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Satake compactification and the great Picard theorem

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§ 1. Introduction.

Let Δ be the unit disk $\{z \in \boldsymbol{C};\ |z|<1\}$ in the complex plane and Δ^* the punctured disk $\{z \in \boldsymbol{C};\ 0<|z|<1\}$. Let $P_1(\boldsymbol{C})$ be the 1-dimensional complex projective space, $P_1(\boldsymbol{C}) = \boldsymbol{C} \cup \{\infty\}$. Delete three points, say, $0, 1, \infty$, from $P_1(\boldsymbol{C})$. The great Picard theorem says that every holomorphic mapping $f: \Delta^* \to P_1(\boldsymbol{C}) - \{0, 1, \infty\}$ can be extended to a holomorphic mapping $f: \Delta \to P_1(\boldsymbol{C})$.

We consider a generalization of the great Picard theorem. Given a complex space M, let d_M be the intrinsic pseudo-distance introduced in [3]. We say that M is hyperbolic if d_M is a distance on M. For example, $P_1(\boldsymbol{C}) - \{0, 1, \infty\}$ is hyperbolic. Consider the following question.

"Let Y be a complex space and M a complex hyperbolic subspace of Y such that its closure $\bar{M}$ is compact. Does every holomorphic mapping $f: \Delta^* \to M$ extend to a holomorphic mapping $f: \Delta \to Y$?"

The answer is, in general, negative as shown by Kiernan [2] (see also [4, Ch. VI, § 1]). On the other hand, we have the following result, [4].

THEOREM 1. *Let Y be a complex space and M a complex subspace of Y satisfying the following conditions:*

(1) *M is hyperbolic;*

(2) *the closure $\bar{M}$ of M is compact;*

(3) *Given a point p on the boundary $\partial M = \bar{M} - M$ and a neighborhood $\mathcal{U}$ of p, there exists a smaller neighborhood $\mathcal{V}$ of p in Y such that*

$$d_M(M \cap (Y - \mathcal{U}),\ M \cap \mathcal{V}) > 0 .$$

Let X be a complex manifold and A a locally closed complex submanifold of X. Then every holomorphic mapping $X - A \to M$ extends to a holomorphic mapping $X \to Y$.

It has been shown in [4; Ch. VI, § 6] that if $Y = P_2(\boldsymbol{C})$ and $M = P_2(\boldsymbol{C}) - Q$, where Q is a complete quadrilateral, then the three conditions of Theorem 1 are satisfied. Hence, every holomorphic mapping of $X - A$ into $P_2(\boldsymbol{C}) - Q$ extends to a holomorphic mapping of X into $P_2(\boldsymbol{C})$. This may be considered

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as a generalized great Picard theorem.

The purpose of this paper is to give another example of $M \subset Y$ satisfying the three conditions of Theorem 1.

THEOREM 2. *Let D be a symmetric bounded domain in C^N and Γ an arithmetically defined discrete subgroup of the largest connected group G of holomorphic automorphisms of D. Let Y be the Satake compactification of $M = D/\Gamma$. Then M and Y satisfy the three conditions of Theorem* 1, *provided that Γ acts freely on D.*

We shall make comments in Remark 1 below on the technical assumption that Γ acts freely on D.

From Theorems 1 and 2, we obtain immediately the following

COROLLARY. *Let M and Y be as in Theorem 2. Let X be a complex manifold and A a locally closed complex submanifold of X. Then every holomorphic mapping $X - A \to M$ extends to a holomorphic mapping $X \to Y$.*

REMARK 1. In order to include into our consideration the case where the action of Γ is not free, we have to use a modified intrinsic pseudo-distance d'_M on a V-manifold M. Let D be a complex manifold and Γ a properly discontinuous group of holomorphic automorphisms of D. Put $M = D/\Gamma$. Then M is a V-manifold in the sense of Satake. Since M is a complex space, we have an intrinsic pseudo-distance d_M. In the definition of d_M, use only those holomorphic mappings f from the disk Δ in M which can be lifted to holomorphic mappings $\tilde{f}$ from Δ to D. Then we obtain a modified intrinsic pseudo-distance d'_M. This pseudo-distance may be defined also by

$$(*) \qquad d'_M(p, q) = d_D(\eta^{-1}(p), \eta^{-1}(q)) \qquad p, q \in M,$$

where $\eta : D \to D/\Gamma = M$ is the projection. For details, see [4; Ch. VII, § 6]. Of course, if Γ acts freely on D, then $d_M = d'_M$. Then Theorem 1 can be modified as follows:

