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# On the generation of semigroups of nonlinear contractions

By Shinnosuke OHARU

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## Introduction

Let $X$ be a real or complex Banach space and $S$ be a subset of $X$. Let $\{T(t); t \geqq 0\}$ be a one-parameter family of (possibly nonlinear) contractions from $S$ into itself satisfying the following conditions:

(i) $T(0)=I$ (the identity mapping), $T(t)T(s)=T(t+s)$ on $S$ for $t, s \geqq 0$;

(ii) for each $x \in S$, $T(t)x$ is strongly continuous in $t \geqq 0$. Then the family $\{T(t)\}$ is called a *semigroup* (*of contractions*) *on* $S$. And we define the *infinitesimal generator* $A_0$ of a semigroup $\{T(t)\}$ by $A_0 x = \lim_{h \to +0} h^{-1}\{T(h)x-x\}$ and the *weak infinitesimal generator* $A'$ by $A'x = w\text{-}\lim_{h \to +0} h^{-1}\{T(h)x-x\}$, if the right sides exist, the notation "lim" (or "$w$-lim") means the strong limit (or the weak limit) in $X$.

The purpose of the present paper is to construct the semigroup of contractions determined by a (nonlinear) operator given in a Banach space. Our results consist of sufficient conditions for a (multi-valued) operator in $X$ or a pseudo-resolvent of contractions in $X$ to determine a semigroup of contractions. Also, we are concerned with the generation of semigroups of differentiable operators.

We find other interesting results on the generation of semigroups of contractions in [2]-[5], [8]-[14], in which (multi-valued) maximal dissipative, $m$-accretive or $m$-dissipative operators are treated as the infinitesimal generators. In this paper we extend these generation theorems to the case of a (multi-valued) dissipative operator $A$ such that the range $R(I-\lambda A)$ of $I-\lambda A$ contains $D(A)$ for every $\lambda > 0$. Recently, Brezis and Pazy [1] considered similar problems in Hilbert spaces. A result related to their generation theorem will be given in § 6.

Section 0 gives the notion of a dissipative operator and some of its basic properties.

Section 1 contains the statements of main results and some remarks.

Section 2 concerns the abstract Cauchy problem.

Section 3 deals with the pseudo-resolvent.

Section 4 deals with the approximation of operators.

Section 5 contains the construction of the semigroup determined by the dissipative operator.

In Section 6, the differentiability of the constructed semigroup is discussed.

Finally, Section 7 deals with the construction of semigroups of differentiable operators.

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## § 0. Preliminaries

In this section we introduce some notions and notations which will be used in this paper.

An *operator* means a single-valued operator or a multi-valued operator when we do not specify it. For the notion of multi-valued operator, we refer to Kato [**11**; § 2]: let $A$ be an operator in $X$; then the *domain* $D(A)$ of $A$ is the set of all $x\in X$ such that $Ax\neq\emptyset$; the *range* $R(A)$ of $A$ is given by $\bigcup_{x\in X} Ax$; here, $Ax=\emptyset$ if $x\notin D(A)$; and we write $AS$ (or $A(S)$) for $\bigcup_{x\in S} Ax$. For $S_1$, $S_2\subset X$, $S_1+S_2$ denotes the set $\{x+y;\ x\in S_1,\ y\in S_2\}$, where $S_1+S_2=\emptyset$ if $S_1=\emptyset$ or $S_2=\emptyset$. For a scalar $\lambda$ and $S\subset X$, $\lambda S$ denotes $\{\lambda x;\ x\in S\}$. And we write $y+S$ for $\{y\}+S$. Accordingly, we define the sum $A+B$ of two operators $A$ and $B$ on $D(A)\cap D(B)$ by $(A+B)x=Ax+Bx$, the scalar multiplication $\lambda A$ on $D(A)$ by $(\lambda A)x=\lambda Ax$, and the product $AB$ of two operators $A$ and $B$ by $ABx=(AB)x=A(Bx)$. We write $\gamma+\lambda A$ for the operator $\gamma I+\lambda A$. And we denote by $A^{-1}$ the inverse operator of an operator $A$. Let $G(A)$ denote the graph of an operator $A$. Then $G(A^{-1})=\{(y, x)\,;\,(x, y)\in G(A)\}$. A single-valued operator $A$ with domain and range in $X$ is regarded as a special case of a multi-valued operator in $X$. Let $A$ be a single-valued operator such that $R(A)\subset D(A)$. Then for any positive integer $k$, we may define the iterations $A^k$ on $D(A)$ by $A^k x=A(A^{k-1}x)$. Here, we define $A^0=I$.

Let $A$, $\tilde{A}$ be two operators in $X$. Then we say that $\tilde{A}$ is an *extension* of $A$, and $A$ is a *restriction* of $\tilde{A}$ (denoted $\tilde{A}\supset A$, $A\subset\tilde{A}$), if $Ax\subset\tilde{A}x$ for $x\in X$. Thus $D(A)\subset D(\tilde{A})$. Let $A$ be an operator in $X$ and $S\subset X$. Then by a *restriction of* $A$ *to* $S$, denoted $A|_S$, we mean an operator such that $D(A|_S)=D(A)\cap S$ and $A|_S x=Ax$ if $x\in S$.

Let $S\subset X$. Then we denote the closure of $S$ in $X$ by $\bar{S}$. Let $A$ be an operator in $X$. Then $B$ is called the *closure* of $A$ if $G(B)=\bar{G}(A)$ in $X\times X$; and we write $B=\bar{A}$.

Let $X^*$ be the dual space of $X$. Then we denote by $\langle x, f\rangle$ the pairing between $x \in X$ and $f \in X^*$, and the *duality mapping* $F$ *of* $X$ is the (multi-valued) mapping from $X$ into $X^*$ defined by

$$F(x)=\{f \in X^* ;\ \mathrm{re}\langle x, f\rangle = \|x\|^2 = \|f\|^2\}\ .$$

DEFINITION 1. A *dissipative* operator in $X$ is an operator $A$ such that for every $x, y \in D(A)$ and $x' \in Ax$, $y' \in Ay$, there exists an $f \in F(x-y)$ such that

$$\mathrm{re}\langle x'-y', f\rangle \leqq 0\ .$$

Let $S \subset X$. If an operator $A$ is dissipative and if any dissipative extension of $A$ coincides on $S$ with $A$, then it is said to be *maximal dissipative on* $S$. And $A$ is said to be *m-dissipative*, if it is dissipative and $R(I-\lambda_0 A)=X$ for some $\lambda_0>0$.

An $m$-dissipative operator $A$ is maximal dissipative on $D(A)$. It is known (see [**11**; Lemma 3.5]) that if $X^*$ is strictly convex and if $A$ is maximal dissipative on $S$, then $Ax$ is convex and closed for $x \in D(A)\cap S$. And it is proved in [**8**] or [**15**] that for an $m$-dissipative operator $A$, $R(I-\lambda A)=X$ for $\lambda>0$.

DEFINITION 2. Let $S \subset X$ and $A$ be a dissipative operator. We denote by $\mathcal{E}[A;S]$ the set of all dissipative operator $B$ such that $D(B)\subset S$ and $Bx \supset Ax$ for $x \in S$. $B \in \mathcal{E}[A;S]$ is called a *maximal dissipative extension of* $A$ *in* $S$, if it is maximal dissipative on $S$.

DEFINITION 3. An operator $A$ is said to be *demi-closed*, if the following condition is satisfied: if $x_n \in D(A)$, $x_n \to x \in X$ strongly and if there are $y_n \in Ax_n$ such that $y_n \to y \in X$ weakly, then $x \in D(A)$ and $y \in Ax$. And $A$ is said to be *closed*, if the graph $G(A)$ is closed in $X \times X$.

A demi-closed operator is closed. Closed linear operators, $m$-dissipative operators in a Banach space with the uniformly convex dual, and the operators treated in [**16**] are all demi-closed. In general, it is proved (see [**11**; Lemma 3.7]) that if $X^*$ is uniformly convex and if $A$ is maximal dissipative on $\overline{D(A)}$, then $A$ is demi-closed. This notion plays a central role in the argument of differentiability of semigroups.

DEFINITION 4. Let $A$ be an operator in $X$. Then we define a (multi-valued) operator $A^0$ by

$$A^0 x=\{y \in Ax ;\ \|y\| = \inf[\|u\| ;\ u \in Ax]\}\ .$$

We call this operator the *canonical restriction of* $A$.

Let $X^*$ be uniformly convex and $A$ be maximal dissipative on $S$. Then $A^0$ is defined on $D(A)\cap S$ and $A^0 x$ is convex and closed for each $x \in D(A)\cap S$. Assume that both $X$ and $X^*$ are uniformly convex and that $Ax$ is convex and closed for $x \in D(A)$. Then the infimum of $\{\|y\| ; y \in Ax\}$ is always attained by a unique element. Thus $A^0$ is single-valued and $D(A^0)=D(A)$. For details on the canonical restriction we refer to Kato [**11**; § 3].

Finally, in order to get shorter statements, we introduce the following notations.

(1) For any non-empty set $S\subset X$, we write $|||S|||$ for the infimum of $\{\|x\|\,;\,x\in S\}$. Thus for any operator $A$ in $X$, $|||Ax|||$ is defined for all $x\in D(A)$.

(2) Let $G$ be a single-valued operator in $X$ and $B\subset D(G)$. Then we mean by $\|G\|_{\mathrm{Lip}(B)}$ the smallest Lipschitz constant for $G$ on $B$. And we denote the family of all contractions on a fixed set $S\subset X$ by $\mathrm{Cont}(S)$.

(3) We write $J_\lambda$ for the resolvent $(I-\lambda A)^{-1}$ if it is well defined. Also, we write $R_\lambda$ for the range $R(I-\lambda A)=\{x-\lambda y\,;\,y\in Ax,\,x\in D(A)\}$ of $I-\lambda A$.

(4) Let $K\subset X$. Then $co\,K$ denotes the convex hull of $K$ and $\overline{co}\,K$ for the convex closure of $K$. Let $A$ be a multi-valued operator in $X$. Then we write $A_c$ for the operator which is defined on $D(A)$ by $A_cx=\overline{co}\,(Ax)$. For instance, if $A$ is maximal dissipative on $D(A)$, then $A=A_c$.

## § 1. Main Results

In this section we state our main results and make some remarks. The detailed statements and their proofs will be given later (see §§ 5, 6).

