

## The ranges of certain isometries of tensor products of Banach spaces

By Sadahiro SAEKI

(Received March 9, 1970)

### § 1. The main result.

Let  $X$  be a compact (Hausdorff, non-empty) space and  $B$  a Banach space. We then denote by  $C(X: B)$  the Banach space of all continuous  $B$ -valued functions on  $X$ , by  $C(X) \otimes B$  the algebraic tensor product of  $C(X)$  and  $B$ , and by  $C(X) \widehat{\otimes} B$  the completion of  $C(X) \otimes B$  with the projective  $\otimes_\pi$  norm.  $C(X) \widehat{\otimes} B$  can and will be regarded in the natural manner as a linear subspace of  $C(X: B)$  [4].

Suppose that  $Y$  is another compact space, and that  $p$  is a continuous mapping from  $X$  onto  $Y$ . The mapping  $p$  induces a linear isometry  $P$  from  $C(Y)$  into  $C(X)$  defined by

$$(P(f))(x) = f(p(x)) \quad (f \in C(Y), x \in X).$$

Consider now the operator

$$P_B = P \widehat{\otimes} I_B: C(Y) \widehat{\otimes} B \longrightarrow C(X) \widehat{\otimes} B,$$

where  $I_B$  denotes the identity operator on  $B$ . As is shown in § 2,  $P_B$  is an isometry.

In this paper we are concerned with the problem of determining the range of the operator  $P_B$ . This problem was solved by N. Th. Varopoulos [7; pp. 65-70] under some assumptions on  $X$  and  $Y$ . To state our main result, let us introduce the notation

$$[C(X) \widehat{\otimes} B; p] = \{F \in C(X) \widehat{\otimes} B: F = G \circ p \text{ for some } G \in C(Y: B)\},$$

which is obviously a closed linear subspace of  $C(X) \widehat{\otimes} B$  containing the range of  $P_B$ .

**THEOREM 1.1.** *The range of  $P_B$  is exactly the space  $[C(X) \widehat{\otimes} B; p]$ .*

In § 2 we give some auxiliary theorems. The proof of Theorem 1.1 is given in § 3. § 4 is devoted to generalize Theorem 1.1. Finally, § 5 contains some remarks on the problem of spectral synthesis for tensor algebras and group algebras.

## § 2. Some theorems on tensor products of Banach spaces.

THEOREM 2.1. *The operator  $P_B = P \widehat{\otimes} I_B$  defined in § 1 is an isometry.*

PROOF. We construct an approximating inverse of the operator  $P$  as follows (for the definition, see [7; p. 53]). Let  $u = \{U_k\}_1^n$  be any finite open covering of  $Y$ . Then there exist functions  $\{\varphi_k\}_1^n$  in  $C(Y)$  such that [5; (3.1)]

$$(1) \quad \sum_{k=1}^n \varphi_k(y) = 1 \quad \text{for all } y \in Y;$$

$$(2) \quad 0 \leq \varphi_k \leq 1 \quad \text{and} \quad \varphi_k(U_k^c) = \{0\} \quad \text{for all } k = 1, 2, \dots, n.$$

Let  $\{x_k \in p^{-1}(U_k)\}_1^n$  be any choice of points, and define an operator  $Q_u: C(X) \rightarrow C(Y)$  by

$$Q_u(g) = \sum_{k=1}^n g(x_k) \varphi_k \quad (g \in C(X)).$$

Then, for any  $f \in C(Y)$  and  $y \in Y$  we have

$$\begin{aligned} |((Q_u \circ P)(f))(y) - f(y)| &\leq \sum_{k=1}^n |f(p(x_k)) - f(y)| \varphi_k(y) \\ &\leq \sum_{k=1}^n \sup \{|f(y') - f(y'')| : y', y'' \in U_k\} \cdot \varphi_k(y) \\ &\leq \sup_k \sup \{|f(y') - f(y'')| : y', y'' \in U_k\}. \end{aligned}$$

Therefore it is easy to see that the family  $\{Q_u : u \in \mathcal{U}\}$  of operators is an approximating inverse of  $P$ , where  $\mathcal{U}$  is the directed set consisting of all finite open coverings of  $Y$  (for  $u$  and  $v$  in  $\mathcal{U}$ ,  $u < v$  if and only if  $v$  is a refinement of  $u$ ). It follows from [7; p. 54] that  $P_B$  is an isometry.

REMARK. Using approximating inverses constructed as above, we can improve the result in [7; p. 63] as follows.

Let  $\{X_j\}_1^N$  and  $\{Y_j\}_1^N$  be two families of  $N$  compact spaces, let  $\{p_j: X_j \rightarrow Y_j\}_1^N$  be  $N$  continuous "onto" mappings, and denote by

$$p = p_1 \times \dots \times p_N: X_1 \times \dots \times X_N \longrightarrow Y_1 \times \dots \times Y_N$$

the product mapping of  $\{p_j\}_1^N$ . Then the algebra homomorphism

$$P: C(Y_1) \widehat{\otimes} \dots \widehat{\otimes} C(Y_N) \longrightarrow C(X_1) \widehat{\otimes} \dots \widehat{\otimes} C(X_N),$$

naturally induced by  $p$ , has a local approximating inverse (for the definition, see [7; p. 57]).