THEOREM 1′. *Let $M = D/\Gamma$ be a complex subspace of a complex space Y. Assume*

(1′) *the pseudo-distance d'_M is a distance;*

(2) *the closure $\bar{M}$ of M is compact;*

(3′) *Given a point $p \in \partial M$ and a neighborhood $\mathcal{U}$ of p in Y, there exists a smaller neighborhood $\mathcal{V}$ of p in Y such that*

$$d'_M(M \cap (Y - \mathcal{U}), M \cap \mathcal{V}) > 0 .$$

Let X be a complex manifold and A a locally closed complex submanifold of X. Then every locally liftable holomorphic mapping $X - A \to M$ extends to a holomorphic mapping $X \to Y$.

A holomorphic mapping $f: X - A \to M$ is said to be *locally liftable* if, for each point x of $X - A$, there exist a neighborhood N_x and a holomorphic mapping $f_x: N_x \to D$ such that $\eta \circ f_x = f$ on N_x.

Theorem 2 can be modified as follows:

THEOREM 2′. *Let D, Γ, $M = D/\Gamma$ and Y be as in Theorem 2 (but without the condition that Γ acts freely on D). Then M and Y satisfy the three conditions of Theorem 1′.*

Accordingly, Corollary can be also modified. In the proof of Theorem 2 or Theorem 2′, we have only to verify the condition (3) or (3′). The remaining conditions are trivially satisfied. In the proof of Theorem 2′, the equality (*) above will be used as the definition of the distance d'_M. Actually, the proof will be written in terms of d_D. Although it may be possible to prove Theorem 2′ using the distance defined by an invariant hermitian metric of D, the intrinsic distance d_D allows us to prove our main proposition (Proposition 2.5) even for non-homogeneous Siegel domains.

REMARK 2. In connection with Theorem 1, we mention the following result of Kwack [5], (see also [4]).

Let M be a hyperbolic complex space, X a complex manifold and A a locally closed complex subspace of X. Then every holomorphic mapping $X - A \to M$ extends to a holomorphic mapping $X \to M$ if one of the following conditions is satisfied:

(1) *M is compact;*

(2) *M is complete with respect to d_M and* codim $A \geqq 2$.

She proved this result in her attempt to prove Corollary above.

REMARK 3. We have been informed that Corollary has been proved recently by A. Borel by a different method. During the spring quarter of 1970, W. Schmid presented his own proof of Corollary for the case where D is a generalized upper-halfplane of Siegel in his seminar in Berkeley.

REMARK 4. For the compactification of D/Γ, we have used the method of Pyatetzki-Shapiro [6]. One can easily check that this is equivalent to that of [1] (See W. L. Baily, Fourier-Jacobi Series, Proc. Symp. Pure Math., Vol. IX, Amer. Math. Soc., 1966).

§ 2. Siegel domains of the third kind and cylindrical subsets [6] [7] [9].

Let V be an n-dimensional real vector space. A *convex cone* Ω in V is an open convex subset such that

i) if $y \in \Omega$ and $t > 0$, then $ty \in \Omega$;

ii) Ω contains no straight line.

The open subset T_Ω of $V_C = V + iV$ defined by

$$T_\Omega = \{x + iy \in V_C\,;\ y \in \Omega\}$$

is called the *tube domain* associated to Ω. It is well known that the tube domain T_Ω is analytically equivalent to a bounded domain. The domain T_Ω

is also called the *Siegel domain of the first kind* associated to Ω.

An Ω-*hermitian form* on an m-dimensional complex vector space W is a mapping $H: W\times W\to V_C$ such that

i) $H(\alpha u+\beta v, w)=\alpha H(u, w)+\beta H(v, w)$ for $u, v, w\in W$, $\alpha, \beta\in \boldsymbol{C}$;

ii) $H(u, v)=\overline{H(v, u)}$ for $u, v\in W$,

where $\overline{H(v, u)}$ is the natural complex conjugate of $H(u, v)$ in V_C;

iii) $H(u, u)\in\bar{\Omega}$ for $u\in W$,

where $\bar{\Omega}$ denotes the topological closure of Ω;

iv) $H(u, u)=0$ only if $u=0$.

