J. Math. Soc. Japan
Vol. 23, No. 1, 1971

Dimension for σ-metric spaces

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(Received May 15, 1970)

§ 1. Introduction.

The class of metric spaces is well designed for dimension theory. At the present stage we have no class of non-metric spaces which can be said harmonious for dimension theory. The aim of this paper is to define new spaces which we name σ-metric spaces and to show that the class of paracompact σ-metric spaces is pretty effective for harmonious dimension theory. We note here that all spaces in this paper are Hausdorff and all mappings are continuous. As for undefined terminology and notations refer to Nagami [**15**] or to Nagata [**17**].

DEFINITION 1. A space X is said *σ-metric* if it is the countable sum of closed metric subsets X_i, $i=1, 2, \cdots$, where $\{X_i: i=1, 2, \cdots\}$ is said a *scale* of X. A scale is said *monotone* if $X_1 \subset X_2 \subset \cdots$.

Every σ-metric space has of course a monotone scale. Every CW-complex is σ-metric. Even every chunk complex in Ceder [**3**] is also σ-metric. As Dowker [**4**] pointed out, the product of two CW-complexes need not be CW. Thus such a product offers an example which is not a CW-complex but a σ-metric space.

Many dimension theoretical theorems for metric spaces are trivially true for normal σ-metric spaces. The following are some of them: i) Coincidence theorem; dim = Ind. ii) Decomposition theorem. iii) Two kinds of monotone decomposition theorems due to the author [**14**]. iv) The existence of equi-dimensional G_δ envelope. v) Dimension preserving property by exactly k-to-one closed mappings. vi) Product theorem. For some cases we need the assumption of paracompact σ-metric spaces. Theorems 4 and 5 are examples of many propositions which are trivially true for paracompact σ-metric spaces. Theorem 6 below is the main result of this paper. By this theorem dimension raising theorems for metric spaces are automatically transferred to those for paracompact σ-metric spaces, as can be seen in Corollaries 1 to 4. It is to be noted that Corollary 3 gives the first positive support of Arhangelskii's theorem [**1**] as well as of its generalization due to Okuyama [**19**]. We can say that the class of normal σ-metric spaces has almost no meaning for dimension theory, while the class of paracompact σ-metric spaces has an interesting feature because of effectiveness for dimension raising theorems.

§ 2. Some properties of σ-metric spaces.

Since every σ-metric space is a σ-space, every collectionwise normal σ-metric space is paracompact and the countable product of paracompact σ-metric spaces is again paracompact by Okuyama [18]. As Hodel mentioned to the author (see [7]), the space H in Bing [2] and its simplification G due to Michael [9] offer examples of normal σ-metric spaces which are not collectionwise normal. The finite product of σ-metric spaces is evidently σ-metric, while the countable product of them need not be σ-metric as the following theorem shows.

THEOREM 1. *Let X be the countable product of non-metric spaces X_i, $i=1, 2, \cdots$. Then X is not σ-metric.*

PROOF. Assume that X is a σ-metric space with a scale $\{Y_i\}$. Since Y_1 is metric, $X-Y_1 \neq \emptyset$. Pick a point $y_1=\langle y_{11}, y_{12}, \cdots\rangle$ from $X-Y_1$. Since Y_1 is closed, there exists a number n_1 such that $D_1 \cap Y_1=\emptyset$ where

$$D_1=\langle y_{11}, \cdots, y_{1n_1}\rangle \times \prod \{X_j : j>n_1\}.$$

Since Y_2 is metric, $D_1-Y_2 \neq \emptyset$. Pick a point $y_2=\langle y_{21}, y_{22}, \cdots\rangle$ from D_1-Y_2. Let n_2 be a number greater than n_1 such that $D_2 \cap Y_2=\emptyset$ where