Let now  $\{B_j \supset K_j\}_1^N$  be Banach spaces and closed linear subspaces and let  $\{Q_j: B_j \rightarrow B_j/K_j\}_1^N$  be the quotient mappings. It is then trivial that the kernel of the mapping

$$Q = Q_1 \widehat{\otimes} \cdots \widehat{\otimes} Q_N : B_1 \widehat{\otimes} \cdots \widehat{\otimes} B_N \longrightarrow B_1/K_1 \widehat{\otimes} \cdots \widehat{\otimes} B_N/K_N$$

contains the linear subspace

$$\mathcal{K} = K_1 \otimes B_2 \otimes \cdots \otimes B_N + \cdots + B_1 \otimes \cdots \otimes B_{N-1} \otimes K_N$$

of  $B_1 \otimes \cdots \otimes B_N$ .

**THEOREM 2.2.** (cf. [7; pp. 55-56]) *Using the above notations, we have:*

- (a)  $\mathcal{K} = (B_1 \otimes \cdots \otimes B_N) \cap \text{Ker } Q$ ;
- (b)  $\mathcal{K}$  is dense in  $\text{Ker } Q$ ;
- (c) If we denote by

$$\tilde{Q} : (B_1 \widehat{\otimes} \cdots \widehat{\otimes} B_N) / \text{Ker } Q \longrightarrow B_1/K_1 \widehat{\otimes} \cdots \widehat{\otimes} B_N/K_N$$

the mapping naturally induced by  $Q$ , then  $\tilde{Q}$  is an "onto" isometry.

**PROOF.** Every element  $F \in B_1 \otimes \cdots \otimes B_N$  can be expressed in the form

$$(1) \quad F = \sum_{l_1} \cdots \sum_{l_N} a(l_1, \dots, l_N) f_1^{(l_1)} \otimes \cdots \otimes f_N^{(l_N)} + G,$$

where  $\{f_j^{(l_j)}\}$  are finite elements of  $B_j$  linearly independent mod  $K_j$  ( $j=1, 2, \dots, N$ ) and  $G \in \mathcal{K}$ . If  $F$  is in  $\text{Ker } Q$ , then we have

$$0 = \sum_{l_1} \cdots \sum_{l_N} a(l_1, \dots, l_N) Q_1(f_1^{(l_1)}) \otimes \cdots \otimes Q_N(f_N^{(l_N)}).$$

Since the elements  $Q_1(f_1^{(l_1)}) \otimes \cdots \otimes Q_N(f_N^{(l_N)})$  are linearly independent in  $B_1/K_1 \otimes \cdots \otimes B_N/K_N$ , this implies that

$$a(l_1, \dots, l_N) = 0$$

for all indices  $l_1, \dots, l_N$ . It follows from (1) that  $F = G \in \mathcal{K}$ , which proves (a).

To prove statements (b) and (c), it suffices to verify that for any element  $F \in B_1 \otimes \cdots \otimes B_N$  we have

$$(2) \quad \|Q(F)\|_\pi = \|F + \mathcal{K}\| = \inf \{\|F + G\|_\pi : G \in \mathcal{K}\}.$$

The inequality  $\|Q(F)\|_\pi \leq \|F + \mathcal{K}\|$  is trivial. Take now an arbitrary  $\varepsilon > 0$ .  $Q(F)$  has an expansion of the form

$$Q(F) = \sum_{i=1}^n Q_1(f_1^{(i)}) \otimes \cdots \otimes Q_N(f_N^{(i)}),$$

where  $f_j^{(i)}$  are elements of  $B_j$  ( $j=1, 2, \dots, N$ ;  $i=1, 2, \dots, n$ ) such that

$$\|Q(F)\|_\pi + \varepsilon > \sum_{i=1}^n \|Q_1(f_1^{(i)})\| \cdots \|Q_N(f_N^{(i)})\|.$$

We can find then  $h_j^{(i)} \in f_j^{(i)} + K_j$  such that

$$\|Q(F)\|_\pi + \varepsilon > \sum_{i=1}^n \|h_1^{(i)}\| \cdots \|h_N^{(i)}\|.$$

Putting

$$H = \sum_{i=1}^n h_1^{(i)} \otimes \cdots \otimes h_N^{(i)} \in B_1 \otimes \cdots \otimes B_N,$$

we see that  $Q(F) = Q(H)$  and so  $H \in F + \mathcal{K}$  by (a). It follows that

$$\|Q(F)\|_{\pi+\varepsilon} > \|H\|_{\pi} \geq \|F + \mathcal{K}\|.$$

Since  $\varepsilon > 0$  was arbitrary, this yields the required equality (2).

The proof is complete.

**COROLLARY 2.2.1.** *Let  $X$  be a compact space,  $E$  a closed subset of  $X$ , and  $B$  a Banach space. Denote by*

$$I(E: C(X)) = \{f \in C(X) : f = 0 \text{ on } E\};$$

$$I(E: C(X)) \widehat{\otimes} B = \{F \in C(X) \widehat{\otimes} B : F = 0 \text{ on } E\}.$$

Then the natural imbedding

$$\mathcal{I} : I(E: C(X)) \widehat{\otimes} B \longrightarrow C(X) \widehat{\otimes} B$$

is an isometry and the range of  $\mathcal{I}$  is  $I(E: C(X)) \widehat{\otimes} B$ .