The open subset $D(H, \Omega)$ of $V_C\times W$ defined by

$$D(H, \Omega)=\{(x+iy, w)\in V_C\times W\ ;\ y-H(w, w)\in\Omega\}$$

is called the *Siegel domain of the second kind* associated to H and Ω. It is also analytically equivalent to a bounded domain. The domain $D(H, \Omega)$ always has analytic automorphisms of the following type:

$$(1)\qquad \begin{cases} z\mapsto z+a+2iH(w, b)+iH(b, b) \\ w\mapsto w+b\,, \end{cases}$$

where $a\in V$ and $b\in W$.

In order to define the Siegel domains of the third kind following [7], we consider the set $\mathcal{K}$ of all complex antilinear mappings $p: W\to W$ such that

i) $H(pu, v)=H(pv, u)$ for $u, v\in W$;

ii) $H(u, u)-H(pu, pu)\in\bar{\Omega}$ for $u\in W$;

iii) $H(u, u)\neq H(pu, pu)$ if $u\neq 0$.

The totality of complex antilinear mappings $p: W\to W$ satisfying only (i) forms a complex vector space in which $\mathcal{K}$ is a bounded domain. We need the following lemma.

LEMMA 2.1. *If $p\in\mathcal{K}$, then $I+p$ is a real linear isomorphism of W onto itself, where I denotes the identity transformation of W.*

PROOF. Suppose $(I+p)w=0$. Then $H(pw, pw)=H(-w, -w)=H(w, w)$. From (iii) above, we obtain $w=0$. QED.

For $p\in\mathcal{K}$, we define $L_p: W\times W\to V_C$ by

$$L_p(u, v)=H(u, (I+p)^{-1}v)\qquad \text{for}\quad u, v\in W\,.$$

Now, let $\mathcal{D}$ be a bounded domain in a complex vector space U and φ an analytic mapping from $\mathcal{D}$ into $\mathcal{K}$. We define a domain $D(H, \Omega, \mathcal{D}, \varphi)$ of $U\times V_C\times W$ by

$$D(H, \Omega, \mathcal{D}, \varphi)=\{(t, z, w)\in U\times V_C\times W\ ;\ t\in\mathcal{D},\ \mathrm{Im}\,(z)-\mathrm{Re}\,(L_{\varphi(t)}(w, w))\in\Omega\}\,.$$

This domain is called the *Siegel domain of the third kind* associated to $H, \Omega, \mathcal{D}$, and φ. This domain admits automorphisms of the following type:

$$(2)\qquad \begin{cases} t \mapsto t \\ z \mapsto z+a+2iH(w, b)+iH((I+\varphi(t))b, b) \\ w \mapsto w+b+\varphi(t)b\,, \end{cases}$$

where $a \in V$, $b \in W$.

LEMMA 2.2. *$\mathrm{Re}\,(L_p(w, w)) \in \bar{\Omega}$ for $p \in \mathcal{K}$ and $w \in W$.*

PROOF. Put $c = I+p$. From the definition of L_p, we have

$$L_p(cv, cv) = H(cv, v) \qquad \text{for} \quad v \in W\,.$$

Hence,

$$\begin{aligned} &2\,\mathrm{Re}\,(L_p(cv, cv)) - H(cv, cv) \\ &\quad = 2\,\mathrm{Re}\,(H(cv, v)) - \{H(v, v)+H(pv, pv)+H(v, pv)+H(pv, v)\} \\ &\quad = 2H(v, v)+2\,\mathrm{Re}\,(H(pv, v)) - \{H(v, v)+H(pv, pv)+2\,\mathrm{Re}\,(H(pv, v))\} \\ &\quad = H(v, v)-H(pv, pv) \in \bar{\Omega} \qquad \text{(from the definition of } \mathcal{K})\,. \end{aligned}$$

Since c is surjective by Lemma 2.1, we obtain

$$2\,\mathrm{Re}\,(L_p(w, w)) - H(w, w) \in \bar{\Omega} \qquad \text{for} \quad w \in W\,.$$

Since $H(w, w) \in \bar{\Omega}$ by the definition of H and since $\bar{\Omega}$ is convex, we obtain

$$\mathrm{Re}\,(L_p(w, w)) = \frac{1}{2}\{H(w, w)+(2\,\mathrm{Re}\,(L_p(w, w))-H(w, w))\} \in \bar{\Omega}\,. \qquad \text{QED.}$$

For $r \in \Omega$, we define a subdomain D_r of $D = D(H, \Omega, \mathcal{D}, \varphi)$ by

$$D_r = \{(t, z, w) \in D\,;\ \mathrm{Im}\,(z) - \mathrm{Re}\,(L_{\varphi(t)}(w, w)) - r \in \Omega\}\,.$$

More generally, for an open set $\mathcal{O}$ in $\mathcal{D}$, the set

$$D_r(\mathcal{O}) = \{(t, z, w) \in D_r\,;\ t \in \mathcal{O}\}$$

is called a *cylindrical set* with base $\mathcal{O}$. In particular, $D_r = D_r(\mathcal{D})$.