$$D_2=\langle y_{21}, \cdots, y_{2n_2}\rangle \times \prod \{X_j : j>n_2\}.$$

Repeating this procedure we get a sequence $n_1<n_2<\cdots$ of numbers and a sequence $y_i=\langle y_{i1}, y_{i2}, \cdots\rangle$, $i=1, 2, \cdots$, of points of X such that $D_i \cap Y_i=\emptyset$ for each i, where

$$D_i=\langle y_{i1}, \cdots, y_{in_i}\rangle \times \prod \{X_i : j>n_i\},$$

and such that $y_{i+1} \in D_i$ for each i. Set

$$y=\langle y_{11}, \cdots, y_{1n_1}, y_{2n_1+1}, \cdots, y_{2n_2}, y_{3n_2+1}, \cdots\rangle.$$

Then y is not contained in any Y_i and hence not in X, which is a contradiction. The proof is completed.

DEFINITION 2. Let X be a σ-metric space. A metric space ρX with a metric ρ defined on the set X is said a *replica* of the space X if the following conditions are satisfied.

i) The identity transformation ρ of X to ρX is continuous.

ii) There exists a common monotone scale $\{X_i\}$ of X and of ρX with respect to which $\rho | X_i$ is a homeomorphism for each i.

Such a replica is said one with respect to $\{X_i\}$.

THEOREM 2. *Let X be a paracompact σ-metric space and $\{X_i\}$ its monotone scale. Then there exists a replica ρX with respect to $\{X_i\}$.*

PROOF. Let $\mathfrak{U}_{ij}$, $j=1, 2, \cdots$, be a sequence of locally finite open coverings of X such that the restriction of it to X_i is a base of X_i. The existence of such a sequence is assured by Hanner [6]. Let U be an arbitrary element of $\mathfrak{U}_{ij}$. Since every paracompact σ-metric space is perfectly normal, there exists a non-negative continuous function f_U defined on X such that $U=\{x: f_U(x)>0\}$. Set

$$\rho_{ij}(x, y)=\sum\{|f_U(x)-f_U(y)| : U\in\mathfrak{U}_{ij}\} .$$

Then ρ_{ij} is a pseudo-metric on X with respect to which every element of $\mathfrak{U}_{ij}$ is open. Set

$$\rho(x, y)=\sum\sum(\rho_{ij}(x, y)/2^{i+j}(1+\rho_{ij}(x, y))) .$$

Then ρ is a metric on the set X. Let ρX be the metric space thus obtained. This ρX satisfies the following conditions.

i) $\cup\mathfrak{U}_{ij}$ is a base of ρX. Especially every element of $\cup\mathfrak{U}_{ij}$ is open in ρX.

ii) Every X_i is closed in ρX.

iii) Every X_i in X is homeomorphic to itself in ρX.

Thus we know that ρX is a replica of X with respect to $\{X_i\}$ and the theorem is proved.

This argument contains a very simple and direct proof of Nagata-Smirnov's metrization theorem. In the sequel ρ thus defined is said the subordinate metric to $\cup\mathfrak{U}_{ij}$.

THEOREM 3. *A paracompact σ-metric space X is the inverse limit of all replicas.*

PROOF. Let $\rho_1 X$ or $\rho_2 X$ be a replica of X with respect to a monotone scale $\{X_{1i}\}$ or $\{X_{2i}\}$ respectively. Let $\mathfrak{U}_1$ or $\mathfrak{U}_2$ be a σ-locally finite base of $\rho_1 X$ or $\rho_2 X$ respectively. If $\cup\mathfrak{U}_{ij}$ in the preceding proof is replaced by $\rho_1^{-1}(\mathfrak{U}_1)\cup\rho_2^{-1}(\mathfrak{U}_2)$, we get the subordinate metric ρ_3 corresponding to it. Then $\rho_3 X$ is a replica with respect to $\{X_{1i}\}$, $\{X_{2i}\}$ and $\{X_{1i}\cap X_{2i}\}$ at the same time. The projections of $\rho_3 X$ to $\rho_i X$, $i=1, 2$, are continuous. Thus the system of replicas is directed and forms an inverse system. Let ρX, having a σ-locally finite base $\mathfrak{U}$, be an arbitrary replica of X with respect to $\{X_i\}$. Let U be an arbitrary open set of X. Set $\mathfrak{V}=\rho^{-1}(\mathfrak{U})\cup\{U\}$. Then for each i, $\mathfrak{V}|X_i$ is a σ-locally finite base of X_i. Let $\rho' X$ be the metric space subordinate to this $\mathfrak{V}$. Then $\rho' X$ is a replica with respect to $\{X_i\}$ and $\rho'(U)$ is open in $\rho' X$. Thus X is essentially the inverse limit of all replicas and the theorem is proved.

