when an ambient manifold is of positive curvature and compact [5, 6, 17]. In this context, we shall study a compact minimal submanifold lying in some special position in a compact Riemannian manifold of positive curvature.

Throughout this note, we shall employ definitions and notation as those of [3]. In § 1, a proof of a theorem of Hermann will be given with some corrections to his original one. In § 2, we shall consider Hermann's condition in the case where the ambient manifold is of non-negative curvature and of product type $V \times R^m$. In § 3, the concavity condition for a compact minimal hypersurface in a Riemannian manifold of positive curvature will be considered.

§ 1. Hermann's theorem.

Let M be a connected and complete Riemannian m-manifold of non-positive curvature, B and N a k-dimensional totally geodesic submanifold and an n-dimensional compact oriented minimal submanifold with oriented boundary ∂N such that $\partial N \subset B$. Hermann's theorem states that *if there is a smooth map*[1] $X : N \to TM$ *in such a way that for each point* $p \in N$ *the geodesic* $a : [0, 1] \to M$ *defined by* $a(t) = \exp_p tX(p)$ *has an end point* $a(1) \in B$ *at which* a *is orthogonal to* B, *then we have* $N \subset B$.

In the following, we shall state an outline of the proof for later use. Putting

$$f(p) = \|X(p)\|^2, \qquad p \in N,$$

we shall show that

$$\Delta^N f |_p = 0, \qquad \text{for any } p \text{ in } N,$$

where Δ^N is the Laplace-Beltrami operator with respect to the Riemannian metric on N. Take an arbitrary but fixed point p in N and a normal geodesic $c : [-\varepsilon, \varepsilon] \to N$ with $c(0) = p$. We then consider a 1-parameter variation defined by

$$\alpha(u, t) = \exp_{c(u)} t \frac{X(c(u))}{\|X(p)\|}, \qquad u \in (-\varepsilon, \varepsilon), \quad t \in [0, \|X(p)\|],$$

where we may assume $X(p) \neq 0$, for in general we have $\Delta^N f |_q = 0$ if $X(q) = 0$. Putting $l = \|X(p)\|$, $X(p) = X'(p) + X''(p)$, where $X'(p) \in N_p$ and $X''(p) \in N_p^{\perp}$, and $Y(t) = \alpha_*(\partial / \partial u)(0, t)$, $T(t) = \alpha_*(\partial / \partial t)(0, t)$, we get

$$\frac{1}{2} \frac{d}{du} f \circ c(u) |_{u=0} = \langle Y, T \rangle |_0^l,$$

$$\frac{1}{2} \frac{d^2}{du^2} f \circ c(u) |_{u=0} = \int_0^l (\langle Y', Y' \rangle - K(Y, T) \|Y_1\|^2) dt$$

$$+ \frac{1}{l} S^N_{X'(p)}(Y(0), Y(0)),$$

where S^N_V is the shape operator of N with respect to the normal vector V and $K(Y, T)$ is the sectional curvature and $Y_\perp = Y - \langle Y, T \rangle T$. Therefore if we introduce normal coordinates of N with origin $p \in N$ and $c_i : [-\varepsilon, \varepsilon] \to N$, with $c_i(0) = p$, are normal geodesics such that $\langle \dot{c}_i(0), \dot{c}_j(0) \rangle = \delta_{ij}$, we have

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1) The condition (c) in the assumption of Theorem in [7] must be rewritten as stated above. In fact, end points of the family of geodesics may vary smoothly with their starting points, even if the map X is not continuous. For example, on the Clifford torus we can easily construct such a non-continuous X, where B is a closed geodesic and N is a geodesic segment whose extremals are in B.

$$\frac{1}{2}\varDelta^N f\,|_p = \sum_{i=0}^{n} \int_0^l (\langle Y_i', Y_i'\rangle - K(Y_i, T)\|Y_{i\perp}\|^2)dt$$

$$+\frac{1}{l}\sum_{i=1}^{n} S^N_{X\cdot(p)}(Y_i(0),\, Y_i(0)),$$

where we use the variations α_i, Y_i, and $Y_{i\perp}$ in the same way as α, Y, $Y_\perp$. Since N is minimal, M is of non-positive curvature and p is arbitrary, we get $\varDelta^N f\,|_p \geq 0$ for every p in N. Assume that $X|_{\partial N}\equiv 0$. Then by Green-Stokes theorem, we get $f\equiv 0$ and hence $X\equiv 0$ on N. It follows $N\subset B$.

REMARK. In the original statement of the theorem by Hermann, the condition $X|_{\partial N}\equiv 0$ is missing, without which a counter example will be constructed on the Clifford torus.