LEMMA 2.3. *The cylindrical set $D_r(\mathcal{O})$ is invariant under the transformations of the type* (2).

PROOF. If $(t, z, w) \to (t', z', w')$ is a transformation of the type (2), then

$$(2)\qquad \begin{cases} t' = t \\ z' = z+a+2iH(w, b)+iH((I+\varphi(t))b, b) \\ w' = w+b+\varphi(t)b\,. \end{cases}$$

It suffices therefore to prove that D_r is invariant by a transformation of the type (2). We have

$$\begin{aligned} &\mathrm{Im}\,(z') - \mathrm{Re}\,(L_{\varphi(t)}(w', w')) - r \\ &\quad = \mathrm{Im}\,(z) + 2\,\mathrm{Re}\,(H(w, b)) + \mathrm{Re}\,(H((I+\varphi(t))b, b)) \end{aligned}$$

$$-\mathrm{Re}\,\{H(w+(1+\varphi(t))b,\ (I+\varphi(t))^{-1}(w+(I+\varphi(t))b))\}-r$$
$$=\mathrm{Im}\,(z)+2\,\mathrm{Re}\,(H(w,b))+\mathrm{Re}\,(H((I+\varphi(t))b,b))$$
$$-\mathrm{Re}\,\{H(w+(I+\varphi(t))b,\ b+(I+\varphi(t))^{-1}w)\}-r$$
$$=\mathrm{Im}\,(z)+2\,\mathrm{Re}\,(H(w,b))+\mathrm{Re}\,(H((I+\varphi(t))b,b))$$
$$-\mathrm{Re}\,\{H(w,b)+H(w,(I+\varphi(t))^{-1}w)$$
$$+H((I+\varphi(t))b,b)+H((I+\varphi(t))b,(I+\varphi(t))^{-1}w)\}-r$$
$$=\mathrm{Im}\,(z)-\mathrm{Re}\,(L_{\varphi(t)}(w,w))-r$$
$$+\mathrm{Re}\,\{H(w,b)-H((I+\varphi(t))b,(I+\varphi(t))^{-1}w)\}\,.$$

It suffices therefore to prove

$$\mathrm{Re}\,\{H(w,b)-H((I+\varphi(t))b,(I+\varphi(t))^{-1}w)\}=0\,.$$

We have, for $e\in W$,

$$H((I+\varphi(t))b,e)=H(b,e)+H(\varphi(t)b,e)$$
$$=H(b,e)+H(\varphi(t)e,b)\qquad \text{(definition of } \mathcal{K}\text{, (i))}$$
$$=H(b,e)+\overline{H(b,\varphi(t)e)}\qquad (H\text{: hermitian})\,.$$

Hence,

$$\mathrm{Re}\,(H((I+\varphi(t))b,e))=\mathrm{Re}\,(H(b,e))+\mathrm{Re}\,(\overline{H(b,\varphi(t)e)})$$
$$=\mathrm{Re}\,(H(b,e))+\mathrm{Re}\,(H(b,\varphi(t)e))$$
$$=\mathrm{Re}\,(H(b,(I+\varphi(t))e))\,.$$

If we set $e=(I+\varphi(t))^{-1}w$ in the equality above, then

$$\mathrm{Re}\,(H((I+\varphi(t))b,(I+\varphi(t))^{-1}w))=\mathrm{Re}\,(H(b,w))=\mathrm{Re}\,(H(w,b))\,,$$

thus proving the desired equality. QED.

The following lemma is evident.

LEMMA 2.4.

$$D_r(\mathcal{O})\supset D_{tr}(\mathcal{O})\qquad if\quad t>1\,.$$

We state the main proposition of this section.

PROPOSITION 2.5. *Let $D=D(H,\Omega,\mathcal{D},\varphi)$ be a Siegel domain of the third kind. Then*

$$d_D(a,b)\geqq\log t\qquad for\quad a\in D-D_r,\ b\in D_{tr},\ t>1,\ r\in\Omega\,,$$

where d_D denotes the intrinsic distance of D explained in § 1.

We prove the proposition in several steps.