Throughout this paper we make the basic assumption that *Banach space $X$ has the uniformly convex dual.* This assumption implies each of the following ([**10**; Lemma 1.2]):

(a) the duality mapping $F$ of $X$ is single-valued and uniformly continuous on every bounded set of $X$;

(b) $X$ is reflexive.

The following result is well known ([**10**; Lemma 1.1]).

PROPOSITION 1.1. *An operator $A$ is dissipative if and only if $J_\lambda=(I-\lambda A)^{-1}$ is defined as a single-valued operator on $R_\lambda=R(I-\lambda A)$ and $J_\lambda\in\mathrm{Cont}(R_\lambda)$, for $\lambda>0$.*

In view of this we consider the dissipative operator $A$ such that

$$(R)\qquad R(I-\lambda A)\supset D(A)\qquad \text{for every } \lambda>0\,.$$

REMARK 1.1. (a) If $A$ is a closed dissipative operator satisfying $(R)$, then we have that

$$(R_{cl})\qquad R(I-\lambda A)\supset \overline{D(A)}\qquad \text{for every } \lambda>0\,.$$

(b) Brezis and Pazy [**1**] treat a closed dissipative operator $A$ in a Hilbert space satisfying

$$(R_{co})\qquad R(I-\lambda A)\supset \overline{co}\,D(A)\qquad \text{for every } \lambda>0\,.$$

EXAMPLE. Let $X=L^2(a, b)$. Let $A$ be an operator with domain and range in $X$, defined as follows. We denote by $\mathcal{L}^2(a, b)$ the class of all functions $u(s)$ on $[a, b]$ such that $u(s)$ is measurable and square summable over $[a, b]$. Let $D(A)$ be the class of those $x\in X$ for which a representative function $x(s)\in\mathcal{L}^2(a, b)$ can be found such that $x(s)$ is monotone non-decreasing on $[a, b]$, $|x(s)-x(s')|\leqq|s-s'|$, and $x(a)=0$. This means that

$$x(s)=\int_a^s x'(\sigma)d\sigma,\quad x'(s)\in\mathcal{L}^2(a, b),\quad \text{and}\quad 0\leqq x'(s)\leqq 1\quad \text{a. e.}$$

We then define $Ax$ as an element of $X$ for which $-x(s)x'(s)$ is a representative function. We shall demonstrate that $A$ is a demi-closed, dissipative operator satisfying $(R)$. Let $x, y\in D(A)$. Then $-2\langle Ax-Ay, x-y\rangle$ can be written in the following forms:

$$\int_a^b(x(s)+y(s))'(x(s)-y(s))^2ds+\int_a^b(x(s)^2-y(s)^2)(x(s)-y(s))'ds\,,$$

$$-\int_a^b(x(s)-y(s))(x(s)-y(s))'ds+\int_a^b[(x(s)^2-y(s)^2)(x(s)-y(s))]'ds\,.$$

Therefore, we have

$$\langle Ax-Ay, x-y\rangle=-2^{-1}\int_a^b(x'(s)+y'(s))(x(s)-y(s))^2ds$$
$$-2^{-1}(x(s)-y(s))^2(x(s)+y(s))\,|_a^b\leqq 0\,,$$

which means that $A$ is dissipative. Next, we show that $A$ is demi-closed. Assume that $x_n\in D(A)$, $x, y\in X$, $x_n\to x$ strongly and $Ax_n\to y$ weakly. We can write $2^{-1}x_n(s)^2=\int_a^s x_n(\sigma)x_n'(\sigma)d\sigma$. But $\int_a^s x_n(\sigma)x_n'(\sigma)d\sigma\to-\int_a^s y(\sigma)d\sigma$ for each $s\in[a, b]$; the convergence on $[a, b]$ is uniform with respect to $s$. Thus we see that $x_n(s)^2$ converges uniformly to the limit $z(s)$, where $2^{-1}z(s)=-\int_a^s y(\sigma)d\sigma$. On the other hand, a subsequence $\{n'\}$ can be found such that $x_{n'}(s)\to x(s)$ a. e. Also, by Arzera-Ascoli's theorem there is a subsequence $\{n''\}$ of $\{n'\}$ such that $x_{n''}(s)\to u(s)$ uniformly with respect to $s$. Clearly $u(s)$ is monotone non-decreasing, $|u(s)-u(s')|\leqq|s-s'|$, and $u(a)=0$. Since $u(s)=x(s)$ a. e., $u(s)$ is a representative function of $x$, and so $x\in D(A)$. Since $x_{n''}(s)^2\to u(s)^2$, $u(s)^2=z(s)$ a. e. But $u(s)^2$ and $z(s)$ are continuous, so $u(s)^2\equiv z(s)$. Consequently, $2^{-1}u(s)^2=-\int_a^s y(\sigma)d\sigma$ and thus $-u(s)u'(s)=y(s)$ a. e. This means that $Ax=y$. It is clear that $D(A)$ is convex. And $D(A)$ is closed, because, as mentioned above, $x_n\in D(A)$, and $x_n\to x\in X$ strongly imply that $x\in D(A)$. Finally, we show that $R_\lambda\supset D(A)$ for every $\lambda>0$. For this purpose, we quote Dorroh [**5**; Example 4.10]: Let $\lambda>0$ and $v(s)$ be monotone nondecreasing on

$[a, b]$, $|v(s)-v(s')| \leqq |s-s'|$, and $v(a)=0$. Then by the same argument as in [**5**; Example 4.10] there is a unique solution $x(s)$ of the differential equation $x(s)+x(s)x'(s)=v(s)$ such that $x(s)$ is monotone nondecreasing on $[a, b]$, $0 \leqq x'(s) \leqq 1$, and $x(a)=0$. This means that $x \in D(A)$ and $(I-\lambda A)x=v$. It then follows that $R_\lambda \supset D(A)$.

In the sequel we shall study that a dissipative operator satisfying $(R)$ determines a semigroup of contractions. Our main results are now stated as follows.

Let $A$ be a dissipative operator satisfying $(R)$. Then we have the following assertions.

I. There is a semigroup $\{T(t)\}$ of contractions on $\overline{D(A)}$ such that $T(t)x = \lim_{\lambda \to +0} (I-\lambda A)^{-[t/\lambda]}x$ for $t \geqq 0$ and $x \in D(A)$. If furthermore, $A$ satisfies $(R_{cl})$, then for any $\tilde{A} \in \mathcal{E}[A; \overline{D(A)}]$, $T(t)x = \lim_{\lambda \to +0} (I-\lambda \tilde{A})^{-[t/\lambda]}x$ holds for $t \geqq 0$ and $x \in \overline{D(A)}$.

II. Let $\tilde{A} \in \mathcal{E}[A; \overline{D(A)}]$ be maximal dissipative on $\overline{D(A)}$. Then there is a uniquely determined semigroup $\{T(t)\}$ of contractions on $D(\tilde{A})$ such that for each $x \in D(\tilde{A})$, $(d/dt)T(t)x \in \tilde{A}^0 T(t)x$ for almost all $t \geqq 0$. If furthermore, $X$ is uniformly convex, then $A^0$ is the infinitesimal generator.

III. Let $A$ be single-valued and $\tilde{A} \in \mathcal{E}[A; \overline{D(A)}]$. If $\tilde{A}$ is also single-valued and demi-closed, then it is the weak infinitesimal generator of a uniquely determined semigroup $\{T(t)\}$ of contractions on $D(\tilde{A})$.

IV. Let $X$ be uniformly convex. Let $\tilde{A} \in \mathcal{E}[A; \overline{D(A)}]$. If $\tilde{A}$ is closed, then $\tilde{A}^0$ is single-valued and is the infinitesimal generator of a uniquely determined semigroup $\{T(t)\}$ of contractions on $D(\tilde{A})$. In particular, $\bar{A}^0$ is the infinitesimal generator of a semigroup on $D(\bar{A})$.

These theorems I—IV correspond to Theorems 5.1, 5.2, 6.1, 6.2 and 6.3.

REMARK 1.2. The results mentioned above can be extended to the case of the semigroup $\{T(t); t \geqq 0\}$ of Lipschitzians on a subset $S$ such that there is a real number $\omega > 0$ with

$$e^{-\omega t}T(t) \in \mathrm{Cont}(S) \qquad \text{for } t \geqq 0.$$

This kind of semigroup is called a semigroup of *local type*. In this case we consider the operator $A$ satisfying the following conditions:

$$A-\omega \text{ is dissipative}, \tag{1.1}$$

$$R(I-\lambda A) \supset D(A) \qquad \text{for } \lambda \in (0, 1/\omega). \tag{1.2}$$

Then, by similar arguments to the analysis in the sequel we can obtain a semigroup $\{T(t)\}$ of local type on $\overline{D(A)}$ and quite similar conclusions as above.

## § 2. Abstract Cauchy Problem

A semigroup of contractions is closely related to the abstract Cauchy problem, formulated as follows:

$(CP)$ Given an operator $A$ in $X$ and an element $x \in X$, find a function $y(t; x)$ such that

(i) $y(t; x)$ is strongly absolutely continuous on every finite subinterval of $[0, \infty)$;

(ii) $y(0; x) = x$ and $(d/dt)y(t; x) \in Ay(t; x)$ for a. a. $t \geqq 0$; here, if $A$ is single-valued, then "$\in$" in the above problem is replaced by "$=$".

We call this an *abstract Cauchy problem formulated for A.* The multi-valued operator has been introduced so that each range $R(I-\lambda A)$ may include $\overline{D(A)}$ (see also Kōmura [**8**]). We are led to such a kind of problem in this paper. $(CP)$ is related to the notion of semigroup in the following manner.

PROPOSITION 2.1. *Let $A$ be a dissipative operator in $X$. And suppose that for each $x \in D(A)$ there is a solution of the $(CP)$ formulated for $A$. Then there is a uniquely determined semigroup $\{T(t)\}$ of contractions on $\overline{D(A)}$ in such a way that $y(t; x) = T(t)x$ for all $t \geqq 0$ and $x \in D(A)$. In particular, if $A$ is single-valued, then $A$ coincides with the weak infinitesimal generator of the $\{T(t)\}$ on a dense subset of $D(A)$. Conversely, if $A$ is the weak infinitesimal generator of a semigroup $\{T(t)\}$ of contractions, then $A$ is dissipative and for each $x \in D(A)$, $T(t)x$ is a unique solution of the $(CP)$.*

The proof is in Miyadera and Oharu [**14**]. We note that the first half of this Proposition 2.1 is still true for an arbitrary Banach space.