**PROOF.** Since the imbedding  $I(E: C(X)) \subset C(X)$  has an approximating inverse,  $\mathcal{I}$  is an isometry [7; p. 54]. To show that  $I(E: C(X)) \widehat{\otimes} B = I(E: C(X) \widehat{\otimes} B)$  (or more precisely that  $\mathcal{I}[I(E: C(X)) \widehat{\otimes} B] = I(E: C(X) \widehat{\otimes} B)$ ), consider the restriction mapping

$$Q_E(f) = f|_E \in C(E) \quad (f \in C(X)).$$

The mapping  $Q_E$  may be also regarded as the quotient mapping

$$Q_E : C(X) \longrightarrow C(E) = C(X)/I(E: C(X)).$$

Thus the preceding theorem applies, and we see that  $I(E: C(X)) \widehat{\otimes} B$  is dense in the kernel of the mapping

$$Q = Q_E \widehat{\otimes} I_B : C(X) \widehat{\otimes} B \longrightarrow C(E) \widehat{\otimes} B.$$

Since the closure of  $I(E: C(X)) \widehat{\otimes} B$  in  $C(X) \widehat{\otimes} B$  is  $I(E: C(X)) \widehat{\otimes} B$ , it follows that  $\text{Ker } Q = I(E: C(X)) \widehat{\otimes} B$ . On the other hand, it is easy to see that  $Q$  is realized as the restriction mapping

$$Q(F) = F|_E \in C(E) \widehat{\otimes} B \quad (F \in C(X) \widehat{\otimes} B),$$

from which we conclude that

$$I(E: C(X) \widehat{\otimes} B) = \text{Ker } Q = I(E: C(X)) \widehat{\otimes} B.$$

This completes the proof.

DEFINITION. Let  $X$  and  $B$  be as before, and let  $\Phi$  be any bounded linear functional on the Banach space  $C(X) \widehat{\otimes} B$  ( $\Phi \in (C(X) \widehat{\otimes} B)'$ ). The  $X$ -support of  $\Phi$ , denoted by  $X\text{-supp}(\Phi)$ , is the intersection of all closed subsets  $E$  of  $X$  satisfying the following condition:

(#) If  $F \in C(X) \widehat{\otimes} B$ , and if  $F=0$  on some neighborhood of  $E$ , then  $\Phi(F)=0$ .

THEOREM 2.3. Let  $\Phi \in (C(X) \widehat{\otimes} B)'$  and  $S = X\text{-supp}(\Phi)$ . Then we have

(#)'  $F \in I(S; C(X) \widehat{\otimes} B) \Rightarrow \Phi(F) = 0$ .

Therefore the  $X$ -support of  $\Phi$  is the smallest closed subset  $S$  for which (#)' holds.

PROOF. It is easy to see that the family of the closed subsets  $E$  satisfying condition (#) has the finite intersection property, and so  $S$  belongs to that family. In order to establish (#)', consider an arbitrary  $F$  in  $I(S; C(X) \widehat{\otimes} B)$ . Since  $I(S; C(X) \widehat{\otimes} B) = I(S; C(X)) \widehat{\otimes} B$  by Corollary 2.2.1, it follows that for any  $\varepsilon > 0$  we can find  $f_\varepsilon \in C(X)$  such that  $\|F - f_\varepsilon F\|_\pi < \varepsilon$  and  $f_\varepsilon = 0$  on some neighborhood of  $S$ . Since  $S$  satisfies condition (#), we conclude that  $\Phi(f_\varepsilon F) = 0$ , and so

$$|\Phi(F)| = |\Phi(F - f_\varepsilon F)| \leq \|\Phi\| \varepsilon.$$

Since  $\varepsilon$  is arbitrary, this yields  $\Phi(F) = 0$ , and consequently (#)' holds.

This completes the proof.

Let us now take  $\Phi \in (C(X) \widehat{\otimes} B)'$  and  $b \in B$  arbitrarily, and observe that

$$|\Phi(f \otimes b)| \leq \|\Phi\| \cdot \|b\|_B \cdot \|f\|_\infty \quad (f \in C(X)).$$

Thus, by F. Riesz's representation theorem [5], there exists a unique measure  $\mu[\Phi, b] \in M(X)$  such that

$$\Phi(f \otimes b) = \int_X f d\mu[\Phi, b] \quad (f \in C(X)).$$

Let  $\mathcal{B}(X)$  be the space of all bounded Borel measurable functions on  $X$ , and for any  $\varphi \in \mathcal{B}(X)$ , let  $\varphi\Phi$  be the functional in  $(C(X) \widehat{\otimes} B)'$  uniquely defined by the requirement

$$(\varphi\Phi)(f \otimes b) = \int_X \varphi f d\mu[\Phi, b] \quad (f \in C(X), b \in B).$$