LEMMA 2.6. *Let $V=\boldsymbol{R}$, $\Omega=\{a\in\boldsymbol{R}\,;\ a>0\}$ and $D=T_\Omega=\{z\in\boldsymbol{C}\,;\ \mathrm{Im}\,(z)>0\}$. Then*

$$d_D(a,b)\geqq\log t\qquad for\quad a\in D-D_r,\ b\in D_{tr},\ t>1,\ r\in\Omega\,.$$

PROOF. The intrinsic distance d_D is identical in this case with the distance defined by the Bergman metric $(dx^2+dy^2)/y^2$. Hence,

$$d_D(a, b) \geqq d_D(\mathrm{Im}\,(a), \mathrm{Im}\,(b)) \geqq d_D(ir, itr) = d_D(i, it) = \log t . \qquad \text{QED.}$$

LEMMA 2.7. *Let* $V = \boldsymbol{R}^n$, $\Omega = \{(y^1, \cdots, y^n) \in \boldsymbol{R}^n ; y^1 > 0, \cdots, y^n > 0\}$ *and* $D = T_\Omega = \{(z^1, \cdots, z^n) \in \boldsymbol{C}^n ; \mathrm{Im}\,(z^1) > 0, \cdots, \mathrm{Im}\,(z^n) > 0\}$. *Then*

$$d_D(a, b) \geqq \log t \qquad \text{for} \quad a \in D - D_r,\ b \in D_{tr},\ t > 1,\ r \in \Omega .$$

PROOF. Let $a = (a^1, \cdots, a^n)$, $b = (b^1, \cdots, b^n)$ and $r = (r^1, \cdots, r^n)$. Then

$$\mathrm{Im}\,(a^j) \leqq r^j \qquad \text{for some} \quad j,\ 1 \leqq j \leqq n ,$$
$$\mathrm{Im}\,(b^i) > tr^i \qquad \text{for all} \quad i,\ 1 \leqq i \leqq n .$$

We can write $D = D_1 \times \cdots \times D_1$, where D_1 is the domain defined by $D_1 = \{z \in \boldsymbol{C} ; \mathrm{Im}\,(z) > 0\}$. Let $p_j : D \to D_1$ be the projection to the j-th factor. Since p_j is holomorphic and hence distance-decreasing, we have

$$d_D(a, b) \geqq d_{D_1}(p_j a, p_j b) = d_{D_1}(a^j, b^j) .$$

Applying Lemma 2.6 to the domain D_1, we obtain

$$d_{D_1}(a^j, b^j) \geqq \log t .$$

Hence,

$$d_D(a, b) \geqq \log t . \qquad \text{QED.}$$

LEMMA 2.8. *Let Ω be a convex cone in an n-dimensional real vector space V. Let* $D = T_\Omega = \{z \in V_C ; \mathrm{Im}\,(z) \in \Omega\}$. *Then*

$$d_D(a, b) \geqq \log t \qquad \text{for} \quad a \in D - D_r,\ b \in D_{tr},\ t > 1,\ r \in \Omega .$$

PROOF. Put $y = \mathrm{Im}\,(a)$. Consider the line $y + sr$, $(-\infty < s < \infty)$; this is a line through y and parallel to the vector r. We shall show that this line meets the boundary $\partial\Omega$ of Ω exactly at one point, say, y_0. Since this line contains a point of Ω, e. g., $y \in \Omega$ and since the convex cone Ω cannot contain a whole straight line, this line meets the boundary $\partial\Omega$. If y_0 is any point where this line meets $\partial\Omega$, we may write $y_0 = y + s_0 r$. If $\varepsilon > 0$, then

$$y + (s_0 + \varepsilon) r = y_0 + \varepsilon r = (1+\varepsilon)\left(\frac{1}{1+\varepsilon} y_0 + \frac{\varepsilon}{1+\varepsilon} r\right).$$

Since $y_0 \in \bar{\Omega}$, $r \in \Omega$ and Ω is convex, it follows that $\frac{1}{1+\varepsilon} y_0 + \frac{\varepsilon}{1+\varepsilon} r$ is in Ω. Since Ω is a cone, $(1+\varepsilon)\left(\frac{1}{1+\varepsilon} y_0 + \frac{\varepsilon}{1+\varepsilon} r\right)$ is in Ω. This shows that the half-line $\{y + sr ; s > s_0\}$ is completely contained in Ω. Hence, y_0 is the unique intersection point.