REMARK 2.1. Let $A$ be a dissipative operator satisfying condition $(R)$. Then, e. g., Theorem II described in § 1 states that the semigroup obtained by iteration of resolvents of $A$ gives a solution operator of $(CP)$ formulated for a maximal dissipative extension $\tilde{A}$ of $A$ in $\overline{D(A)}$. In particular, if $X$ is uniformly convex and if $A$ is closed, then the semigroup gives a solution operator of $(CP)$ formulated for the canonical restriction $A^0$. In view of these facts, we may restate the main results in § 1 in terms of abstract Cauchy problem. See also Kato [**11**; Remark 6.4].

EXAMPLE. Let $X = L^2(a, b)$ and $A$ be the operator defined in Example in § 1. We are concerned with the initial-value problem

$$u_t + uu_s = 0, \quad u(0, s) = x(s) \in \mathcal{L}^2(a, b). \tag{2.1}$$

If we restrict ourselves to the initial functions $x(s)$ such that $x \in D(A)$, then as an equation in the space $X$, (2.1) can be written

$$(d/dt)u(t) = Au(t), \qquad u(0) = x.$$

According to Theorem III or IV stated in § 1, this problem has a unique solution in $X$ if $x \in D(A)$. The solution is $u(t)=T(t)x$, where $\{T(t)\}$ is a semigroup on $D(A)$ generated by the $A$. And for each $f \in X^*$, $\langle u(t), f\rangle$ is continuously differentiable and $\langle u(t), f\rangle_t = \langle Au(t), f\rangle$ (see Theorem 6.2 mentioned later). This means that

$$\int_a^b u(t, \sigma)f(\sigma)d\sigma - \int_a^b x(\sigma)f(\sigma)d\sigma = -\int_0^t \int_a^b u(\xi, \sigma)u_\sigma(\xi, \sigma)f(\sigma)d\sigma d\xi .$$

Since it is easily seen that both $u(t, s)$ and $u_s(t, s)$ are measurable and essentially bounded as functions of two variables $s$ and $t$, using Fubini's theorem we obtain

$$\int_a^b \{u(t, s)-x(s)+\int_0^t u(\xi, s)u_s(\xi, s)d\xi\}f(s)ds = 0 .$$

This means that for each initial function $x(s) \in \mathcal{L}^2(a, b)$ with $x \in D(A)$, (2.1) has a weak solution $u(t, s)$ in the sense that

$$u_t(t, s)+u(t, s)u_s(t, s)=0 \qquad \text{for a. a. } (t, s);$$

$$\lim_{t\to +0}\int_a^b |u(t, s)-x(s)|^2 ds = 0 .$$

In order to show the unicity of semigroup of contractions, we restate Proposition 2.1 in the following form.

PROPOSITION 2.2. *Suppose that $A$ is dissipative. Then there is at most one semigroup $\{T(t)\}$ of contractions on $\overline{D(A)}$ such that for $x \in D(A)$,*

(a) *$T(t)x \in D(A)$ for a. a. $t$; and*

(b) *there is a Bochner measurable, locally integrable function $f(t; x)$ such that $f(t; x) \in AT(t)x$ for almost all $t$ and $(d/dt)T(t)x = f(t; x)$ almost everywhere.*

## § 3. Pseudo-Resolvents

In this section we consider pseudo-resolvents of nonlinear operators in $X$. Let $\{I_\lambda; \lambda > 0\}$ be a one-parameter family of contractions with the following properties: for every $\lambda$, $\mu > 0$ and $x \in D(I_\lambda)$,

(i) $$R\left[\frac{\mu}{\lambda} + \left(1-\frac{\mu}{\lambda}\right)I_\lambda\right] \subset D(I_\mu);$$

(ii) $$I_\lambda x = I_\mu\left[\frac{\mu}{\lambda}x + \left(1-\frac{\mu}{\lambda}\right)I_\lambda x\right].$$

We call this $\{I_\lambda; \lambda > 0\}$ a *pseudo-resolvent of contractions.* We say that a pseudo-resolvent $\{I_\lambda\}$ has the property $(DM)$, if there exists an $\eta > 0$ such that $I_\eta$ has the following property: if $D(I_\eta) \ni x_n \to x$ weakly and $I_\eta x_n \to y$

strongly imply that $x \in D(I_\eta)$ and $y = I_\eta x$.

REMARK 3.1. $R(I_\lambda)$ is constant, for the resolvent equation (ii) implies that $R(I_\lambda) \subset R(I_\mu)$ for every $\lambda, \mu > 0$.

PROPOSITION 3.1. *Let $A$ be a dissipative operator in $X$ and let $J_\lambda = (I - \lambda A)^{-1}$ for $\lambda > 0$. Then we have:*

(a) *$\{J_\lambda; \lambda > 0\}$ defines a pseudo-resolvent of contractions;*

(b) *if $A$ is closed, then each $R_\lambda$ is closed; if $A$ is demi-closed, then $\{J_\lambda\}$ has the property (DM);*

(c) *if $A$ satisfies (R), then for $x \in D(A)$, $J_\lambda x$ is strongly continuous in $\lambda > 0$ and $\lim_{\lambda \to +0} J_\lambda x = x$.*

PROOF. (a) Take any $x \in R_\lambda$. Then we have that

$$R_\mu \supset (I - \mu A) J_\lambda x = J_\lambda x - \frac{\mu}{\lambda} \lambda A J_\lambda x \ni \frac{\mu}{\lambda} x + \left(1 - \frac{\mu}{\lambda}\right) J_\lambda x,$$

from which it follows that $R\left[\frac{\mu}{\lambda} + \left(1 - \frac{\mu}{\lambda}\right) J_\lambda\right] \subset R_\mu$ and

$$J_\lambda x = J_\mu \left[\frac{\mu}{\lambda} + \left(1 - \frac{\mu}{\lambda}\right) J_\lambda\right] x. \tag{3.1}$$

(b) Fix any $\lambda > 0$. Suppose that $A$ is closed, $y_n \in R_\lambda$, and $y_n \to y \in X$ strongly. Then $x_n = J_\lambda y_n \to x \in X$, and so, $A x_n \ni (x_n - y_n)/\lambda \to (x - y)/\lambda$. Thus $(x - y)/\lambda \in Ax$, which means that $y \in R_\lambda$. Next, assume that $A$ is demi-closed, $y_n \in R_\lambda$, $y_n \to y$ weakly, and $x_n = J_\lambda y_n \to x$ strongly. Then $A x_n \ni (x_n - y_n)/\lambda \to (x - y)/\lambda$ weakly. Thus $x \in D(A)$ and $(x - y)/\lambda \in Ax$, that is, $J_\lambda y = x$.

(c) Take any $x \in D(A)$. Since $J_\mu \in \mathrm{Cont}(R_\mu)$ for $\mu > 0$, and since $\{J_\lambda x; 0 < \lambda < \varepsilon\}$ is bounded, $J_\lambda x \to J_\mu x$ as $\lambda \to \mu$ by (3.1). The last part is clear from $\| J_\lambda x - x \| \leqq \lambda |||Ax|||$. Q. E. D.

We obtain the converse of the above Proposition 3.1.

PROPOSITION 3.2. *Let $\{I_\lambda; \lambda > 0\}$ be a pseudo-resolvent of contractions. Then there is a dissipative operator $A$, defined on $D \equiv R(I_\lambda)$, such that $I_\lambda = (I - \lambda A)^{-1}$ for every $\lambda > 0$. If a $D(I_\lambda)$ is closed, then $A$ is closed. And if the $\{I_\lambda; \lambda > 0\}$ has the property (DM), then $A$ is demi-closed.*

PROOF. Since $R(I_\lambda)$ is constant by Remark 3.1, we write $D$ for it. We first show that for each $\lambda, \mu > 0$ the following relation holds:

$$\lambda^{-1}(x - I_\lambda^{-1} x) = \mu^{-1}(x - I_\mu^{-1} x) \quad \text{for } x \in D. \tag{3.2}$$

For this purpose we show that

$$\lambda^{-1}(x - I_\lambda^{-1} x) \subset \mu^{-1}(x - I_\mu^{-1} x) \tag{3.3}$$

for each $x \in D$ and $\lambda, \mu > 0$. Fix any $\lambda > 0$ and $\mu > 0$. Then, operating $I_\mu^{-1}$ on both sides of the resolvent equation (ii), we have that $I_\mu^{-1} I_\lambda y \ni (\mu/\lambda) y +$

$(1-\mu/\lambda)I_\lambda y$ for $y \in R_\lambda$, that is, $\lambda^{-1}(I_\lambda y-y) \in \mu^{-1}(I-I_\mu^{-1})I_\lambda y$. Hence, for any $x \in D$ and $y \in D(I_\lambda)$ with $I_\lambda y=x$, we have $\lambda^{-1}(x-y) \in \mu^{-1}(I-I_\mu^{-1})x$, which implies (3.3). We then define an operator $A$ on $D$ by $Ax=\lambda^{-1}(x-I_\lambda^{-1}x)$ for $x \in D$. It is easy to see that $I_\lambda=(I-\lambda A)^{-1}$. Since $(I-\lambda A)^{-1} \in \mathrm{Cont}\,(D(I_\lambda))$ for $\lambda>0$, Proposition 1.1 yields that $A$ is dissipative. The remaining part of the assertions is easily seen. Q.E.D.

REMARK 3.2. Assume that each $I_\lambda$ of a pseudo-resolvent $\{I_\lambda\,;\lambda>0\}$ belongs to $\mathrm{Cont}\,(X)$. Then the operator $A$ obtained by Proposition 3.2 is, by definition, an $m$-dissipative operator. So, $A$ is demi-closed. Hence, $\{I_\lambda\}$ has the property $(DM)$. Also, $A=A_c$ in this case.

Next, we present an extension of Proposition 3.2 (see also Brezis and Pazy [**1**; Theorem 2.3]).