The condition automatically holds when $N\cap C(B)=\phi$, where $C(B)$ is the cut locus of B. Furthermore it is true when M is simply connected as well. However this is not true when M is not simply connected, as shown on the Clifford torus.

§2. Applications of Hermann's theorem.

In this section, we shall first consider the case where M is of non-negative curvature and B is a compact totally geodesic hypersurface. If M is non-compact, a position of B in M is determined in [16]. In fact, by Theorem 2 of [16], every point of B has a neighborhood U in which a unit normal vector field Z is defined smoothly such that for every point $q \in U$ the geodesic defined by $c: [0, \infty)\to M$, $c(t)=\exp_q tZ(q)$ is a ray from B to ∞. Hence if B has no unit normal vector field defined globally over B, every normal geodesic $\lambda_x: [0, \infty)\to M$, $\lambda_x(0)=x\in B$, such that $\dot\lambda_x(0)$ is normal to B at x, is a ray from M to ∞. Then, for each $t>0$ the set $B_t=\{\lambda_x(\pm t)\mid x\in B,\ \dot\lambda_x(0)\in B_x^\perp\}$ is a compact totally geodesic hypersurface, and the map $\phi: B_t\to B$ defined by $\phi(\lambda_x(\pm t))=x$ is a local isometry, from which B_t is isometric to $B_{t'}$ for any $t, t'>0$. Then we see that M is isometric to $B_t\times R/\phi$, where $\psi: B_t\times R\to B_t\times R$ is an isometric involution defined by $\psi(\lambda_x(t), v)=(\lambda_x(-t), -v)$, $\lambda_x(t)\in B_t$, $v\in R$. Then we get

THEOREM 2.1. *Let M be a complete and non-compact Riemannian m-manifold of non-negative curvature, B a compact totally geodesic hypersurface. Assume that B has no unit normal vector field defined globally over B. Then every compact minimal submanifold N is contained either in B, or else in some B_t, $t>0$.*

PROOF. Let $\tilde M$ be the double covering of $B_t\times R$ of M for a fixed $t>0$, π be the covering projection. Then $\pi^{-1}(N)$ is also a minimal submanifold of M. Let V be a connected component of $\pi^{-1}(N)$, which is compact and minimal

in M. Let (y, t) be a local coordinate of M. There is a large $t_0 \in R$ such that $V \cap B_{t_0} = \phi$. Then, for any $p \in V$, the smooth function f defined by $f(p)$ $= t(p) - t_0$ and the smooth map $X: V \to T\tilde{M}$, $X(p) = (t(p) - t_0)\dfrac{\partial}{\partial t}(y(p), t(p))$ are well defined. Since we have $K(\partial/\partial t, Z) = 0$ for any $Z \in TM$, we obtain

$$\Delta^N f|_p = \sum_{i=1}^{m-1} \int_0^{\|X(p)\|} \langle Y_i', Y_i' \rangle \, dt \geqq 0 .$$

This implies $f = c$, therefore V is contained in B_{c-t_0}.

Now, under the situation of Theorem 2.1, if B has a unit normal vector field defined globally over B, then B devides M into two parts $B \times (0, \infty)$ and the bounded part $D = M - B \times [0, \infty)$ each of which has boundary B. (We need not consider the case where D is unbounded, because this implies $M = B \times R$ and the argument is essentially covered in Theorem 2.1.) If N intersects with D, we know nothing about the position of N. If $N \cap D = \phi$ holds, then N is contained in some $B \times t_0$.

According to [2], there is a *soul* which is a compact totally geodesic submanifold without boundary and a totally convex set in a complete and non-compact Riemannian manifold of non-negative curvature. It seems to the authors that the relation between souls of M and compact minimal submanifolds is not yet investigated, in which we will be particularly interested. If M is compact, there will be no information about the positions of compact minimal submanifolds in M.

We shall next consider the case where M is of product type.

THEOREM 2.2. *Let B be a compact Riemannian k-manifold and put $M = B \times R^{m-k}$. If N is a compact minimal submanifold (without boundary) of M, then $N \subset B \times t_0$ holds for a certain $t_0 \in R^{m-k}$. Hence if the dimension of N is not less than k, N is isometric to B.*

PROOF. Since N is compact, there is $t_1 \in R^{m-k}$ such that $N \cap B \times t_1 = \phi$. Then N is locally expressed by $(y, t^1, \cdots, t^{m-k})$. Let $X^\alpha: N \to TM$ be defined by $X^\alpha(p) = (t^\alpha(p) - t_1^\alpha)\dfrac{\partial}{\partial t^\alpha}(y(p), t^1(p), \cdots, t^{m-k}(p))$ and $f^\alpha: N \to R$ by $f^\alpha(p) = t^\alpha(p) - t_1$. We see that $f^\alpha = c^\alpha$ (constant) for $\alpha = 1, 2, \cdots, m-k$. This implies $N \subset B \times (t_0 - c)$, where $c = (c^1, \cdots, c^{m-k})$, from which we obtain $\dim N \leqq k$.