We claim that there exists a basis $e_1, \cdots, e_n$ of V such that the open convex cone $\Omega_n = \{\sum y^i e_i \in V ; y^1 > 0, \cdots, y^n > 0\}$ contains Ω and $y_0 \in \partial\Omega_n$. In order to prove our claim, we use the following well known fact on the dual cone. Let V^* be the dual space of V and define the dual cone Ω^* of Ω by

$$\Omega^* = \{y^* \in V^* ; \langle y^*, y\rangle > 0 \text{ for all nonzero } y \in \bar{\Omega}\} .$$

Then $\Omega^{**} = \Omega$. In particular, Ω^* is an open convex cone in V^*. It is easy to see that there exists a nonzero element e_1^* in the closure of Ω^* such that $\langle e_1^*, y_0\rangle = 0$. Choose $e_2^*, \cdots, e_n^*$ in Ω^* so that $e_1^*, \cdots, e_n^*$ is a basis for V^*; this is possible because Ω^* is an open cone. Then the dual basis $e_1, \cdots, e_n$ for V possesses the desired property.

Put $D_n = T_{\Omega_n} = \{z \in V_C ; \operatorname{Im}(z) \in \Omega_n\}$. Since $D_{tr} = \{z \in D ; \operatorname{Im}(z) - tr \in \Omega\}$ and $D_{n,tr} = \{z \in D_n ; \operatorname{Im}(z) - tr \in \Omega_n\}$, we have $D_{tr} \subset D_{n,tr}$. Hence $b \in D_{tr}$ implies $b \in D_{n,tr}$. We shall now show that $a \in D_n - D_{n,r}$. Since $y = \operatorname{Im}(a)$ and $a \notin D_r$, it follows that $y - r \notin \Omega$. Since $y + sr$ is in Ω if and only if $s > s_0$ as we saw above, we may conclude that $-1 \leqq s_0$. Since the line $y + sr$, $(-\infty < s < \infty)$, meets $\partial\Omega_n$ also exactly at one point $y_0 = y + s_0 r$, we see that $y + sr$ is in Ω_n if and only if $s > s_0$. Hence, $y - r$ is not in Ω_n. This shows that $a \notin D_{n,r}$.

Since the injection $h : D \to D_n$ is holomorphic and hence distance-decreasing, we have

$$d_D(a, b) \geqq d_{D_n}(ha, hb) = d_{D_n}(a, b) .$$

Applying Lemma 2.7 to the domain D_n, we have

$$d_{D_n}(a, b) \geqq \log t .$$

Hence,

$$d_D(a, b) \geqq \log t . \qquad \text{QED.}$$

PROOF OF PROPOSITION 2.5. Let $D = D(H, \Omega, \mathcal{D}, \varphi)$, $a \in D - D_r$ and $b \in D_{tr}$ with $t > 1$. Put $a = (\tilde{t}, \tilde{z}, \tilde{w}) \in U \times V_C \times W$. Since $I + \varphi(\tilde{t})$ is a real automorphism of W by Lemma 2.1, the Siegel domain D of the third kind admits an automorphism of the type (2) which sends $a = (\tilde{t}, \tilde{z}, \tilde{w})$ into $(\tilde{t}, \tilde{z}, 0)$. Since such an automorphism of D leaves the distance d_D invariant and, by Lemma 2.3, leaves the domains D_r and D_{tr} invariant, we may assume without loss of generality that $a = (\tilde{t}, \tilde{z}, 0)$.

Let $\rho : U \times V_C \times W \to V_C$ be the natural projection. We claim that ρ maps D into $D' = T_\Omega = \{z \in V_C ; \operatorname{Im}(z) \in \Omega\}$. In fact, if (t, z, w) is in D so that $\operatorname{Im}(z) - \operatorname{Re}(L_{\varphi(t)}(w, w)) \in \Omega$, then $\operatorname{Im}(z) \in \Omega$ because $\operatorname{Re}(L_{\varphi(t)}(w, w))$ is in $\bar{\Omega}$ by Lemma 2.2. Hence, z is in D', proving our claim. In particular, $\rho(a)$ is in D'. Since $a = (\tilde{t}, \tilde{z}, 0)$ is not in D_r, it follows that $\operatorname{Im}(\tilde{z}) - r$ is not in Ω. Hence $\tilde{z} = \rho(a)$ is not in D'_r, thus proving $\rho(a) \in D' - D'_r$. From Lemma 2.2 it follows easily that $b \in D_{tr}$ implies $\rho(b) \in D'_{tr}$. Since ρ is holomorphic and hence distance-decreasing, we have

$$d_D(a, b) \geqq d_{D'}(\rho(a), \rho(b)) .$$

Applying Lemma 2.8 to the domain D', we have

$$d_{D'}(\rho(a), \rho(b)) \geqq \log t .$$

Hence,

$$d_D(a, b) \geqq \log t . \qquad \text{QED.}$$

§ 3. Boundary components of symmetric bounded domains [1], [6], [7], [8], [9].