PROPOSITION 3.3. *Let $\{I_\lambda\,;\lambda>0\}$ be a one-parameter family of single-valued operators such that the relations* (i) *and* (ii) *hold for each $\lambda$, $\mu$ with $0<\mu\leqq\lambda$ and the following condition is satisfied:*

(iii) *for each $\lambda>0$ and $x, y \in D(I_\lambda)$,*

$$\|I_\lambda x-I_\lambda y\|^2 \leqq \mathrm{re}\,\langle x-y,\, F(I_\lambda x-I_\lambda y)\rangle\,.$$

*Then there is a dissipative operator $A$ such that $D(A)=\bigcup_{\lambda>0} R(I_\lambda)$ and $(I-\lambda A)^{-1}x =I_\lambda x$ for $x \in D(I_\lambda)$ and $\lambda>0$.*

PROOF. For each $\lambda>0$ we define an operator $A(\lambda)$ by $A(\lambda)x=\lambda^{-1}(x-I_\lambda^{-1}x)$ for $x \in R(I_\lambda)$. Then by the same argument as in the proof of (3.3), we see that

$$R(I_\lambda)\subset R(I_\mu) \quad \text{and} \quad A(\lambda)\subset A(\mu) \qquad \text{for } 0<\mu\leqq\lambda\,. \tag{3.4}$$

Let $A=\bigcup_{\lambda>0} A(\lambda)$. Then it follows from (3.4) and (iii) that $A$ is dissipative. Also, for $\lambda>0$, we have

$$x-\lambda Ax \supset x-\lambda A(\lambda)x=I_\lambda^{-1}x \qquad \text{for } x \in R(I_\lambda)\,, \tag{3.5}$$

so it follows that $(I-\lambda A)^{-1}y=I_\lambda y$ for $y \in D(I_\lambda)$. Q.E.D.

COROLLARY 3.1. *Let $K$ be a convex set and let $\{I_\lambda\,;\lambda>0\}$ be a one-parameter family of single-valued operators such that $I_\lambda K\subset K$ for every $\lambda>0$ and* (ii) *and* (iii) *hold for every pair $\lambda$, $\mu$ with $0<\mu\leqq\lambda$ and $x \in K$. Then there is a dissipative operator $A$ such that $D(A)=\bigcup_{\lambda>0} I_\lambda K$, $R(I-\lambda A)\supset K\supset co\,D(A)$ for $\lambda>0$, and $(I-\lambda A)^{-1}x=I_\lambda x$ for $\lambda>0$ and $x \in K$.*

PROOF. Set $\hat{I}_\lambda=I_\lambda|_K$ for $\lambda>0$. Then $\{\hat{I}_\lambda\,;\lambda>0\}$ satisfies all of the assumptions of Proposition 3.3. We then let $A$ be the dissipative operator obtained by Proposition 3.3. Then (3.5) states that $R(I-\lambda A)\supset R(I-\lambda A(\lambda))\supset K \supset \bigcup R(I_\lambda)=D(A)$ for $\lambda>0$. Q.E.D.

## § 4. Approximation of Operators

Let $A$ be a dissipative operator in $X$. Then by Proposition 1.1, $J_\lambda=(I-\lambda A)^{-1}$ exists on $R_\lambda=R(I-\lambda A)$ for each $\lambda>0$. We then define a single-valued operator $A_\eta$ on $R_\eta$ for $\eta>0$, by

$$A_\eta x=\eta^{-1}[J_\eta x-x], \qquad x\in R_\eta. \tag{4.1}$$

It is easy to see that $\|A_\eta\|_{\mathrm{Lip}(R_\eta)}\leqq 2/\eta$ and that each $A_\eta$ is dissipative. In this section we shall give some properties of $A_\eta$ and some of the basic properties of closed or demi-closed dissipative operators. First, we present the next lemma which is proved in a similar way to Kato [**11**; § 4] or Crandall and Pazy [**2**; Lemma 2.3].

Proposition 4.1. *Let $A$ be a demi-closed operator in $X$, $\{x_n\}\subset D(A)$, and $x_n\to x_0$ strongly as $n\to\infty$. Then we have:*

(a) *Let $y_n\in Ax_n$ for each $n$ and $\{y_n\}$ be bounded. Let $V$ be the set of all weak cluster points of $\{y_n\}$. Then $x_0\in D(A)$, $V\neq\emptyset$ and $V\subset Ax_0$. In particular, if $A$ is single-valued, then $Ax_0=w\text{-}\lim_{n\to\infty} y_n$.*

(b) *Assume that $X$ is uniformly convex and that the canonical restriction $A^0$ is single-valued. If $y_n\in Ax_n$ for each $n$, $x_0\in D(A^0)$ and* $\limsup\|y_n\|\leqq\|A^0x_0\|$, *then $A^0x_0=\lim y_n$.*

Next, referring to Brezis and Pazy [**1**; § 2], we obtain

Proposition 4.2. *Let $X$ be uniformly convex and $A$ be a closed dissipative operator satisfying (R). Let $\tilde{A}$ be a maximal dissipative extension of $A$ in $\overline{D(A)}$. Then $D(\tilde{A})=D(A)=D(A^0)$ and $\tilde{A}^0=A^0_c=A^0$.*

Proof. We first note that $\tilde{A}$ is demi-closed, $\tilde{A}^0$ is single-valued and that $J_\lambda x=(I-\lambda\tilde{A})^{-1}x\to x$ strongly as $\lambda\to+0$ for $x\in D(\tilde{A})$. And $A^0_c$ is also single-valued. Let $\eta_k\downarrow 0$, $\tilde{J}_k\equiv\tilde{J}_{\eta_k}$, and $\tilde{A}_k\equiv\eta_k^{-1}(\tilde{J}_k-I)$. Then by Proposition 4.1 (b),

$$\tilde{A}^0x=\lim_{k\to\infty}\tilde{A}_kx \qquad \text{for } x\in D(\tilde{A}). \tag{4.2}$$

Now take any $z\in D(\tilde{A})$. Then by the assumption there are $x_k\in D(A)$ and $y_k\in Ax_k$ such that $z=x_k-\eta_k y_k$. Since $y_k=\tilde{A}_kz\in\tilde{A}x_k$, it follows that $x_k=\tilde{J}_kz\to z$ strongly. Combining with (4.2), the closedness of $A$ implies that $z\in D(A)$ and $\tilde{A}^0z\in Az$. But then $Az\subset A_cz\subset\tilde{A}z$, so we see that $\tilde{A}^0z=A^0_cz\in A^0z$. Also, $|||A^0z|||\leqq\|\tilde{A}^0z\|$. Hence the relation $v\in A^0z\subset\tilde{A}z$ states that $v=\tilde{A}^0z$, because $\tilde{A}^0$ is single-valued. This means that $A^0$ is also single-valued. Q. E. D.

That $A^0$ is single-valued was suggested by Mr. J. Chambers.

Let $\eta_k\downarrow 0$, $J_k\equiv J_{\eta_k}$, and $A_k\equiv A_{\eta_k}$. Then the sequence $\{A_k\}$ approximates $A$ in the following sense.

Corollary 4.1. *Let $A$ be a dissipative operator satisfying (R). For each $x\in D(A)$ let $V(x)$ be the set of all weak cluster points of $\{A_kx\}$. Then we have:*

(a) *If $A$ is demi-closed, then $V(x)\neq\emptyset$ and $V(x)\subset A^0x$. If $A$ is demi-closed and if $A^0$ is single-valued, then $A^0x=w\text{-}\lim A_kx$.*

(b) *Let $X$ be uniformly convex. If $A$ is closed, then $A^0x=\lim A_kx$ for $x\in D(A)$.*

PROOF. (a) For $x\in D(A)$, $\lim J_kx=x$ and $\|A_kx\|\leqq|||Ax|||$. It then follows from Proposition 4.1 (a) that $V(x)\subset Ax$. Let $A_{k'}x\to y$ weakly. Then we see that $\|y\|\leqq|||Ax|||$, which means that $V(x)\subset A^0x$. If $A^0$ is single-valued, then each $V(x)$ is a singleton, and so the whole sequence $\{A_kx\}$ converges weakly to $A^0x$ for $x\in D(A)$. (b) is an immediate consequence of Proposition 4.2 and the convergence (4.2), since $\tilde{A}_kx=A_kx$ for each $x\in D(A)$. Q. E. D.

REMARK 4.1. As proved in Proposition 4.4, $(I-\lambda A_k)^{-1}x\to J_\lambda x$ strongly as $k\to\infty$, for $\lambda>0$ and $x\in D(A)$. This shows that $\{A_k\}$ approximates $A$ in a generalized sense.

REMARK 4.2. Let $A$ be a demi-closed dissipative operator satisfying $(R)$. Let $\tilde{A}$ be any maximal dissipative extension of $A$ in $\overline{D(A)}$, and $\tilde{J}_\lambda=(I-\lambda\tilde{A})^{-1}$ for $\lambda>0$. Then $\tilde{J}_\lambda\supset J_\lambda$ and $\tilde{A}\supset A_c\supset A$. Now fix an $x\in D(A)$, then $\|A_\lambda x\|=\lambda^{-1}\|J_\lambda x-x\|=\lambda^{-1}\|\tilde{J}_\lambda x-\tilde{J}_\lambda(x-\lambda y)\|\leqq\|y\|$ for $y\in\tilde{A}x$ and $\lambda>0$. Hence, $\|A_\lambda x\|\leqq|||\tilde{A}x|||\leqq|||\overline{co}\,Ax|||\leqq|||Ax|||$ for $x\in D(A)$ and $\lambda>0$. Let $J_{\lambda_i}x\to x$ strongly and $A_{\lambda_i}x\to y$ weakly. Then Corollary 4.1 (a) yields that $y\in Ax$, and $\|y\|\leqq\liminf\|A_{\lambda_i}x\|$. This means that $|||Ax|||=|||\tilde{A}x|||=|||A_cx|||=\lim_{\lambda\to+0}\|A_\lambda x\|$. Since $Ax\subset\overline{co}\,Ax\subset\tilde{A}x$ for $x\in D(A)$, it follows that $A^0x\neq\emptyset$ and $A^0x\subset A_c^0x\subset\tilde{A}^0x$ for $x\in D(A)$.

In the following, we consider the resolvent of $A_\eta$.

