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Correction to my paper: Conjugate classes of Cartan subalgebras in real semisimple Lie algebras

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In our previous paper [1], we stated in Theorem 7 (p. 415) that if $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ is a real semisimple Lie algebra of the first category, then for any maximal admissible (i.e. strongly orthogonal) subset $F=\{\alpha_1, \cdots, \alpha_l\}$ of $R_{\mathfrak{p}}$ (the set of non compact roots), the subspace

$$\mathfrak{m}(F)=\sqrt{-1}\sum_{i=1}^{l} R(E_{\alpha_i}+E_{-\alpha_i})$$

is a maximal abelian subalgebra in $\mathfrak{p}$. However this statement is false as the example at the end of this note shows. We shall prove in this note that Theorem 7 in [1] remains valid if we replace a maximal admissible subset of $R_{\mathfrak{p}}$ by an admissible subset of $R_{\mathfrak{p}}$ having the maximal number of elements. In the remaining part of [1] § 4, we used Theorem 7 to construct a maximal abelian subalgebra in $\mathfrak{p}$. As a matter of fact, all the maximal admissible sets in $R_{\mathfrak{p}}$ used in [1] have the maximal number of elements. Therefore all the results in § 4 of [1] remain valid. We use the notation in § 4 of [1]. In particular let $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ be the real semisimple Lie algebra and its Cartan decomposition, $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ contained in $\mathfrak{k}$, R be the set of all roots with respect to $\mathfrak{h}^C$, E_α $(\alpha\in R)$ be a Weyl base corresponding to the compact form $\mathfrak{g}_u=\mathfrak{k}+i\mathfrak{p}$, that is, $\mathfrak{g}_u=\mathfrak{h}+\sum_{\alpha\in R}\{R(E_\alpha+E_{-\alpha})+R\sqrt{-1}(E_\alpha-E_{-\alpha})\}$.

LEMMA 1. *The sum of three non compact positive roots is not a root.*

PROOF. Let $B=\{\alpha_1, \cdots, \alpha_r\}$ be the set of all simple roots with respect to the given linear order in R and $\beta=\sum_{i=1}^{r} m_i\alpha_i$ be the maximal root in R. Any root $\alpha=\sum_{i=1}^{r} n_i\alpha_i$ satisfies the inequality

$$n_i \leqq m_i \quad (1\leqq i\leqq r). \tag{1}$$

We can assume that $\mathfrak{g}$ is simple. In this case there exists only one non compact root, say α_j, in B. The coefficients of α_j in β satisfies

$$1\leqq m_j\leqq 2. \tag{2}$$

The set $R_{\mathfrak{p}}^+$ of all non compact positive roots is given by

(3) $$R_{\mathfrak{p}}^{+}=\{\alpha=\sum_{i=1}^{r}n_i\alpha_i \mid n_j=1\}\ .$$

(For the proof of (1), (2) and (3), see e.g., Murakami [2].) (1), (2) and (3) prove Lemma 1.

Two roots α and β in $\boldsymbol{R}$ are said to be strongly orthogonal if neither $\alpha+\beta$ nor $\alpha-\beta$ belongs to $\boldsymbol{R}\cup\{0\}$.

Using Lemma 1, we can generalize a result of Harish-Chandra ([3] Lemma 7 and Lemma 8) to general semisimple Lie algebra of the first category. He proved the result for the Lie algebras of automorphism groups of symmetric bounded domains. For any subset Q in $\boldsymbol{R}_{\mathfrak{p}}^{+}$, put

$$\mathfrak{p}(Q)=\sum_{\gamma\in Q}(\mathfrak{g}^{\gamma}+\mathfrak{g}^{-\gamma})\ .$$

If β is the lowest root in Q, we define $Q(\beta)$ as

$$Q(\beta)=\{\gamma\in Q \mid \gamma\neq\beta,\ \gamma\pm\beta\notin\boldsymbol{R}\}\ .$$