Let D be a symmetric bounded domain in C^N in the so-called Harish-Chandra realization. Let $\bar{D}$ be the topological closure of D and put $\partial D = \bar{D} - D$. A subset $\mathcal{F}$ of ∂D is called a *boundary component* of D if (i) $\mathcal{F}$ is an analytic subset of C^N and (ii) $\mathcal{F}$ is minimal with respect to the property that any analytic arc contained in ∂D and having a point in common with $\mathcal{F}$ must be entirely contained in $\mathcal{F}$. Then each boundary component $\mathcal{F}$ is again a bounded symmetric domain. And if $\mathcal{F}'$ is another boundary component of D and if $\mathcal{F}' \subset \partial \mathcal{F}$, then $\mathcal{F}'$ is a boundary component of $\mathcal{F}$ also. For each boundary component $\mathcal{F}$ of D, there exists a Siegel domain of the third kind $D(H, \Omega, \mathcal{F}, \varphi)$ which is biholomorphic to D. In the following, we fix such a realization $D(H, \Omega, \mathcal{F}, \varphi)$ once and for all for each D and $\mathcal{F}$.

Let G be the identity component of the group of automorphisms of D. Then each element of G extends to an automorphism of a neighborhood of $\bar{D}$. Let Γ be a discrete subgroup of G defined arithmetically. We consider only those boundary components $\mathcal{F}$ which are called the rational boundary components with respect to Γ. Let B denote the union of all rational boundary components of D and set

$$D^* = D \cup B .$$

The action of Γ on D extends to D^* in a natural manner. With a topology described below, $D^*/\Gamma = (D/\Gamma) \cup (B/\Gamma)$ is the so-called Satake compactification of D/Γ. Let $\eta : D^* \to D^*/\Gamma$ denote the natural projection. For each point of D/Γ, a basis of its neighborhood system is given by its neighborhood system in D/Γ with the usual quotient topology. For a point p in B/Γ, we construct a basis of its neighborhood system as follows. Assume $p \in \eta(\mathcal{F})$ and let $\tilde{p} \in \mathcal{F}$ be a point such that $\eta(\tilde{p}) = p$. Consider the family of all rational boundary components $\mathcal{E}$ of D such that $\mathcal{F} \subset \partial \mathcal{E}$. It is known that there are only a finite number of Γ-equivalence classes in this family. Let $\mathcal{F}_1, \cdots, \mathcal{F}_m$ be a system of representatives for these Γ-equivalence classes. Thus the family $\{\gamma(\mathcal{F}_i);\ \gamma \in \Gamma$ and $i = 1, \cdots, m\}$ exhausts the rational boundary components $\mathcal{E}$ of D such that $\mathcal{F} \subset \partial \mathcal{E}$. Let $\mathcal{O}$ be an open neighborhood of $\tilde{p}$ in $\mathcal{F}$. Considering D as a Siegel domain $D(H, \Omega, \mathcal{F}, \varphi)$ of the third kind, we consider a cylindrical set $D_r(\mathcal{O})$ in D (as defined in § 2), where r is an element of the open convex cone Ω. Each $\mathcal{F}_i$ is also a Siegel domain $\mathcal{F}_i = D(H_i, \Omega_i, \mathcal{F}, \varphi_i)$ of the third kind. We choose a cylindrical set $\mathcal{F}_{i,r_i}(\mathcal{O})$ in $\mathcal{F}_i$, where $r_i \in \Omega_i$. Put

$$\tilde{\mathcal{U}} = \mathcal{O} \cup D_r(\mathcal{O}) \cup \mathcal{F}_{1,r_1}(\mathcal{O}) \cup \cdots \cup \mathcal{F}_{m,r_m}(\mathcal{O})$$

and

$$\mathcal{U} = \eta(\tilde{\mathcal{U}}) .$$

We take the family of $\mathcal{U}$ with varying $\mathcal{O}, r, r_1, \cdots, r_m$ as a basis for the open neighborhood system for $\tilde{p}$.