PROPOSITION 4.3. *Let $A$ be a dissipative operator, $J_\lambda=(I-\lambda A)^{-1}$ for $\lambda>0$ and $A_\eta$ be defined by* (4.1). *Then for every $\eta$, $h>0$, $(I-hA_\eta)^{-1}$ exists as an operator from $R_{\eta+h}$ onto $R_\eta$, and*

$$(4.3)\qquad (I-hA_\eta)^{-1}x=\frac{h}{\eta+h}J_{\eta+h}x+\frac{\eta}{\eta+h}x,\qquad \textit{for } x\in R_{\eta+h}.$$

PROOF. Fix any pair $\eta$, $h>0$. Then for $x\in R_{\eta+h}$,

$$\frac{h}{\eta+h}J_{\eta+h}x+\frac{\eta}{\eta+h}x=J_{\eta+h}x-\frac{\eta}{\eta+h}[J_{\eta+h}x-x]\in(I-\eta A)J_{\eta+h}x\subset R_\eta,$$

and for $x\in R_\eta$,

$$\begin{aligned}(I-hA_\eta)x&=\frac{\eta+h}{\eta}x-\frac{h}{\eta}J_\eta x\\&=J_\eta x-\frac{\eta+h}{\eta}[J_\eta x-x]\in(I-(\eta+h)A)J_\eta x\subset R_{\eta+h},\end{aligned}$$

where we used the relation $h^{-1}[J_hx-x]\in AJ_hx$. Using these relations, we obtain

$$(I-hA_\eta)\Big[\frac{h}{\eta+h}J_{\eta+h}x+\frac{\eta}{\eta+h}x\Big]$$

$$=\frac{\eta+h}{\eta}\Big[\frac{h}{\eta+h}J_{\eta+h}x+\frac{\eta}{\eta+h}x\Big]-\frac{h}{\eta}J_\eta(I-\eta A)J_{\eta+h}x=x$$

for $x\in R_{\eta+h}$; and

$$\Big[\frac{h}{\eta+h}J_{\eta+h}+\frac{\eta}{\eta+h}I\Big](I-hA_\eta)x$$

$$=\frac{h}{\eta+h}J_{\eta+h}(I-(\eta+h)A)J_\eta x+\frac{\eta}{\eta+h}\Big[\frac{\eta+h}{\eta}x-\frac{h}{\eta}J_\eta x\Big]=x,$$

for $x\in R_\eta$. Thus $(I-hA_\eta)^{-1}$ exists as an operator from $R_{\eta+h}$ onto $R_\eta$ and hence (4.3) holds. Q. E. D.

PROPOSITION 4.4. *Let $A$ be a dissipative operator in $X$ satisfying* $(R_{co})$ *(stated in Remark 1.1). Let $U_\eta=A_\eta|_{\overline{co}D(A)}$ and $J_h(\eta)=(I-hU_\eta)^{-1}$ for $\eta$, $h>0$. Then $R(I-hU_\eta)\supset\overline{co}\,D(A)=D(U_\eta)$ and $J_h(\eta)\in\mathrm{Cont}\,(R(I-hU_\eta))$ for $\eta$, $h>0$. Furthermore, for $x\in D(A)$, $h>0$ and $n$,*

$$\lim_{\eta\to+0}J_h(\eta)^n x=J_h^n x. \tag{4.4}$$

PROOF. Under $(R_{co})$, (4.3) implies that $J_h(\eta)[\overline{co}\,D(A)]\subset\overline{co}\,D(A)$ for $\eta$, $h>0$. Hence, we see that $R(I-hU_\eta)\supset\overline{co}\,D(A)$. Since $U_\eta$ is dissipative, $J_h(\eta)\in\mathrm{Cont}\,(R(I-hU_\eta))$. Hence, the iterations $J_h(\eta)^n$, $n=1, 2, \cdots$, are well-defined on $R(I-hU_\eta)$ and

$$\|U_\eta J_h(\eta)^n x\|\leqq|||Ax||| \qquad \text{for } x\in D(A). \tag{4.5}$$

Take $x\in D(A)$, then by (4.3), $J_h(\eta)x-J_hx=\frac{h}{\eta+h}J_{\eta+h}x+\frac{\eta}{\eta+h}x-J_hx$ for $h$, $\eta>0$. Since the right side is estimated by $\frac{h}{\eta+h}\|J_{\eta+h}x-J_hx\|+\frac{\eta}{\eta+h}\|x-J_hx\|$, Proposition 3.1 (c) implies that $\lim_{\eta\to+0}J_h(\eta)x=J_hx$. Since $J_h(\eta)\in\mathrm{Cont}\,(R(I-hU_\eta))$, (4.4) holds for each positive integer $n$. Q. E. D.

## § 5. Construction of the Semigroups

In this section, we construct the semigroup determined by a dissipative operator $A$ which satisfies the condition $(R)$.

LEMMA 5.1. *Let $A$ be a dissipative operator in $X$ satisfying $(R)$. If $x\in D(A)$ and $T>0$, then*

$$y(t\,;x)=\lim_{\lambda\to+0}(I-\lambda A)^{-[t/\lambda]}x \tag{5.1}$$

*exists uniformly for $t\in[0, T]$.*

PROOF. Set $J_\lambda=(I-\lambda A)^{-1}$ and $A_\lambda=\lambda^{-1}(J_\lambda-I)$, $\lambda>0$. Let $x\in D(A)$ and

$T>0$. We note that

(5.2) $$\|A_h J_h^m x\| \leqq |||Ax||| \quad \text{for } h>0 \text{ and } m.$$

Now, assume that $n\lambda \leqq h$ and $hm \leqq T$, where $\lambda$, $h>0$ and $m$, $n$ are integers. Let $k \leqq m$. Since $J_\lambda^{nk}x - J_\lambda^{n(k-1)}x = \lambda \sum_{p=0}^{n-1} A_\lambda J_\lambda^p J_\lambda^{n(k-1)}x$, we have that $(J_\lambda^{nk}x - J_\lambda^{n(k-1)}x) - (J_h^k x - J_h^{k-1}x) = \lambda \sum_{p=0}^{n-1} \{A_\lambda J_\lambda^p J_\lambda^{n(k-1)}x - A_h J_h^{k-1}x\} + (n\lambda - h) A_h J_h^{k-1}x$. Thus we can write

$$\begin{aligned}
&\langle (J_\lambda^{nk}x - J_\lambda^{n(k-1)}x) - (J_h^k x - J_h^{k-1}x),\ F(J_\lambda^{nk}x - J_h^k x)\rangle \\
&= \lambda \sum_{p=0}^{n-1} \langle A_\lambda J_\lambda^{n(k-1)+p}x - A_h J_h^{k-1}x,\ F(J_\lambda^{n(k-1)+p+1}x - J_h^k x)\rangle \\
&\quad + \lambda \sum_{p=0}^{n-1} \langle A_\lambda J_\lambda^{n(k-1)+p}x - A_h J_h^{k-1}x,\ F(J_h^{nk}x - J_h^k x) - F(J_\lambda^{n(k-1)+p+1}x - J_h^k x)\rangle \\
&\quad + (n\lambda - h)\langle A_h J_h^{k-1}x,\ F(J_\lambda^{nk}x - J_h^k x)\rangle \\
&\equiv I_1 + I_2 + I_3 .
\end{aligned}$$

We now estimate each term. Since $A$ is dissipative, $I_1 \leqq 0$. Also, by using (5.2) we have that

$$\|I_2\| \leqq 2|||Ax|||\lambda \sum_{p=0}^{n-1} \|F(J_\lambda^{nk}x - J_h^k x) - F(J_\lambda^{n(k-1)+p+1}x - J_h^k x)\| .$$

Employing the uniform continuity of $F$ on bounded sets, we can find a function $\varepsilon(h) \equiv \varepsilon(h; x, T)$ such that $\varepsilon(h) \to 0$ as $h \downarrow 0$, and

$$\sup_{n\lambda \leqq h, hk \leqq T} \|F(J_\lambda^{nk}x - J_h^k x) - F(J_\lambda^{n(k-1)+p+1}x - J_h^k x)\| \leqq \varepsilon(h) .$$

Note that $\|J_\lambda^{n(k-1)+p+1}x - J_\lambda^{nk}x\| = 0(h)$ as $h \downarrow 0$. Also,

$$\|I_3\| \leqq |n\lambda - h|\, |||Ax|||\, \|J_\lambda^{nk}x - J_h^k x\| .$$

Consequently,

$$\begin{aligned}
\|J_\lambda^{nm}x - J_h^m x\|^2 &= \sum_{k=1}^m \{\|J_\lambda^{nk}x - J_h^k x\|^2 - \|J_\lambda^{n(k-1)}x - J_h^{k-1}x\|^2\} \\
&\leqq \sum_{k=1}^m 2\,\mathrm{re}\,\langle (J_\lambda^{nk}x - J_h^k x) - (J_\lambda^{n(k-1)}x - J_h^{k-1}x),\ F(J_\lambda^{nk}x - J_h^k x)\rangle \\
&\leqq \psi(\lambda, h),
\end{aligned}$$

where $\psi(\lambda, h) \equiv \mathrm{const}\,(x, T)(h + \varepsilon(h) + m|\lambda n - h|)$ and note that $\|x\|^2 - \|y\|^2 \leqq 2\,\mathrm{re}\,\langle x-y, F(x)\rangle$. Hence, for each $t \in [0, T]$,

$$\|J_\lambda^{n[t/h]}x - J_h^{[t/h]}x\|^2 \leqq \psi(\lambda, h) .$$

First, take $\lambda = \varepsilon_\mu = 2^{-\mu}$, $h = \varepsilon_\nu = 2^{-\nu}$, $m = [t/\varepsilon_\nu]$ and $n = 2^{\mu-\nu}$. In this case $\psi(\varepsilon_\mu, \varepsilon_\nu) = \mathrm{const}\,(x, T)(\varepsilon_\nu + \varepsilon(\varepsilon_\nu)) \to 0$ as $\nu \to \infty$ and $|[t/\varepsilon_\mu] - 2^{\mu-\nu}[t/\varepsilon_\nu]| \leqq 2^{\mu-\nu}$. So, we see that (by (5.2)) $\|J_{\varepsilon_\mu}^{[t/\varepsilon_\mu]}x-J_{\varepsilon_\mu}^{2^{\mu-\nu}[t/\varepsilon_\nu]}x\|=0(\varepsilon_\nu)$, and hence $\|J_{\varepsilon_\mu}^{[t/\varepsilon_\mu]}x-J_{\varepsilon_\nu}^{[t/\varepsilon_\nu]}x\| \leqq 0(\varepsilon_\nu)+\sqrt{\phi(\varepsilon_\mu, \varepsilon_\nu)}$. This means that $\{J_{\varepsilon_\nu}^{[t/\varepsilon_\nu]}x\}$ is a Cauchy sequence. We then set

$$y(t\,;x)=\lim_{\nu\to\infty} J_{\varepsilon_\nu}^{[t/\varepsilon_\nu]}x\,, \qquad t\in[0, T]\,. \tag{5.3}$$