It is then trivial that  $\|\varphi\Phi\| \leq \|\varphi\|_\infty \cdot \|\Phi\|$ , and that  $\text{supp}(\mu[\Phi, b]) \subset X\text{-supp}(\Phi)$  for all  $b \in B$ . Note also that if  $\varphi \in C(X)$  we have

$$(\varphi\Phi)(f \otimes b) = \int_X \varphi f d\mu[\Phi, b] = \Phi(\varphi f \otimes b) \quad (f \in C(X), b \in B).$$

THEOREM 2.4. For any  $\Phi \in (C(X) \widehat{\otimes} B)'$  we have:

(a) If  $\varphi \in \mathcal{B}(X)$ , then

$$X\text{-supp}(\varphi\Phi) \subset \text{supp } \varphi \cap X\text{-supp}(\Phi).$$

(b) If  $\varphi \in \mathcal{B}(X)$ ,  $F \in C(X) \widehat{\otimes} B$ , and if  $\varphi F = 0$  on  $X\text{-supp}(\Phi)$ , then  $(\varphi\Phi)(F) = 0$ .

(c) Let  $\{\varphi_n\}_1^\infty$  and  $\{F_n\}_1^\infty$  be any sequences in  $\mathcal{B}(X)$  and  $C(X) \widehat{\otimes} B$ , respectively, such that

$$\sum_{n=1}^{\infty} \|\varphi_n\|_\infty \cdot \|F_n\|_\pi < +\infty$$

and suppose that

$$\sum_{n=1}^{\infty} \varphi_n(x) F_n(x) = 0 \quad (x \in X\text{-supp}(\Phi)).$$

Then

$$\sum_{n=1}^{\infty} (\varphi_n\Phi)(F_n) = 0.$$

PROOF. We easily see that the part (a) is an immediate consequence of the part (b).

In order to prove the part (b), let  $\varphi$  and  $F$  be as in (b), and set  $E = \{x \in X : F(x) = 0\}$ . It then follows from Corollary 2.1.1 that for any  $\varepsilon > 0$ , there exists  $f_\varepsilon \in C(X)$  such that  $f_\varepsilon = 0$  on some open set  $U$  containing  $E$  and such that

$$\|f_\varepsilon\|_\infty \leq 1, \quad \text{and} \quad \|F - f_\varepsilon F\|_\pi < \varepsilon.$$

Let

$$F = \sum_{i=1}^{\infty} f_i \otimes b_i \quad (f_i \in C(X), b_i \in B)$$

be any fixed expansion of  $F$  with

$$\sum_{i=1}^{\infty} \|f_i\|_\infty \cdot \|b_i\|_B < +\infty.$$

Putting  $\mu_i = \mu[\Phi, b_i]$ , we then have

$$(\varphi\Phi)(F) = \sum_{i=1}^{\infty} \int_X \varphi f_i d\mu_i.$$

Take any natural number  $n = n_\varepsilon$  so that

$$\sum_{i=n}^{\infty} \|f_i\|_\infty \cdot \|b_i\|_B < \varepsilon.$$

Using Luzin's theorem, we see that there exists a closed subset  $K_\varepsilon$  of  $X$  such that

$$\varphi|_{K_\varepsilon} \in C(K_\varepsilon), \quad \text{and} \quad \sum_{i=1}^n \|f_i\|_\infty \cdot |\mu_i|(K_\varepsilon^c) < \varepsilon.$$

Since  $\varphi = 0$  on the closed set  $U^c \cap X\text{-supp}(\Phi)$ , we may assume that

$$U^c \cap X\text{-supp}(\Phi) \subset K_\varepsilon.$$

Let  $\tilde{\varphi} \in C(X)$  be any continuous extension of  $\varphi|_{K_\varepsilon}$  such that  $\|\tilde{\varphi}\|_\infty \leq \|\varphi\|_\infty$ . Then  $\tilde{\varphi}f_\varepsilon = 0$  on  $X\text{-supp}(\Phi)$  and so by Theorem 2.2 we have

$$\Phi(\tilde{\varphi}f_\varepsilon F) = 0.$$

It follows that

$$\begin{aligned} |(\varphi\Phi)(f_\varepsilon F)| &= |(\varphi\Phi)(f_\varepsilon F) - \Phi(\tilde{\varphi}f_\varepsilon F)| \\ &\leq \sum_{i=1}^{\infty} \left| \int_X (\varphi - \tilde{\varphi}) f_\varepsilon f_i d\mu_i \right| \\ &\leq 2\|\varphi\|_\infty \left\{ \sum_{i=1}^n \|f_i\|_\infty \cdot |\mu_i|(K_\varepsilon^c) + \sum_{i=n}^{\infty} \|f_i\|_\infty \cdot \|b_i\|_B \cdot \|\Phi\| \right\} \\ &\leq 2\|\varphi\|_\infty (1 + \|\Phi\|) \varepsilon. \end{aligned}$$

Consequently we have

$$\begin{aligned} |(\varphi\Phi)(F)| &\leq |(\varphi\Phi)(f_\varepsilon F)| + \|\varphi\|_\infty \cdot \|\Phi\| \cdot \varepsilon \\ &\leq 3\|\varphi\|_\infty \cdot (1 + \|\Phi\|) \varepsilon. \end{aligned}$$

Letting  $\varepsilon \rightarrow 0$ , we have  $(\varphi\Phi)(F) = 0$  and so the part (b) is proved.