LEMMA 2. *The centralizer* $\mathfrak{z}$ *of* $E_{\beta}+E_{-\beta}$ *in* $\mathfrak{p}(Q)$ *is equal to*

$$C(E_{\beta}+E_{-\beta})+\mathfrak{p}(Q(\beta))\ .$$

PROOF. Let X be an element in $\mathfrak{p}(Q)$ and $Q'=Q-\{\beta\}$. Then X can be written as

$$X=c_{\beta}E_{\beta}+c_{-\beta}E_{-\beta}+\sum_{\gamma\in Q'}(c_{\gamma}E_{\gamma}+c_{-\gamma}E_{-\gamma})\ .$$

Since the $\mathfrak{h}^c$-component of $[X, E_{\beta}+E_{-\beta}]$ in the root space decomposition $\mathfrak{g}^c=\mathfrak{h}^c+\sum_{\alpha\in\boldsymbol{R}}\mathfrak{g}^{\alpha}$ is $(c_{\beta}-c_{-\beta})[E_{\beta}, E_{-\beta}]$, we have

(4) $$c_{\beta}=c_{-\beta} \quad \text{if} \quad X\in\mathfrak{z}\ .$$

Moreover if $X\in\mathfrak{z}$, then

$$Y=\sum_{\gamma\in Q'}(c_{\gamma}E_{\gamma}+c_{-\gamma}E_{-\gamma})$$

also belongs to $\mathfrak{z}$, and we have

(5) $$\begin{aligned}0&=[Y, E_{\gamma}+E_{-\gamma}]\\&=\sum_{\gamma\in Q'}\{c_{\gamma}[E_{\gamma}, E_{\beta}]+c_{\gamma}[E_{\gamma}, E_{-\beta}]+c_{-\gamma}[E_{-\gamma}, E_{\beta}]+c_{-\gamma}[E_{-\gamma}, E_{-\beta}]\}\ .\end{aligned}$$

$c_{\gamma}[E_{\gamma}, E_{\beta}]$ is the only term in the right hand side of (5) belonging to $\mathfrak{g}^{\gamma+\beta}$. Because, if we assume that $\gamma+\beta=\delta-\beta$ or $-\delta+\beta$ or $-\delta-\beta$ for some $\delta\in Q'$, then $\gamma+\beta+\beta=\delta$ or $\gamma+\beta+\delta=\beta$ or $\gamma+\delta+\beta=-\beta$ is a root. A contradiction to Lemma 1. Therefore by (2) we have

(6) $$c_{\gamma}[E_{\gamma}, E_{\beta}]=0 \qquad \text{for all} \quad \gamma\in Q'\ .$$

Similarly we have

(7) $$c_{-\gamma}[E_{-\gamma}, E_{-\beta}]=0 \qquad \text{for all} \quad \gamma\in Q'\ .$$

For the term $c_\gamma[E_\gamma, E_{-\beta}]$ and $c_{-\gamma}[E_{-\gamma}, E_\beta]$, the above argument is not valid. Nevertheless we have

$$(8) \qquad c_\gamma[E_\gamma, E_{-\beta}] = c_{-\gamma}[E_{-\gamma}, E_\beta] = 0 \quad \text{for all } \gamma \in Q'.$$

Suppose $c_\gamma[E_\gamma, E_{-\beta}] \neq 0$. Then there exists a root $\delta \in Q'$ such that $\gamma-\beta = -\delta+\beta = \alpha \in \boldsymbol{R}$. (Note that there exists no root δ in Q' such that $\gamma-\beta = \delta+\beta$ or $\gamma-\beta = -\delta-\beta$ by Lemma 1.) But this implies $0 < \delta = \beta-\alpha < \beta$, which contradicts the fact that β is the lowest root in Q. So we have proved (8). (6), (7) and (8) imply that if X belongs to $\mathfrak{z}$, then c_γ and $c_{-\gamma}$ must be equal to zero for all $\gamma \in Q'$ satisfying either $\gamma+\beta \in \boldsymbol{R}$ or $\gamma-\beta \in \boldsymbol{R}$. Therefore the centralizer $\mathfrak{z}$ of $E_\beta+E_{-\beta}$ in $\mathfrak{p}(Q)$ is contained in $C(E_\beta+E_{-\beta})+\mathfrak{p}(Q(\beta))$. Since obviously we have $C(E_\beta+E_{-\beta})+\mathfrak{p}(Q(\beta)) \subset \mathfrak{z}$, Lemma 2 is proved.