Finally, we show that the limit is independent of the sequence chosen. Let $0\leqq t<T$, and $0<\lambda\leqq h<T-t$. Taking, this time $m=[t/h]+1$ and $n=\left[\frac{[t/\lambda]}{[t/h]+1}\right]$, we observe that

$$\begin{cases} mh\leqq t+h, \quad n\lambda\leqq h, \quad |t-n\lambda m|\leqq 2\lambda+T\lambda/h\,, \\ |[t/\lambda]-nm|\lambda\leqq 3\lambda+T\lambda/h, \quad m|n\lambda-h|\leqq 2h+2\lambda+T\lambda/h\,. \end{cases} \tag{5.4}$$

Again take $\lambda=\varepsilon_\nu$. Using (5.4) and letting $\nu\to\infty$, we see that $\phi(\varepsilon_\nu, h) \to \mathrm{const}\,(x, T)(3h+\varepsilon(h))$. Therefore, (5.3) and (5.4) imply that

$$\|y(t\,;x)-J_h^{[t/h]}x\|\leqq \mathrm{const}\,(x, T)\sqrt{h+\varepsilon(h)}\,. \tag{5.5}$$

Q. E. D.

LEMMA 5.2. *Let $A$ be a dissipative operator in $X$ satisfying $(R)$.*

(a) $\|y(t\,;x_1)-y(t\,;x_2)\|\leqq\|x_1-x_2\|$ *for* $t\geqq 0$ *and* $x_1, x_2\in D(A)$.

(b) *For every* $x\in D(A)$,

$$\|y(t\,;x)-y(t'\,;x)\|\leqq|t-t'|\,|||Ax|||\qquad \textit{for } t,\ t'\geqq 0\,.$$

PROOF. (a) Since $J_\lambda\in \mathrm{Cont}\,(R_\lambda)$ for $\lambda>0$, (a) follows from (5.3).

(b) For $x\in D(A)$, (5.2) yields that $\|J_h^{[t/h]}x-J_h^{[t'/h]}x\|\leqq|t-t'+h|\,|||Ax|||$ for $t$, $t'\geqq 0$. Letting $h\to+0$, we have (b).

Consequently, we have the following main theorem.

THEOREM 5.1. *If $A$ is a dissipative operator in $X$ satisfying $(R)$, then there exists a semigroup $\{T(t)\}$ on $\overline{D(A)}$ such that*

$$T(t)x=\lim_{\lambda\to+0}(I-\lambda A)^{-[t/\lambda]}x \qquad \textit{for } t\geqq 0 \textit{ and } x\in D(A) \tag{5.6}$$

*and the convergence is uniform with respect to $t$ in every finite interval.*

PROOF. In view of Lemma 5.1, set $T(t)x=y(t\,;x)$ for $t\geqq 0$ and $x\in D(A)$. First, by using Lemma 3.2 (a), we can obtain a unique extension of $T(t)$ to $\overline{D(A)}$ by continuity, we denote this extension by the same symbol $T(t)$. Then each $T(t)$ maps $\overline{D(A)}$ into itself and $T(t)\in\mathrm{Cont}\,(\overline{D(A)})$. To establish the semigroup property; first take $x\in D(A)$ and $t$, $s\geqq 0$ with $t+s\leqq T$. Let $\varepsilon>0$ and take $z\in D(A)$ such that $\|z-T(s)x\|<\varepsilon$. Then, $\|T(t+s)x-T(t)T(s)x\|\leqq\|T(t+s)x -J_\lambda^{[(t+s)/\lambda]}x\|+\|J_\lambda^{[(t+s)/\lambda]}x-J_\lambda^{[t/\lambda]+[s/\lambda]}x\|+\|J_\lambda^{[t/\lambda]+[s/\lambda]}x-J_\lambda^{[t/\lambda]}z\|+\|J_\lambda^{[t/\lambda]}z-T(t)z\|+\|T(t)z -T(t)T(s)x\|\leqq\|T(t+s)x-J_\lambda^{[(t+s)/\lambda]}x\|+2\lambda|||Ax|||+\|J_\lambda^{[s/\lambda]}x-z\|+\|J_\lambda^{[t/\lambda]}z-T(t)z\|+\|z-$

$T(s)x\|$. Therefore, letting $\lambda \to +0$, we have $\|T(t+s)x-T(t)T(s)x\| \leqq 2\varepsilon$. This means that $\{T(t); t \geqq 0\}$ has the semigroup property. (5.6) was established in Lemma 5.1. Q. E. D.

REMARK 5.1. Let $A$ be a dissipative operator in $X$ satisfying $(R_{cl})$. Then $J_\lambda^{[t/\lambda]} \in \mathrm{Cont}(\overline{D(A)})$ for $\lambda > 0$ and $t \geqq 0$. Hence, the convergence (5.6) holds for all $t \geqq 0$ and $x \in \overline{D(A)}$.

REMARK 5.2. (a) Let $A$ be a dissipative operator satisfying $(R)$, and $\{T(t)\}$ be the corresponding semigroup obtained by Theorem 5.1. Let $\tilde{A} \in \mathcal{E}[A; \overline{D(A)}]$ and put $\tilde{R}_\lambda = R(I-\lambda\tilde{A})$ and $\tilde{J}_\lambda = (I-\lambda\tilde{A})^{-1}$, then $\tilde{J}_\lambda \supset J_\lambda$ for $\lambda > 0$. Now, if $\tilde{A}$ is maximal dissipative on $\overline{D(A)}$, then by Remark 1.1 (a) we have that $\tilde{R}_\lambda \supset \overline{D(A)} \supset D(\tilde{A})$ for $\lambda > 0$. So that $\tilde{A}$ satisfies $(R_{cl})$. Thus, Lemma 5.1 and Remark 5.1 yield that

$$T(t)x = \lim_{\lambda \to +0} (I-\lambda\tilde{A})^{-[t/\lambda]}x \tag{5.7}$$

for $t \geqq 0$ and $x \in \overline{D(A)}$.

(b) Let $A$ be a dissipative operator satisfying $(R_{cl})$. Then for $\tilde{A} \in \mathcal{E}[A; \overline{D(A)}]$, $\tilde{R}_\lambda \supset \overline{D(A)} \supset D(\tilde{A})$ for $\lambda > 0$, and hence we have the same conclusion (5.6) for $t \geqq 0$ and $x \in \overline{D(A)}$.

In view of these results, we obtain the following

THEOREM 5.2. *Let $A$ be a dissipative operator in $X$ satisfying $(R_{cl})$. Then there is a semigroup $\{T(t)\}$ of contractions on $\overline{D(A)}$ such that the convergence* (5.7) *holds for $\tilde{A} \in \mathcal{E}[A; \overline{D(A)}]$.*

Next, in terms of pseudo-resolvent, we can obtain some variations of Theorem 5.1.

COROLLARY 5.1. *Let $S \subset X$, and $\{I_\lambda; \lambda > 0\}$ be a pseudo-resolvent of contractions from $S$ into itself. Then there is a semigroup $\{T(t)\}$ of contractions on $\overline{R(I_\lambda)}$ such that $T(t)x = \lim_{\lambda \to +0} I_\lambda^{[t/\lambda]}x$ for each $t \geqq 0$ and $x \in R(I_\lambda)$.*

The proof is a simple consequence of Theorem 5.1. For, Proposition 3.2 implies that $\{I_\lambda\}$ determines a dissipative operator $A$ on $D \equiv R(I_\lambda)$ in such a way that $I_\lambda = (I-\lambda A)^{-1}$. The assumptions yield that $A$ satisfies condition $(R)$ and therefore, by Theorem 5.1 we have the assertions.

COROLLARY 5.2. *Let $K \subset X$ be a convex set and $\{I_\lambda; \lambda > 0\}$ be a one-parameter family of single-valued operators such that $I_\lambda K \subset K$ for every $\lambda > 0$ and such that*

$$I_\lambda x = I_\mu\Big[\frac{\mu}{\lambda}x + \Big(1-\frac{\mu}{\lambda}\Big)I_\lambda x\Big], \tag{5.8}$$

$$\|I_\lambda x - I_\lambda y\|^2 \leqq \mathrm{re}\,\langle x-y, F(I_\lambda x - I_\lambda y)\rangle, \tag{5.9}$$

*for $\lambda$, $\mu$ with $0 < \mu \leqq \lambda$ and $x \in K$. Then there exists a semigroup $\{T(t)\}$ of* *contractions on* $\overline{\bigcup_{\lambda>0} I_\lambda K}$ *such that* $T(t)x=\lim_{\lambda\to+0} I_\lambda^{[t/\lambda]}x$ *for* $t\geqq 0$ *and* $x\in\bigcup_{\lambda>0} I_\lambda K$.

The proof is easily seen from Corollary 3.1 and Theorem 5.1, in the similar way to the proof of Corollary 5.1.

Finally, we present a result on the semigroups determined by the approximate operators $A_\eta$.

COROLLARY 5.3. *Let* $A$ *be a dissipative operator in* $X$ *satisfying* $(R_{co})$, *and* $\{T(t)\}$ *be the corresponding semigroup on* $\overline{D(A)}$. *Let* $A_\eta$, $\eta>0$, *be the operators defined by* (4.1). *Then we have:*

(a) *Let* $U_\eta\equiv A_\eta|_{\overline{co}D(A)}$ *for* $\eta>0$, *then*

$$T_\eta(t)x=\lim_{\lambda\to+0}(I-\lambda U_\eta)^{-[t/\lambda]}x$$

*exists for* $t\geqq 0$, $x\in\overline{co}\,D(A)$ *and* $\eta>0$, *and* $\{T_\eta(t);\ t\geqq 0\}$ *forms a semigroup of contractions on* $\overline{co}\,D(A)$. *Furthermore, if* $x\in D(A)$, *then the convergence is uniform with respect to* $\eta>0$ *and* $t$ *in every finite interval.*

(b) *For* $x\in\overline{D(A)}$ *and* $t\geqq 0$,

$$\lim_{\eta\to+0}T_\eta(t)x=T(t)x\,,$$

*where the convergence is uniform with respect to* $t$ *in every finite interval.*

PROOF. (a) The first assertion follows from Proposition 4.4 and Remark 5.1. To prove the second assertion; let $x\in D(A)$ and $T>0$, then by (4.5) we see that the set $\{J_h(\eta)^n x;\ h,\eta>0,\ nh\leqq T\}$ is bounded and $\sup_{0\leqq p\leqq n-1}\|J_\lambda(\eta)^{n(k-1)+p+1}x-J_\lambda(\eta)^{nk}x\|\leqq n\lambda|||Ax|||$. Therefore, by the same argument as in Lemma 5.1, we have the following estimate:

$$\|T_\eta(t)x-J_h(\eta)^{[t/h]}x\|\leqq \mathrm{const}\,(x,T)\sqrt{h+\varepsilon(h)}\,, \tag{5.10}$$

where the constant const$\,(x,T)$ and the function $\varepsilon(h)$ are independent of $\eta>0$ and $t\in[0,T]$.