The part (c) is obviously a generalization of the part (b). Let  $\{\varphi_n\}_1^\infty$  and  $\{F_n\}_1^\infty$  be as in (c). In order to show that  $\sum_{n=1}^{\infty} (\varphi_n\Phi)(F_n) = 0$ , we can and will assume without loss of generality that

$$(1) \quad \|\varphi_n\|_\infty = 1, \quad F_n = f_n \otimes b_n, \quad \text{and} \quad \|f_n\|_\infty = 1$$

for some  $f_n \in C(X)$  and  $b_n \in B$  ( $n = 1, 2, \dots$ ). Then

$$\sum_{n=1}^{\infty} \|b_n\|_B < +\infty,$$

and hence

$$(2) \quad \mu = \sum_{n=1}^{\infty} |\mu_n| \in M(X)$$

converges in the norm of  $M(X)$ , where  $\mu_n = \mu[\Phi, b_n]$ . By the Radon-Nikodým theorem, we can find  $w_n \in \mathcal{B}(X)$  so that

$$(3) \quad d\mu_n = w_n d\mu \quad (n = 1, 2, \dots).$$

Then the series

$$(4) \quad g = \sum_{n=1}^{\infty} \varphi_n f_n w_n \in L^1(d\mu)$$

absolutely converges in the norm of  $L^1(d\mu)$  by (1), (2), and (3). Let  $\varepsilon > 0$  be arbitrary. From the absolute continuity of indefinite integrals, there is a  $\delta = \delta(\varepsilon) > 0$  such that for every Borel subset  $E$  of  $X$  we have

$$(5) \quad \mu(E) < \delta \Rightarrow \int_E |g| d\mu < \varepsilon.$$

Using Luzin's theorem as before, we see that there is a closed subset  $K=K_\delta$  of  $X$  such that

$$(6) \quad \mu(K^c) < \delta, \quad \text{and} \quad \varphi_n|_K \in C(K) \quad (n = 1, 2, \dots).$$

We then have by (1), (3), (4), (5), and (6)

$$(7) \quad \left| \sum_{n=1}^{\infty} (\varphi_n \Phi)(F_n) - \sum_{n=1}^{\infty} \int_K \varphi_n f_n d\mu_n \right| = \left| \int_{K^c} g d\mu \right| < \varepsilon.$$

Let now  $\tilde{\varphi}_n$  be any continuous extension of  $\varphi_n|_K$  such that  $\|\tilde{\varphi}_n\|_\infty \leq 1$  for all  $n = 1, 2, \dots$ , let

$$(8) \quad G = \sum_{n=1}^{\infty} \tilde{\varphi}_n f_n \otimes b_n \in C(X) \widehat{\otimes} B,$$

and observe that

$$\chi_K(x)G(x) = \chi_K(x) \sum_{n=1}^{\infty} \varphi_n(x)F_n(x) = 0 \quad (x \in X\text{-supp } \Phi)$$

by assumption. It follows from the part (b) and (8) that

$$\begin{aligned} 0 &= (\chi_K \Phi)(G) \\ &= \sum_{n=1}^{\infty} \int_K \tilde{\varphi}_n f_n d\mu_n = \sum_{n=1}^{\infty} \int_K \varphi_n f_n d\mu_n. \end{aligned}$$

This combined with (7) shows that

$$\left| \sum_{n=1}^{\infty} \varphi_n \Phi(F_n) \right| < \varepsilon.$$

Since  $\varepsilon$  can be taken as small as one pleases, we have the desired conclusion. The proof is now complete.

### § 3. The proof of Theorem 1.1.

Let  $X, Y$ , etc., be as in §1. We first prove the following.

(I). If both  $X$  and  $Y$  are metrizable, then Theorem 1.1 holds.

To show this, take any  $F \in [C(X) \widehat{\otimes} B; p]$  and  $\Phi \in (C(X) \widehat{\otimes} B)'$  such that

$$(1) \quad \Phi(P_B(H)) = 0 \quad (H \in C(Y) \widehat{\otimes} B).$$

Since the range of  $P_B$  is a closed linear subspace of  $C(X) \widehat{\otimes} B$  by Theorem 2.1, the proof will be complete by the Hahn-Banach theorem as soon as we have shown that  $\Phi(F) = 0$ .

First note that (1) implies



$$(2) \quad (\chi_{p^{-1}(E)}\Phi)(P_B(H)) = 0 \quad (H \in C(Y) \widehat{\otimes} B)$$

for all Borel subsets  $E$  of  $Y$ . This may be proved as Theorem 2.4 was proved. Let

$$(3) \quad F = \sum_{n=1}^{\infty} f_n \otimes b_n \quad (f_n \in C(X), b_n \in B)$$

be any fixed expansion of  $F$  such that

$$\|f_n\|_{\infty} \leq 1 \quad (n = 1, 2, \dots), \quad \text{and} \quad \sum_{n=1}^{\infty} \|b_n\|_B < +\infty.$$

Let  $\mu_n = \mu[\Phi, b_n]$ ,

$$\mu_X = \sum_{n=1}^{\infty} |\mu_n| \in M(X)$$

and choose  $w_n \in \mathcal{B}(X)$  so that

$$d\mu_n = w_n d\mu_X \quad (n = 1, 2, \dots).$$

Then we have for any Borel subset  $D$  of  $X$

$$(4) \quad (\chi_D\Phi)(F) = \int_D g d\mu_X$$

where

$$g = \sum_{n=1}^{\infty} f_n w_n \in L^1(d\mu).$$

We now define  $\mu_Y \in M(Y)$  to be the measure obtained by setting

$$(5) \quad \mu_Y(E) = \mu_X(p^{-1}(E))$$

for all Borel subsets  $E$  of  $Y$ .