Once Lemma 2 is established for general real semisimple Lie algebra of the first category, the following Lemma 3 is proved by Lemma 2 in exactly the same way as in the proof of Lemma 8 and its corollary in [3].

LEMMA 3. *There exists a strongly orthogonal subset* $\boldsymbol{F}$ *of* $\boldsymbol{R}_\mathfrak{p}$ such that

$$\sum_{\gamma \in \boldsymbol{F}} C(E_\gamma+E_{-\gamma})$$

is a maximal abelian subalgebra in $\mathfrak{p}^c$ *and*

$$\mathfrak{m}(\boldsymbol{F}) = \sqrt{-1} \sum_{\gamma \in \boldsymbol{F}} R(E_\gamma+E_{-\gamma})$$

is a maximal abelian subalgebra in $\mathfrak{p}$.

Now we have the correct version of Theorem 7 in [1].

THEOREM 7. *Let* $\mathfrak{g} = \mathfrak{k}+\mathfrak{p}$ *be a Cartan decomposition of a real semisimple Lie algebra* $\mathfrak{g}$ *and* $\mathfrak{h}$ *be a Cartan subalgebra of* $\mathfrak{g}$ *contained in* $\mathfrak{k}$. *Let* $\boldsymbol{F} = \{\alpha_1, \cdots, \alpha_l\}$ *be any strongly orthogonal subset of* $\boldsymbol{R}_\mathfrak{p}$ *with the maximal number of elements. Then there exists an automorphism* ρ *of* $\mathfrak{g}^C$ *and a Cartan subalgebra* $\mathfrak{h}_0$ *of* $\mathfrak{g}$ *satisfying the following conditions:* 1) ρ *transforms* $\mathfrak{h}^C$ *onto* $\mathfrak{h}_0^C$. 2) $\mathfrak{h}_0^- = \mathfrak{m}(\boldsymbol{F}) = \sqrt{-1} \sum_{i=1}^{l} \boldsymbol{R}(E_{\alpha_i}+E_{-\alpha_i})$ *is a maximal abelian subalgebra in* $\mathfrak{p}$. 3) $\boldsymbol{R}' = \{\alpha' = {}^t\rho(\alpha) \mid \alpha \in \boldsymbol{R}\}$ *is the root system of* $\mathfrak{g}^C$ *with respect to* $\mathfrak{h}_0^C$, *and* $\{H_{\alpha'_1}, \cdots, H_{\alpha'_l}\}$ *spans* $\mathfrak{h}_0^-$.

PROOF. Let Γ be the set of all strongly orthogonal subsets of $\boldsymbol{R}_\mathfrak{p}$. It is clear that if $\boldsymbol{F}$ belongs to Γ, then $\mathfrak{m}(\boldsymbol{F}) = \sqrt{-1} \sum_{\gamma \in \boldsymbol{F}} R(E_\gamma+E_{-\gamma})$ is an abelian subalgebra in $\mathfrak{p}$. Lemma 3 assures that there exists at least one $\boldsymbol{F}_0$ in Γ such that $\mathfrak{m}(\boldsymbol{F}_0)$ is a maximal abelian subalgebra in $\mathfrak{p}$. Since all the maximal abelian subalgebras in $\mathfrak{p}$ have the same dimension ([1] Proposition 3), $\boldsymbol{F}_0$ is an element of Γ having the maximal number of elements, because, if not, there exists a maximal abelian subalgebra in $\mathfrak{p}$ having the greater dimension than $\mathfrak{m}(\boldsymbol{F}_0)$. Moreover for any $\boldsymbol{F}$ in Γ having the maximal number of elements, $\mathfrak{m}(\boldsymbol{F})$ is a maximal abelian subalgebra in $\mathfrak{p}$, because $\dim \mathfrak{m}(\boldsymbol{F}) = \dim \mathfrak{m}(\boldsymbol{F}_0)$. The proof of the remaining part of Theorem 7 in [1] is valid for the above revised Theorem 7.