(b) Let $x\in D(A)$ and $T>0$. Then (5.5) and (5.10) imply that

$$\|T(t)x-T_\eta(t)x\|\leqq \mathrm{const}\,(x,T)\sqrt{h+\varepsilon(h)}+\|J_h^{[t/h]}x-J_h(\eta)^{[t/h]}x\|\,.$$

Take any $\varepsilon>0$, then there is an $h_0>0$ such that const$\,(x,T)\sqrt{h_0+\varepsilon(h_0)}<\varepsilon/2$. Since $\{[t/h_0];\ 0\leqq t\leqq T\}$ is a finite set, there is an $\eta_0$ such that $\|J_{h_0}^{[t/h_0]}x-J_{h_0}(\eta)^{[t/h_0]}x\|<\varepsilon/2$ for $\eta\in(0,\eta_0)$. Consequently, $\sup_{0\leqq t\leqq T}\|T(t)x-T_\eta(t)x\|\leqq\varepsilon$ for $\eta\in(0,\eta_0)$. Q. E. D.

## § 6. Differentiability of Semigroups

In this section we consider the differentiability of semigroups obtained by Theorem 5.1 and some relations among the semigroups and the infinitesimal generators. The central part of the proof is based on the results by Kato [**10, 11**].

Throughout this section we use the following notations: Let $r_n=2^n$, and $I_r=[0, r]$ for $r>0$. We then set $J_n=(I-r_n^{-1}A)^{-1}$, $A_n=r_n[J_n-I]$, and $T(n;t)=(I-r_n^{-1}A)^{-[tr_n]}$.

LEMMA 6.1. *Let $A$ be a dissipative operator satisfying* (*R*). *Let $x\in D(A)$ and $f_n(t)=A_nT(n;t)x$ for $t\geqq 0$. Then,*

$$f_n(t)\in AJ_nT(n;t)x \quad \text{and} \quad \|f_n(t)\|\leqq |\!|\!| Ax |\!|\!| \qquad \text{for } t\geqq 0\,, \tag{6.1}$$

$$\|[T(n;t)-I]J_nx-\int_0^t f_n(s)ds\|=0(1/r_n)\,. \tag{6.2}$$

PROOF. (6.1) is obvious. (6.2) is obtained by estimating

$$\begin{aligned} \int_0^{[r_nt]/r_n} f_n(s)ds &= r_n^{-1}\sum_{k=1}^{[r_nt]} A_nJ_n^{(k-1)}x \\ &= [T(n;t)-I]J_nx+r_n^{-1}\{A_nx-A_nT(n;t)x\}\,. \qquad \text{Q.E.D.} \end{aligned} \tag{6.3}$$

LEMMA 6.2. *Let $A$ be a demi-closed dissipative operator satisfying* (*R*). *Let $\{T(t)\}$ be the semigroup of contractions on $\overline{D(A)}$, obtained by Theorem* 5.1. *Then we have:*

(a) *For every $x\in D(A)$, $T(t)x\in D(A)$ for $t\geqq 0$ and there is a function $f(\cdot\,;x)$ on $[0,\infty)$ such that $f(t;x)\in A_c^0T(t)x$ for almost all $t\geqq 0$ and*

$$T(t)x-x=\int_0^t f(s;x)ds\,, \qquad \text{for } t\geqq 0\,. \tag{6.4}$$

(b) *If $A$ is single-valued, then for $x\in D(A)$, $AT(t)x$ is weakly continuous in $t\geqq 0$, and*

$$T(t)x-x=\int_0^t AT(s)xds\,, \qquad \text{for } t\geqq 0\,. \tag{6.5}$$

PROOF. (a) Fix any $x\in D(A)$ and any $p$ with $1<p<+\infty$. And set $f_n(t)=A_nT(n;t)x$. Then by (6.1) $\{f_n|_{I_r};n\}$ forms a bounded set of $L^p(I_r;X)$ for each integer $r>0$. Thus, by moving $r$ and by using the diagonal procedure, we may find a subsequence $\{q\}$ of $\{n\}$ and a function $f$, defined on $[0,\infty)$, such that $f_q|I_r$ converges to $f|I_r$ weakly in $L^p(I_r;X)$, for each $r$. Hence, $x^*\int_0^t f_q(s)ds\to x^*\int_0^t f(s)ds$ for $x^*\in X^*$ and $t\geqq 0$. Therefore (6.4) follows from (6.2). Next, we write $V(t)$ for the set of all weak cluster points of $\{f_n(t);n\}$ for each $t$. Then from Lemma 5.1 and Proposition 4.1 (a) it follows that $T(t)x\in D(A)$, $V(t)\neq\emptyset$, and $V(t)\subset AT(t)x$ for $t\geqq 0$. Hence, by the same argument as in Kato [**11**; Lemma 8.2] we see that $f(t)\in\overline{co}\,AT(t)x$ almost everywhere. Moreover, in a similar way to Kato [**11**; Lemma 6.2], we can prove that $\|f(t)\| \leqq |||\overline{co}\, AT(t)x|||$ almost everywhere (where we apply the argument in Remark 4.2). Thus it follows that $f(t) \in A_c^0 T(t)x$ almost everywhere.

(b) Aussume that $A$ is single-valued. Then each $V(t)$ is a singleton, and so, Proposition 4.1 implies that $w\text{-}\lim f_n(t) = AT(t)x$ for all $t \geqq 0$. Also, from the strong continuity of $T(t)x$ and the boundedness of $AT(t)x$ we see that $AT(t)x$ is weakly continuous in $t$. Thus (6.5) follows from (6.2). Q. E. D.

REMARK 6.1. Let $A$ be a demi-closed dissipative operator satisfying $(R)$ and $\{T(t)\}$ the semigroup obtained by Theorem 5.1. Then $\{T(t)|_{D(A)}\,;\ t \geqq 0\}$ forms a semigroup of contractions on $D(A)$ by the above lemma.

REMARK 6.2. By (6.4) we see that the infinitesimal generator $A_0$ of $\{T(t)\}$ is densely defined in $D(A)$.

In view of these results, we obtain the following.

First, combining with Remark 5.2 (a) we obtain

THEOREM 6.1. *Let $A$ be a dissipative operator satisfying $(R)$. Let $\tilde{A}$ be any maximal dissipative extension of $A$ in $\overline{D(A)}$. Then there is a uniquely determined semigroup $\{T(t)\}$ of contractions on $D(\tilde{A})$ such that for each $x \in D(\tilde{A})$, $(d/dt)T(t)x \in \tilde{A}^0 T(t)x$ for almost all $t \geqq 0$.*

Next, for the single-valued case, we obtain

THEOREM 6.2. *Let $A$ be a single-valued, demi-closed dissipative operator satisfying $(R)$. Then $A$ is the weak infinitesimal generator of a unique semigroup $\{T(t)\}$ of contractions on $D(A)$ such that for each $x \in D(A)$, $T(t)x$ is weakly continuously differentiable in $t \geqq 0$ and $T(t)x - x = \int_0^t AT(s)x\,ds$ for $t \geqq 0$.*

In the remainder of this section we consider the case $X$ is uniformly convex. Assume that $X$ is uniformly convex. Let $A$ be a closed dissipative operator satisfying $(R)$, and $\tilde{A}$ be any maximal dissipative extension of $A$ in $\overline{D(A)}$. Then by Theorem 6.1, $\tilde{A}$ generates a semigroup $\{T(t)\}$ of contractions on $D(\tilde{A})$. Also, from Proposition 4.2 it follows that $D(\tilde{A}) = D(A)$ and $\tilde{A}^0 = A^0$. Hence, for $x \in D(A)$, $|||\tilde{A}T(t)x||| = |||AT(t)x|||$ for $t \geqq 0$ and $(d/dt)T(t)x = A^0 T(t)x$ almost everywhere. Therefore, we can say that $\{T(t)\}$ is the semigroup on $D(A)$ which is determined by the $A$. For this semigroup, we have the following

LEMMA 6.3. *For each $x \in D(A)$, we have:*

(a) *$|||AT(t)x|||$ is of bounded variation on every finite interval of the form $[0, T]$ and has no positive jumps;*

(b) *the right derivative $D^+T(t)x$ exists and strongly right-continuous in $t$, and $D^+T(t)x = A^0 T(t)x$ for all $t \geqq 0$;*

(c) *$A^0 T(t)x$ is strongly continuous except possibly at a countable number of points $t$.*

PROOF. (a) Let $\tilde{A}$ be a maximal dissipative extension of $A$ in $\overline{D(A)}$. Let

$x \in D(A)$ and $0 \leqq r < t$. Then by the same argument as in Kato [**11**; Lemma 6.6] we see that $|||AT(t)x||| \leqq |||AT(r)x|||$.

(b) By Proposition 4.2, $\tilde{A}^0$ $(=A^0)$ is a single-valued operator with $D(\tilde{A}^0) = D(A)$. Fix any $x \in D(A)$ and $t \geqq 0$. And choose a sequence $t_k \downarrow t$. Then by the proof of Kato [**11**; Theorem 7.5] we see that $\{\tilde{A}^0 T(t_k)x\}$ contains a subsequence which converges strongly to $\tilde{A}^0 T(t)x$. So, $\tilde{A}^0 T(t)x$ is strongly right-continuous in $t$. But, since $T(t)x - x = \int_0^t \tilde{A}^0 T(s)x\,ds$ by Theorem 6.2, it follows that $D^+T(t)x = \tilde{A}^0 T(t)x = A^0 T(t)x$ for each $t$.