THEOREM 4. *For a paracompact σ-metric space X the following are equi-*

valent.

i) $\dim X \leqq n$.

ii) *There exist a mapping* $f: X \to I^n$ *and a scale* $\{X_i\}$ *of* X *such that* $f|X_i$ *is uniformly* 0*-dimensional for each* i.

PROOF. ii)→i): By Katětov [8] $\dim X_i \leqq n$. Thus $\dim X \leqq n$ by the sum theorem.

i)→ii): Let $\{X_i\}$ be a monotone scale of X and ρX a replica with respect to it. Again by Katětov [8] there exists a uniformly 0-dimensional mapping g of ρX to I^n. Set $f = g\rho$. Then f is the desired one and the theorem is proved.

THEOREM 5. *Let* X *be a paracompact* σ*-metric space and* F *a closed set of* X *with* $\dim(X-F) = m > 0$. *Let* n *be a positive integer with* $n \leqq m$ *and* f *a mapping of* F *to* I^n. *Then there exists a mapping* $g: X \to I^n$ *which has the following properties.*

i) g *is an extension of* f.

ii) $\dim(g^{-1}(y) - F) \leqq m-n$ *for each* $y \in I^n$.

iii) *If moreover* $\dim f^{-1}(y) \leqq m-n$ *for each* $y \in I^n$, *then* $\dim g^{-1}(y) \leqq m-n$ *for each* $y \in I^n$.

PROOF. This is true for the case when X is a metric space as was shown by Sakai [20]. It is to be noted that it stems from the work of Fort, Jr. [5]. Let $\cup \mathfrak{V}_i$ be a σ-locally finite open covering of X stated in the proof of Theorem 2. Let $\mathfrak{U}$ be a σ-locally finite base of I^n and $\mathfrak{W}$ a σ-locally finite open covering of X such that $\mathfrak{W}|F = f^{-1}(\mathfrak{U})$. Set

$$\mathfrak{G} = (\cup \mathfrak{V}_i) \cup \mathfrak{W} \cup \{X-F\}.$$

Let ρX be a replica of X where ρ is a metric subordinate to $\mathfrak{G}$. Let $\hat{f}: \rho F \to I^n$ be the transformation defined by: $\hat{f} = f\rho^{-1}$.

$$\begin{array}{ccc} X \supset F & \xrightarrow{f} & I^n \\ \downarrow \rho & & \| \\ \rho X \supset \rho F & \xrightarrow{\hat{f}} & I^n \end{array}$$

Then the following conditions are satisfied.

a) $\dim(\rho X - \rho F) = m$.

b) $\hat{f}$ is continuous.

c) ρF is closed in ρX.

d) $\dim \hat{f}^{-1}(y) \leqq m-n$ if $\dim f^{-1}(y) \leqq m-n$.

Let $\hat{g}: \rho X \to I^n$ be an extension of f satisfying the conditions i), ii) and iii) where f, g or F is respectively replaced with $\hat{f}$, $\hat{g}$ or ρF. Set $g = \hat{g}\rho$. Then g is the desired extension and the theorem is proved.

§ 3. Dimension-raising perfect mappings.