EXAMPLE. Let $\boldsymbol{R}=\{\pm e_i, \pm e_i \pm e_j ; 1 \leqq i, j \leqq n\}$ be the root system of type B_n. Then

$$B=\{\alpha_1=e_1-e_2, \alpha_2=e_2-e_3, \cdots, \alpha_{n-1}=e_{n-1}-e_n, \alpha_n=e_n\}$$

is a base of $\boldsymbol{R}$. Let $\mathfrak{g}$ be the real simple Lie algebra of type $B_n I_2$. Then α_1 is the only non compact root in B and, by (3), the set $\boldsymbol{R}_{\mathfrak{p}}$ of non compact roots is given by

$$\boldsymbol{R}_{\mathfrak{p}}=\{\pm e_1, \pm(e_1-e_i), \pm(e_1+e_i)\ (i \geqq 2)\},$$

and the set

$$A=\{e_1-e_2, e_1+e_2\}$$

is a strongly orthogonal subset in $\boldsymbol{R}_{\mathfrak{p}}$. On the other hand, the set

$$M=\{e_1\}$$

is a maximal strongly orthogonal subset in $\boldsymbol{R}_{\mathfrak{p}}$.

As a corollary to the above revised Theorem 7, we have obtained the following relation between two methods of classification of real simple Lie algebras given by Murakami [2] and Araki [4].

A pair $(\boldsymbol{R}, \sigma)$ of a root system $\boldsymbol{R}$ and an involutive automorphism σ of $\boldsymbol{R}$ is called a root system with an involution. Two root systems with involutions $(\boldsymbol{R}, \sigma)$ and $(\boldsymbol{R}', \sigma')$ are called isomorphic if there exists an isomorphism φ of $\boldsymbol{R}$ onto $\boldsymbol{R}'$ satisfying $\varphi \circ \sigma=\sigma' \circ \varphi$.

Let $\mathfrak{g}$ be a real semisimple Lie algebra and σ be the conjugation of the complexification $\mathfrak{g}^c$ of $\mathfrak{g}$ with respect to $\mathfrak{g}$. Moreover let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ whose vector part has maximal dimension and $\boldsymbol{R}$ be the root system of $(\mathfrak{g}^c, \mathfrak{h}^c)$. The conjugation σ acts on the dual space $\mathfrak{h}^{c*}$ of $\mathfrak{h}^c$ by $(\sigma\lambda)(H)=\overline{\lambda(\sigma H)}$. Then σ leaves the root system $\boldsymbol{R}$ stable and the pair $(\boldsymbol{R}, \sigma)$ is a root system with an involution. Another choice of a Cartan subalgebra whose vector part has maximal dimension gives a root system with an involution which is isomorphic to $(\boldsymbol{R}, \sigma)$. A root system with an involution $(\boldsymbol{R}, \sigma)$ thus obtained is called the root system with an involution of a real semisimple Lie algebra $\mathfrak{g}$.

COROLLARY TO THEOREM 7. *Let $\mathfrak{g}, \mathfrak{p}, \mathfrak{h}$ and $\boldsymbol{F}$ be the same as in Theorem 7 and let $\mathfrak{h}^+=\sum_{i=1}^{l} RH_{\alpha_i}$ and $\mathfrak{h}^-=\mathfrak{h}^{+\perp} \cap \sqrt{-1}\,\mathfrak{h}$. Then the linear transformation σ on $\sqrt{-1}\,\mathfrak{h}$ with $\mathfrak{h}^+$ and $\mathfrak{h}^-$ as the eigenspaces of eigenvalues $+1$ and -1 respectively is an involutive automorphism of the root system $\boldsymbol{R}$. The root system with an involution $(\boldsymbol{R}, \sigma)$ is isomorphic to the root system with an involution of $\mathfrak{g}$.*

ERRATA of [1]: p. 394, line 20, read $\sum_{i=1}^{k-1} RU_{\alpha_i}$ for $\sum_{i=1}^{k-1} R_{\alpha_i}$. p. 414, line 26, read $V_\alpha = \frac{\pi}{2}\sqrt{-1}(E_\alpha - E_{-\alpha})/(2(\alpha, \alpha))^{\frac{1}{2}}$ for $V_\alpha = \sqrt{-1}(E_\alpha - E_{-\alpha})/(2(\alpha, \alpha))^{\frac{1}{2}}$.

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