Since both  $X$  and  $Y$  are compact and metrizable, and since  $p$  is a continuous mapping from  $X$  onto  $Y$ , there exists a Borel measurable mapping  $q: Y \rightarrow X$  such that

$$(6) \quad p(q(y)) = y \quad (y \in Y).$$

(See [1]). Let  $\varepsilon > 0$  be given. Since  $f_n \circ q \in \mathcal{B}(Y)$ , we can find a closed subset  $K$  of  $Y$  such that

$$(7) \quad \mu_Y(K^c) < \varepsilon \quad \text{and} \quad f_n \circ q|_K \in C(K) \quad (n = 1, 2, \dots).$$

Let  $\tilde{g}_n \in C(Y)$  be any continuous extension of  $f_n \circ q|_K$  such that  $\|\tilde{g}_n\| \leq 1$  and let

$$(8) \quad G = \sum_{n=1}^{\infty} \tilde{g}_n \otimes b_n \in C(Y) \widehat{\otimes} B.$$

Setting  $K_X = p^{-1}(K)$ , we see from (2) that

$$(9) \quad (\chi_{K_X}\Phi)(P_B(G)) = 0.$$

On the other hand, since  $F \in [C(X) \widehat{\otimes} B; p]$ , we have by (3), (6), and (8), for all  $x \in K_X$

$$\begin{aligned} F(x) &= F(q(p(x))) = \sum_{n=1}^{\infty} f_n(q(p(x)))b_n \\ &= \sum_{n=1}^{\infty} \tilde{g}_n(p(x))b_n = (P_B(G))(x). \end{aligned}$$

Therefore Theorem 2.4 combined with (9) shows

$$(\chi_{K_X} \Phi)(F) = 0.$$

It follows from (4) that

$$(10) \quad \Phi(F) = (\chi_{(K_X)^c} \Phi)(F) = \int_{(K_X)^c} g d\mu.$$

We have also by (5) and (7)  $\mu_X(K_X^c) < \varepsilon$ . Since  $\varepsilon$  was arbitrary, (10) implies  $\Phi(F) = 0$ , which proves (I).

We now prove Theorem 1.1 for general  $X$  and  $Y$ . Let  $F \in [C(X) \widehat{\otimes} B; p]$  be as in (3). There exists  $G \in C(Y; B)$  such that

$$(1') \quad F(x) = G(p(x)) \quad (x \in X).$$

Regarding as a single point each closed subset of  $X$  on which every  $f_n$  is constant, we obtain a compact metrizable space  $\tilde{X}$  and a continuous mapping  $q_X$  from  $X$  onto  $\tilde{X}$  such that

$$(2') \quad F(x) = \tilde{F}(q_X(x)) \quad (x \in X)$$

for some  $\tilde{F} \in C(\tilde{X}) \widehat{\otimes} B$ . Similarly regarding as a single point each closed subset of  $Y$  on which  $G$  is constant, we obtain a compact metrizable space  $\tilde{Y}$ , a continuous mapping  $q_Y$  from  $Y$  onto  $\tilde{Y}$ , and a continuous mapping  $\tilde{p}$  from  $\tilde{X}$  onto  $\tilde{Y}$  such that: In the diagram

$$\begin{array}{ccc} X & \xrightarrow{p} & Y \\ q_X \downarrow & \tilde{p} & \downarrow q_Y \\ \tilde{X} & \longrightarrow & \tilde{Y} \end{array}$$

we have

$$(3') \quad q_Y \circ p = \tilde{p} \circ q_X$$

and

$$(4') \quad G(y) = \tilde{G}(q_Y(y)) \quad (y \in Y)$$

for some  $\tilde{G} \in C(\tilde{Y}; B)$ . It follows from (1'), (2'), (3'), and (4') that we have for all  $x \in X$

$$\tilde{F}(q_X(x)) = F(x) = G(p(x)) = (\tilde{G} \circ \tilde{p})(q_X(x)),$$

that is, that  $\tilde{F} = \tilde{G} \circ \tilde{p}$ . Therefore (I) assures that  $\tilde{G} \in C(\tilde{Y}) \widehat{\otimes} B$ , which combined with (4') yields  $G \in C(Y) \widehat{\otimes} B$ . We have thus  $F = G \circ p = P_B(G)$ , being the required conclusion.

The proof of Theorem 1.1 is now established.

#### § 4. Generalizations of Theorem 1.1.

In this section we shall generalize Theorem 1.1. We begin as follows.