Taking account of the theorem above, we see that the next statement is clear.

COROLLARY 2.3. *Let B and M be those of Theorem 2.2 and N a compact oriented minimal submanifold with oriented boundary ∂N such that $\partial N \subset B$. Then $N \subset B$.*

Theorem 2.2 states that every compact minimal submanifold in $M = B \times R^{m-k}$ must satisfy $\dim N \leqq \dim B$. Hence if $\dim N = \dim B$, N is isometric to B.

§ 3. *M* **of positive curvature.**

Throughout this section, let M be a complete Riemannian manifold of positive curvature and N be a compact minimal submanifold without boundary of M. Suppose that M is non-compact. Then, since, in general, there is no minimal immersion of a compact Riemannian manifold N in a non-compact, complete Riemannian manifold [17], M is necessarily compact and hence there exists a positive minimum δ of curvature of M. If N is a hypersurface and M is of positive Ricci curvature, then M must be also compact by [17]. First of all, we shall prove

LEMMA 3.1. *Let r be a point on M such that $d(r, N) = \mathrm{Max}\ \{d(N, x) \mid x \in M\}$. Then we have $d(r, N) \leqq \pi/2\sqrt{\delta}$. In particular, if N is a hypersurface and there exists a point r in M such that $d(r, N) = \pi/2\sqrt{\delta}$, then there exists a point p in N at which all eigenvalues of the shape operator are zero, i.e., p is a geodesic point.*

PROOF. Let $p \in N$ be the point such that $d(r, p) = d(r, N) = l$, and $c : [0, l] \to M$ be a shortest geodesic from r to p. For an arbitrary unit tangent vector X in N_p, let $X(t)$ be the parallel vector field defined by $X(l) = X$. Suppose that l is greater than $\pi/2\sqrt{\delta}$. We have a 1-parameter variation α along c which is associated with the vector field $Y(t) = \sin \frac{\pi t}{2l} X(t)$. Then, Proposition 3 of [1] shows that $L''(0) < 0$, which is a contradiction. If l is equal to $\pi/2\sqrt{\delta}$, we must have $K(X(t), c(t)) = \delta$ for all $t \in [0, \pi/2\sqrt{\delta}]$ and $S^N(X, X) = 0$ for any $X \in N_p$.

Now, we shall consider M admitting a point r in M such that $d(r, N) = \pi/2\sqrt{\delta}$. By virtue of the theorem due to the second named author (Proposition 2.1 of [15]), if the diameter $d(M)$ of M is greater than $\pi/2\sqrt{\delta}$, then M is simply connected. Then we have the following

THEOREM 3.2. *Let M be a complete Riemannian m-manifold whose curvature is everywhere not less than $\delta (> 0)$ and N a compact minimal hypersurface without boundary. Assume that there is a point r in M such that $d(r, N) = \pi/2\sqrt{\delta}$ and M is not simply connected. Then M is isometric to the real projective space $PR^m(\delta)$ of constant curvature δ and N is also isometric to the real projective space $PR^{m-1}(\delta)$.*

PROOF. By Proposition 2.1 in [15] for every point q in N, we have $d(r, q) = \pi/2\sqrt{\delta}$, from which it follows that q is the furthest point from r in M. Therefore we obtain $C(r) \subset N$. For any fixed point q in N and a shortest geodesic $c : [0, \pi/2\sqrt{\delta}] \to M$ from r to q, we see that $K(Z, \dot{c}(t)) = \delta$ for any Z in $M_{c(t)}$ such that $\langle Z, \dot{c}(t) \rangle = 0$, $Z \neq 0$ and any $t \in [0, \pi/2\sqrt{\delta}]$. This fact implies that r is not conjugate to q along c. Hence q and r can be joined