LEMMA 3.1. *Let $D_r(\mathcal{O})$ be a cylindrical set in D with a base $\mathcal{O}$ in a boundary component $\mathcal{F}$. Let $\mathcal{O}'$ be an open set in $\mathcal{F}$ such that $\bar{\mathcal{O}}' \subset \mathcal{O}$ and let $D_{tr}(\mathcal{O}')$ be a cylindrical set in D with a base $\mathcal{O}'$, where $t > 1$. Then*

$$d_D(a, b) \geqq \mathrm{Min}\,\{\log t,\ d_{\mathcal{F}}(\mathcal{F} - \mathcal{O}, \mathcal{O}')\} \qquad \textit{for} \quad a \in D - D_r(\mathcal{O}),\ b \in D_{tr}(\mathcal{O}') .$$

PROOF. Let $\theta : D = D(H, \Omega, \mathcal{F}, \varphi) \to \mathcal{F}$ be the natural projection. If $\theta(a) \in \mathcal{O}$, then $a \in D - D_r$ and Proposition 2.5 implies $d_D(a, b) \geqq \log t$. Suppose $\theta(a) \notin \mathcal{O}$. Since θ is holomorphic and hence distance-decreasing, we have

$$d_D(a, b) \geqq d_{\mathcal{F}}(\theta a, \theta b) \geqq d_{\mathcal{F}}(\mathcal{F} - \mathcal{O}, \mathcal{O}') . \qquad \text{QED.}$$

PROOF OF THEOREM 2′. Let p be a point of B/Γ and $\mathcal{U}$ a neighborhood of p in $D^*/\Gamma = (D/\Gamma) \cup (B/\Gamma)$. We have to prove that there is a smaller neighborhood $\mathcal{V}$ of p in D^*/Γ such that $\bar{\mathcal{V}} \subset \mathcal{U}$ and

$$d_D(a, b) \geqq \delta \quad \text{if} \quad a, b \in D, \quad \eta(a) \notin \mathcal{U} \quad \text{and} \quad \eta(b) \in \mathcal{V} ,$$

where δ is a positive number which depends only on $\mathcal{U}$ and $\mathcal{V}$ but not on a, b. We choose $\tilde{p} \in \mathcal{F}$ such that $\eta(\tilde{p}) = p$.

Without loss of generality, we may assume that $\mathcal{U} = \eta(\tilde{\mathcal{U}})$, where $\tilde{\mathcal{U}}$ is of the form

$$\tilde{\mathcal{U}} = \mathcal{O} \cup D_r(\mathcal{O}) \cup \mathcal{F}_{1,r_1}(\mathcal{O}) \cup \cdots \cup \mathcal{F}_{m,r_m}(\mathcal{O}) ,$$

where $\mathcal{O}$ is an open neighborhood of $\tilde{p} \in \mathcal{F}$, (see the definition of the topology in D^*/Γ above). Let $\mathcal{O}'$ be a smaller neighborhood of $\tilde{p}$ in $\tilde{\mathcal{F}}$ such that $\bar{\mathcal{O}}' \subset \mathcal{O}$ and let $t > 1$. Put

$$\tilde{\mathcal{V}} = \mathcal{O}' \cup D_{tr}(\mathcal{O}') \cup \mathcal{F}_{1,tr_1}(\mathcal{O}') \cup \cdots \cup \mathcal{F}_{m,tr_m}(\mathcal{O}')$$

and

$$\mathcal{V} = \eta(\tilde{\mathcal{V}}) .$$

Let $a, b \in D$, $\eta(a) \notin \mathcal{U}$ and $\eta(b) \in \mathcal{V}$. Since b is equivalent to a point in $\tilde{\mathcal{V}}$ under the group Γ and since d_D is invariant by Γ, we may assume that $b \in \tilde{\mathcal{V}}$. Clearly, $a \notin \tilde{\mathcal{U}}$. Since $a \in D$ and $a \notin \tilde{\mathcal{U}}$, we have $a \in D - D_r(\mathcal{O})$. Since $b \in D$ and $b \in \tilde{\mathcal{V}}$, we have $b \in D_{tr}(\mathcal{O}')$. By Lemma 3.1,

$$d_D(a, b) \geqq \delta ,$$

where

$$\delta = \mathrm{Min}\,\{\log t,\ d_{\mathcal{F}}(\mathcal{F} - \mathcal{O}, \mathcal{O}')\} . \qquad \text{QED.}$$

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