**THEOREM 4.1.** *Let  $\{X_j\}_1^N$  and  $\{Y_j\}_1^N$  be two families of  $N$  compact spaces,  $\{p_j: X_j \rightarrow Y_j\}_1^N$   $N$  continuous "onto" mappings, and  $B$  a Banach space. Denote by*

$$p = p_1 \times \cdots \times p_N: X = X_1 \times \cdots \times X_N \longrightarrow Y = Y_1 \times \cdots \times Y_N$$

the product mapping of  $p_1, \dots, p_N$ , and define the operator

$$P: C(Y_1) \widehat{\otimes} \cdots \widehat{\otimes} C(Y_N) \widehat{\otimes} B \longrightarrow C(X_1) \widehat{\otimes} \cdots \widehat{\otimes} C(X_N) \widehat{\otimes} B$$

by

$$P(G) = G \circ p \quad (G \in C(Y_1) \widehat{\otimes} \cdots \widehat{\otimes} C(Y_N) \widehat{\otimes} B).$$

Then  $P$  is a linear isometry, and its range is

$$\begin{aligned} & [C(X_1) \widehat{\otimes} \cdots \widehat{\otimes} C(X_N) \widehat{\otimes} B; p] \\ & = \{F \in C(X_1) \widehat{\otimes} \cdots \widehat{\otimes} C(X_N) \widehat{\otimes} B: F = G \circ p \text{ for some } G \in C(Y: B)\}. \end{aligned}$$

**PROOF.** We prove this only for the case  $N=2$ . The general proof proceeds similarly. The first statement that  $P$  is a linear isometry is easily seen (cf. Theorem 2.1). Suppose now that  $F \in C(X_1) \widehat{\otimes} C(X_2) \widehat{\otimes} B$  is such that  $F = H \circ p$  for some  $H \in C(Y_1 \times Y_2: B)$ . Regarding  $F$  as a  $C(X_1) \widehat{\otimes} B$ -valued function on  $X_2$ , we then see that  $F \in C(X_2) \widehat{\otimes} (C(X_1) \widehat{\otimes} B)$  and that  $F = G \circ p_2$  for some  $G \in C(Y_2: C(X_1) \widehat{\otimes} B)$ . It follows from Theorem 1.1 that  $G \in C(Y_2) \widehat{\otimes} (C(X_1) \widehat{\otimes} B)$ . Regarding  $G$  as a  $C(Y_2) \widehat{\otimes} B$ -valued functions on  $X_1$ , we also see that  $G \in C(X_1) \widehat{\otimes} (C(Y_2) \widehat{\otimes} B)$  and that  $G = H \circ p_1$  for some  $H \in C(Y_1: C(Y_2) \widehat{\otimes} B)$ . (Note that in general we have linearly and isometrically

$$A \widehat{\otimes} B \widehat{\otimes} C = A \widehat{\otimes} (B \widehat{\otimes} C) = (A \widehat{\otimes} B) \widehat{\otimes} C$$

for any Banach spaces  $A, B$ , and  $C$ .) Therefore we have  $H \in C(Y_1) \widehat{\otimes} C(Y_2) \widehat{\otimes} B$  by Theorem 1.1, and  $F = H \circ p$  as  $B$ -valued functions on  $X_1 \times X_2$ .

This clearly establishes our theorem.

Let now  $A$  be any semi-simple commutative Banach algebra with a unit 1. We denote by  $M_A$  the maximal ideal space of  $A$ , and regard  $A$  as a sub-

algebra of  $C(M_A)$ .

Let  $X$  be any non-empty set, and let  $\mathcal{F}(X)$  be the Banach algebra consisting of the bounded complex-valued functions on  $X$ , the norm of  $f \in \mathcal{F}(X)$  being

$$\|f\|_\infty = \sup \{|f(x)| : x \in X\}.$$

DEFINITION. A  $\mathcal{B}$ -algebra on a non-empty set  $X$  is any closed subalgebra  $A$  of  $\mathcal{F}(X)$  satisfying the following three conditions:

- (a)  $A$  contains the constants.
- (b) The functions of  $A$  separate points of  $X$ .
- (c)  $f \in A$  implies  $\bar{f} \in A$ .

If  $X$  is a compact topological space, then both of the algebras  $C(X)$  and  $\mathcal{B}(X)$  (for the definition, see § 2) are  $\mathcal{B}$ -algebras on  $X$ . For any  $\mathcal{B}$ -algebra  $A$  on a set  $X$ , we have  $A = C(M_A)$  algebraically and isometrically [2], and it is easy to see that  $X$  may be regarded as a dense subset of  $M_A$ .

Let now  $\{X_j\}_1^N$  be a family of  $N$  non-empty sets, and  $A_j$  a  $\mathcal{B}$ -algebra on  $X_j$  for  $j=1, 2, \dots, N$ . We then have  $A_1 \widehat{\otimes} \dots \widehat{\otimes} A_N = C(M_{A_1}) \widehat{\otimes} \dots \widehat{\otimes} C(M_{A_N})$  as Banach algebras and the natural dense imbedding:  $X = X_1 \times \dots \times X_N \subset M_{A_1} \times \dots \times M_{A_N}$ . Therefore for any Banach space  $B$ , we can identify the Banach space  $A_1 \widehat{\otimes} \dots \widehat{\otimes} A_N \widehat{\otimes} B$  with a linear subspace of  $\mathcal{F}(X; B)$ , the space of all bounded  $B$ -valued functions on  $X$ .