by another shortest geodesic $c_1 : [0, \pi/2\sqrt{\delta}\,] \to M$. Because of $\dot{c}(\pi/2\sqrt{\delta}\,)$ $\neq \dot{c}_1(\pi/2\sqrt{\delta}\,)$, we must have $\dot{c}(\pi/2\sqrt{\delta}\,) = -\dot{c}_1(\pi/2\sqrt{\delta}\,)$ by the hypothesis of dimension. We next prove that $C(r)$ is contained in N. Let $S_r^{m-1}(\pi/2\sqrt{\delta}\,)$ be a hypersphere in M_r with center origin and radius $\pi/2\sqrt{\delta}$. We observe that $\exp_r (S_r^{m-1}(\pi/2\sqrt{\delta}\,)) \supset N$. If we put $W = (\exp_r S_r^{m-1}(\pi/2\sqrt{\delta}\,))^{-1}(N)$, then we also see that $\exp_r W$ is locally regular and hence it is an open map. The compactness (without boundary) implies $W = S_r^{m-1}(\pi/2\sqrt{\delta}\,)$. This fact shows that every geodesic starting from r and of length $\pi/\sqrt{\delta}$ is a geodesic loop (or a closed geodesic segment without self-intersection). Then by a theorem due to Nakagawa [11] we have $\pi_1(M) = Z_2$. Making use of the well-known comparison theorem of Rauch, we obtain that M is of constant curvature δ. For details, see [12]. Hence M is isometric to the real projective space $PR^m(\delta)$. By virtue of Lemma 3.1, we see that N is a totally geodesic hypersurface, from which N is isometric to $PR^{m-1}(\delta)$.   Q. E. D.

THEOREM 3.3. *Let M be a complete Riemannian m-manifold whose Ricci curvature is not less than δ (>0) and N a compact minimal hypersurface (without boundary). Assume that there exists a point r in M in such a way that $d(r, N) = \mathrm{Max}\,\{d(r, x)\,|\,x \in M\} = \pi/2\sqrt{\delta}$ is satisfied. Then, both M and N are isometric to real projective spaces $PR^m(\delta)$ and $PR^{m-1}(\delta)$ of constant curvature δ, respectively.*

PROOF. Since every point of N is the furthest point from r, every shortest geodesic joining r to each point of N comes back to the starting point r with length $\pi/\sqrt{\delta}$. Let $c : [0, \pi/\sqrt{\delta}\,] \to M$ be a closed geodesic segment such that $c(0) = c(\pi/\sqrt{\delta}\,) = r$, $c(\pi/2\sqrt{\delta}\,) = q \in N$, and $X_1, X_2, \cdots, X_{m-1}$ in N_q be an orthonormal basis for N_q. Let $X_i(t)$ be the unit parallel vector field along c defined by $X_i(\pi/2\sqrt{\delta}\,) = X_i$, for $i = 1, 2, \cdots, m-1$, and Y_i be defined by $Y_i(t) = \sin \sqrt{\delta}\, t \cdot X_i(t)$. Then the variation formula for each 1-parameter variation α_i associated with Y_i implies that the Ricci curvature in the direction of $\dot{c}(t)$ is constant for $t \in [0, \pi/\sqrt{\delta}\,]$. If c has a conjugate point to 0 in the interval $(0, \pi/\sqrt{\delta}\,)$, we have a vector field Z along c and a 1-parameter variation associated with Z such that $L''(0) < 0$. Then there is a point x in N near to q in N which has the distance $d(x, r) < d(q, r) = \pi/2\sqrt{\delta}$. But this is a contradiction. Therefore we have $\mathrm{Ind}\, c = 0$ and $\mathrm{Ind}_0\, c = m-1$, which implies $\exp_r|\,W$ is locally regular, (W is defined in the proof of Theorem 3.2) and it is an open map. On the other hand, we shall claim that $Y_i(t) = \sin \sqrt{\delta}\, t \cdot X_i(t)$ is a Jacobi field along c. In fact, we obtain

$$I(Y_i, Y_i) = \int_0^{\pi/\sqrt{\delta}} (\langle Y_i', Y_i'\rangle - K(Y_i, \dot{c})\langle Y_i, Y_i\rangle)dt = 0\,,$$

for $i=1, 2, \cdots, m-1$. Since c has no conjugate point to 0 in $(0, \pi/\sqrt{\delta})$, we have $I(Y_i, V)=0$ for any piecewise smooth vector field V along c such that $V(0)=V(\pi/\sqrt{\delta})=0$ and $\langle V, \dot{c}\rangle=0$. This implies Y_i is a Jacobi field. Then it follows from $Y_i''+R(Y_i, \dot{c})\dot{c}=0$ that $K(X_i, \dot{c})(t)=\delta$ for all $t\in[0, \pi/\sqrt{\delta}]$. The rest of the proof is covered in that of Theorem 3.2.

REMARK. Calabi gave an interesting problem [9], asking whether or not a complete minimal hypersurface in a Euclidean space is unbounded, to which Omori [13] gave an affirmative answer. Then we may consider an analogous question for complete minimal submanifolds in a sphere. Is there a complete minimal submanifold in a sphere whose image is contained in a closed (or open) hemisphere? In this case the local concavity seems to be of no use.

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