THEOREM 6. *Let X be a paracompact σ-metric space, ρX its replica, σZ a metric space and $\hat{f}: \sigma Z \to \rho X$ a mapping onto.*

$$\begin{array}{ccc} Z & \xrightarrow{f} & X \\ \downarrow \sigma & \hat{f} & \downarrow \rho \\ \sigma Z & \longrightarrow & \rho X \end{array}$$

Then there exist a paracompact σ-metric space Z and a mapping f of Z to X satisfying the following conditions.

i) *σZ is a replica of Z.*

ii) *$\rho f = \hat{f}\sigma$.*

iii) *If every point-inverse under $\hat{f}$ is compact, then every point-inverse under f is also compact.*

iv) *If $\hat{f}$ is perfect, then f is perfect.*

v) *If $\hat{f}$ is open, then f is open.*

PROOF (suggested by the referee, simplifying the original one). Set $Z = \{\langle z, x\rangle \in \sigma Z \times X : \hat{f}(z) = \rho x\}$. Since $\sigma Z \times X$ is paracompact by Morita [**11**, Theorem 5.1] and Z is closed in it, Z is paracompact. Let $\sigma : Z \to \sigma Z$ and $f: Z \to X$ be the projections. Let $\{X_i\}$ be a monotone scale of X. Set $\sigma Z_i = \hat{f}^{-1}(\rho X_i)$ and $Z_i = \sigma^{-1}(\sigma Z_i)$. Since $Z \cap (\sigma Z_i \times X) = Z \cap (\sigma Z_i \times \rho X_i)$, $\sigma | Z \cap (\sigma Z_i \times X) : Z \cap (\sigma Z_i \times X) \to \sigma Z_i$ is a homeomorphism onto. Hence Z is a σ-metric space with the scale $\{Z_i\}$. The conditions i), ii), iii) and v) are evidently satisfied. To verify the condition iv) let $\beta\hat{f}: \beta(\sigma Z) \to \beta(\rho X)$ or $\beta\rho : \beta X \to \beta(\rho X)$ be respectively the extension of $\hat{f}$ or ρ over the Stone-Čech compactifications. Set $Z' = \{\langle z, x\rangle \in \beta(\sigma Z) \times \beta X : \beta\hat{f}(z) = \beta\rho(x)\}$. Let $\sigma' : Z' \to \beta(\sigma Z)$ and $f' : Z' \to \beta X$ be the projections. Then f' is perfect. Since $\hat{f}$ is perfect, $(f')^{-1}(X) = Z$ and $f' | (f')^{-1}(X)$ is perfect. Since $f' | (f')^{-1}(X) = f$, f is perfect and the theorem is proved.

Since $\dim X = \dim \rho X$ and $\dim Z = \dim \sigma Z$, the following four corollaries are trivially true from our lifting Theorem 6 and known theorems for metric spaces.

COROLLARY 1. *Let X be a nonempty paracompact σ-metric space. Then X is the image of a paracompact σ-metric space Z with* $\dim Z = 0$ *under a perfect mapping.*

Cf. a similar theorem to the metric case due to Morita [**10**].

COROLLARY 2. *Let X be a nonempty paracompact σ-metric space. Then X is the image of a paracompact σ-metric space Z with* $\dim Z = 0$ *under an open mapping such that every point-inverse is compact.*

Cf. a similar theorem to the metric case due to Nagami [**13**].

COROLLARY 3. *Let X be a nonempty paracompact σ-metric space of countable dimension. Then there exist a paracompact σ-metric space Z with* $\dim Z=0$ *and a perfect mapping f of Z onto X such that $f^{-1}(x)$ is finite for each point x in X.*

Cf. a similar theorem to the metric case due to Nagata [**16**].

COROLLARY 4. *Let X be a nonempty paracompact σ-metric space with* $\dim X=n$. *Let k be an arbitrary integer with $0\leqq k\leqq n$. Then there exist a paracompact σ-metric space Z with* $\dim Z=k$ *and a perfect mapping f of Z onto X such that* $\operatorname{ord} f=n-k+1$.

Cf. a similar theorem to the metric case due to Nagami [**12**].

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