THEOREM 4.2. Let  $\{X_j\}_1^N$  and  $\{Y_j\}_1^N$  be two families of  $N$  non-empty sets,  $\{p_j: X_j \rightarrow Y_j\}_1^N$   $N$  "onto" mappings, and  $B$  a Banach space. Let  $\{A_j \subset \mathcal{F}(Y_j)\}_1^N$  be  $\mathcal{B}$ -algebras, and define the operator

$$P: A_1 \widehat{\otimes} \dots \widehat{\otimes} A_N \widehat{\otimes} B \longrightarrow \mathcal{F}(X_1) \widehat{\otimes} \dots \widehat{\otimes} \mathcal{F}(X_N) \widehat{\otimes} B$$

by

$$P(G) = G \circ p \quad (G \in A_1 \widehat{\otimes} \dots \widehat{\otimes} A_N \widehat{\otimes} B),$$

where

$$p = p_1 \times \dots \times p_N: X = X_1 \times \dots \times X_N \longrightarrow Y = Y_1 \times \dots \times Y_N.$$

Then  $P$  is a linear isometry, and any  $F \in \mathcal{F}(X_1) \widehat{\otimes} \dots \widehat{\otimes} \mathcal{F}(X_N) \widehat{\otimes} B$  is in the range of  $P$  if and only if  $F = G \circ p$  for some  $G \in C(M_{A_1} \times \dots \times M_{A_N}; B) \subset \mathcal{F}(Y; B)$ .

PROOF. The proof follows from the above observations and Theorem 4.1. We omit the details.

THEOREM 4.3. Let  $\{X_j\}_1^N$  be  $N$  compact spaces, and  $B$  a Banach space. Then we have

$$C(X_1) \widehat{\otimes} \dots \widehat{\otimes} C(X_N) \widehat{\otimes} B = C(X; B) \cap (\mathcal{F}(X_1) \widehat{\otimes} \dots \widehat{\otimes} \mathcal{F}(X_N) \widehat{\otimes} B),$$

where  $X = X_1 \times \dots \times X_N$ .

PROOF. Setting  $Y_j = X_j$  and  $A_j = C(X_j)$  for  $j = 1, 2, \dots, N$ , we see that this statement is a special case of Theorem 4.2.

### § 5. Some remarks on the problem of spectral synthesis for tensor algebras and group algebras.

This section is independent of the preceding four sections, and contains some rather trivial remarks on the problem of spectral synthesis for tensor algebras and group algebras.

THEOREM 5.1. *Let  $\{X_j\}_1^N$  be  $N$  compact spaces, and let  $V = C(X_1) \widehat{\otimes} \dots \widehat{\otimes} C(X_N)$ . Then spectral synthesis fails in the algebra  $V$  if and only if  $X_j$  contains a perfect subset for at least two values of the  $j$ 's.*

This follows from [6; p. 562] and [7; p. 102].

THEOREM 5.2. *Suppose that  $G$  is a locally compact abelian group, and that  $p$  is a natural number  $\geq 2$ . Let  $\{K_j\}_1^N$  be  $N$   $K_p$ -sets of  $G$ , and  $\{x_j\}_1^N \subset G$ . Then the union  $\bigcup_1^N (x_j + K_j)$  is an SH-set (for the definition, see [6; p. 551]).*

This follows from [6; p. 552] and [3].

THEOREM 5.3. *Suppose that  $G$  is a locally compact abelian group, that  $\{K_j\}_1^N$  are  $N$  quasi-Kronecker sets of  $G$  (for the definition, see [6; p. 549]), and that  $\{x_j\}_1^N \subset G$ . If  $K_j$  is totally disconnected for  $N-1$  values of the  $j$ 's, then the union  $\bigcup_1^N (x_j + K_j)$  is an SH-set.*

This follows from [6; p. 552] and [8; p. 957].

Tokyo Metropolitan University

### References

- [1] N. Bourbaki, *Topologie générale*, Hermann, Paris; A. S. I. 1045, Nouvelle édition.
- [2] N. Dunford and J. Schwartz, *Linear operators*, Vol. 2, Interscience Publishers, New York, 1963.
- [3] S. W. Drury, *Sur les ensembles de type  $\mathcal{K}$  dans les groupes compacts abéliens metrisables d'ordre borné*, C. R. Acad. Sci. Paris, **268** (1969), Sér. A, 775-777.
- [4] A. Grothendieck, *Produits tensoriels topologiques et espaces nucléaires*, Mem. Amer. Math. Soc., **16** (1955).
- [5] E. Hewitt and K. A. Ross, *Abstract harmonic analysis*, Vol. I, Springer-Verlag, Heidelberg, 1963.
- [6] S. Saeki, *Spectral synthesis for the Kronecker sets*, J. Math. Soc. Japan, **21** (1969), 549-563.
- [7] N. Th. Varopoulos, *Tensor algebras and harmonic analysis*, Acta Math., **119** (1967), 51-112.
- [8] N. Th. Varopoulos, *Sur les ensembles de Kronecker*, C. R. Acad. Sci. Paris, **268** (1969), Sér. A, 954-957.