(c) By (a), $\|\tilde{A}^0 T(t)x\| = |||AT(t)x|||$ is continuous except for a countable number of points $t$. In order to prove that $\tilde{A}^0 T(t)x$ is continuous except for those points, it suffices to repeat the same argument as in (b) with $t_k \uparrow t$.

Q. E. D.

Hence, we obtain the following result.

THEOREM 6.3. *Assume that $X$ is uniformly convex. Let $A$ be a closed dissipative operator satisfying (R). Then $A^0$ is the infinitesimal generator of a unique semigroup $\{T(t)\}$ of contractions on $D(A)$ such that for each $x \in D(A)$, $D^+T(t)x = A^0 T(t)x$ for all $t \geqq 0$ and $D^+T(t)x$ is strongly right-continuous in $t \geqq 0$.*

REMARK 6.3. Let $X$ be a Hilbert space. Then there are the following remarkable results (see Crandall and Pazy [**2, 3**], and Kōmura [**9**]):

(I) If $A$ is maximal dissipative in $X$, then $\overline{D(A)}$ is convex.

(II) For any dissipative operator $A$ in $X$ there is a unique $m$-dissipative extension $\tilde{A}$ of $A$ such that $D(\tilde{A}) \subset \overline{co}\, D(A)$.

In view of these results, we see that Theorem 6.3 gives a result of Brezis and Pazy [**1**; Theorem 2.1].

Finally, we present some variations of Theorem 6.3 in terms of pseudo-resolvent:

COROLLARY 6.1. *Assume that $X$ is uniformly convex. Let $S$ be a closed subset of $X$ and $\{I_\lambda;\ \lambda > 0\}$ be a pseudo-resolvent of contractions from $S$ into itself. Then there are a closed dissipative operator $A$ defined on $D \equiv R(I_\lambda)$ and a semigroup $\{T(t)\}$ on $D$, such that $I_\lambda = (I - \lambda A)^{-1}$ for $\lambda > 0$ and $A^0$ is the infinitesimal generator of $\{T(t)\}$.*

PROOF. Let $A$ be the closed dissipative operator obtained by Proposition 3.2. Then it satisfies the conditions of Theorem 6.3.

COROLLARY 6.2. *Let $K \subset X$ be convex, and $\{I_\lambda;\ \lambda > 0\}$ be a family of single-valued operators satisfying all of the assumptions of Corollary* 5.2. *Then there is a closed dissipative operator $A$ such that $\overline{D(A)} = \overline{\bigcup_{\lambda>0} I_\lambda K}$, $I_\lambda x = (I - \lambda A)^{-1} x$ for $\lambda > 0$ and $x \in K$, and such that $A^0$ is the infinitesimal generator of a unique semigroup on $D(A)$.*

PROOF. Let $A$ be the closure of the dissipative operator obtained by

Corollary 3.1. Then it satisfies the assumptions of Theorem 6.3. Q. E. D.

## § 7. Semigroups of Differentiable Operators

The purpose of this section is to construct the semigroups of differentiable operators.

Let $S \subset X$ and $G$ be a single-valued operator in $X$ such that $D(G) \supset S$ and there is an $L(X, X)$-valued continuous function $dG(\cdot)$ on $S$ such that

$$\|G(x+y)-Gx-dG(x)y\| = o(\|y\|)$$

for $x \in S$ and $y \in X$ with $x+y \in S$. Here $L(X, X)$ denotes the Banach algebra of endomorphisms of $X$. Then we denote the family of such operators by $F(S)$. If $S$ is open, then $F(S)$ is the family of operators which are continuously Fréchet ($F$-) differentiable in $S$. For the notions of $F$-differentiable operators, Gateaux ($G$-) differentiable operators, and analytic operators, we refer to Hille-Phillips [**7**; Chap. III] and J. T. Schwartz [**19**]. (By an analytic operator we mean an analytic function on vectors to vectors.)

In the remainder of this section we need to assume that $X$ is a *complex* Banach space. Our result is the following

THEOREM 7.1. *Let $A$ be a single-valued, closed dissipative operator satisfying* ($R$). *Assume that $\overline{D(A)}$ is the closure of an open convex set $D$ (hence* ($R_{co}$) *is satisfied) and that $(I-\lambda A)^{-1}$ is $G$-differentiable in $D$ for every $\lambda > 0$. Let $\{T(t)\}$ be the semigroup on $\overline{D(A)}$ obtained by Theorem* 5.1. *Then we have:*

(a) *for each $t > 0$, $T(t)$ is analytic in $D$ and $T(t) \in F(D)$;*

(b) *for each $x \in D \cap D(A)$, $AT(t)x$ is strongly continuous in $t \geqq 0$ and $(d/dt)T(t)x = AT(t)x = dT(t)(x)Ax$ for all $t \geqq 0$.*

Before proving this theorem, we state some remarks and give an example.

REMARK 7.1. Let $\{T(t)\}$ be a $(C_0)$-semigroup of bounded linear operators and $A$ be the infinitesimal generator. Then for each $x \in D(A)$, $(d/dt)T(t)x = AT(t)x = T(t)Ax$ for $t \geqq 0$. The assertion (b) corresponds to this property and the nonlinearity appears in the term $dT(t)(x)$.

REMARK 7.2. If a single-valued operator $G$ on a complex Banach space $X$ is uniformly Lipschitz continuous and $G$-differentiable on $X$, then by virtue of the Liouville's theorem for vector-valued analytic functions $G$ becomes a combination of a linear operator and a translation by a constant vector. Now, assume that $A$ is a densely defined, $m$-dissipative operator in $X$ and $(I-\lambda A)^{-1}$ is $G$-differentiable in $X$ for $\lambda > 0$. Then $A$ determines a semigroup $\{T(t)\}$ of contractions on $X$, and Theorem 7.1 states that each $T(t)$ is an analytic operator on $X$. Therefore, it is proved that for each $t \geqq 0$, $x \to S(t)x=$

$T(t)x-T(t)0$ is a linear operator on $X$ (see Kōmura [**9**; § 4, p. 397]). Since $\|S(t)x\| \leqq \|x\|$ and $S(t)S(s)x=S(t)T(s)x-S(t)T(s)0=T(t+s)x-T(t+s)0=S(t+s)x$ for $t$, $s\geqq 0$ and $x\in X$, the family $\{S(t); t\geqq 0\}$ forms a $(C_0)$-semigroup of linear contractions on $X$. Therefore $\{T(t)\}$ is an *affine* semigroup (see Crandall and Pazy [**2**; § 5]).

EXAMPLE. Let $X=C^1$ and let us consider an operator $A$ defined by $Az=z^2$, $z\in D(A)=\{z\in C^1; \operatorname{re} z\leqq -|\operatorname{im} z|\}$. Then the interior $D(A)^0=\{z\in C^1; \operatorname{re} z<-|\operatorname{im} z|\}$ is convex. Since $\operatorname{re}(z_1^2-z_2^2)(\overline{z_1-z_2})=|z_1-z_2|^2 \operatorname{re}(z_1+z_2)\leqq 0$ for $z_1$, $z_2\in D(A)$, $A$ is dissipative. And we see that $R_\lambda\supset D(A)$ for every $\lambda>0$. Since $z\to z-\lambda z^2$ is analytic on $D(A)^0$, it follows from the implicit function theorem that $(I-\lambda A)^{-1}$ is analytic in $D(A)^0$.

In the remainder of this section, let $A$ be a single-valued, closed dissipative operator satisfying all of the assumptions of Theorem 7.1. Let $U_\eta=A_\eta|_{\overline{D(A)}}$ for $\eta>0$, and $\{T_\eta(t)\}$ be the semigroup on $\overline{D(A)}$ determined by $U_\eta$ in the sense of Corollary 5.3. And, taking a sequence $\eta_m\downarrow 0$, we write $U_m$ for $U_{\eta_m}$ and $\{T_m(t)\}$ for $\{T_{\eta_m}(t)\}$.

For the proof of Theorem 7.1 we need the theory of analytic functions on vectors to vectors. Let $G$ be an operator in $X$, which is $G$-differentiable in a finitely open set $D$, and if $x\in D$ we may define the $n$-th variation $\delta^n G(x; v)$ of $G$ at $x$ with increment $v$ as

$$\delta^n G(x; v)=\left[\frac{d^n}{d\xi^n}G(x+\xi v)\right]_{\xi=0}.$$

The key fact is the following theorem (see Hille-Phillips [**7**; Theorem 3.18.1]).

THEOREM A. *Let $\{G_n\}$ be a sequence of operators in $X$ analytic and locally uniformly bounded in a fixed domain $D$. If $Gx=\lim G_n x$ exists in $D$, then the limit operator $G$ is analytic in $D$. Furthermore, $\delta^k G(x; v)=\lim \delta^k G_n(x; v)$ for each $k$, $x\in D$ and $v\in X$.*

LEMMA 7.1. (a) *For each $\lambda>0$, $J_\lambda=(I-\lambda A)^{-1}$ is analytic in $D$ and for $x\in D\cap D(A)$ and $y\in X$, $\lim_{\lambda\to+0} dJ_\lambda(x)y=y$.*

(b) *For each $x\in D\cap D(A)$, $\lim_{\eta\to+0} U_\eta x=Ax$.*

PROOF. Since each $J_\lambda$ is locally bounded in D, it is analytic in $D$. Since $\lim_{\lambda\to+0} J_\lambda x=x$ for $x\in D(A)$, Theorem A implies that $dJ_\lambda(x)y=\delta J_\lambda(x; y)\to y$ strongly as $\lambda\to+0$, for $x\in D\cap D(A)$ and $y\in X$.

(b) Fix an $x\in D\cap D(A)$. Then $x-\eta Ax\in D$ for $\eta>0$ sufficiently small. Since $U_\eta x=\eta^{-1}[J_\eta x-J_\eta(x-\eta Ax)]$, it follows from the argument in Hille-Phillips [**7**; Theorem 3.17.1, p. 113] that $\|U_\eta x-dJ_\eta(x)Ax\|=\eta^{-1}o(\|Ax\|\eta)$ as $\eta\to+0$. Therefore, combining with (a) we have the assertion. Q. E. D.

LEMMA 7.2. (a) *For each $t\geqq 0$, $T_m(t)$ and $T(t)$ are analytic in $D$ (hence $\in F(D)$), and $\lim dT_m(t)(x)y=dT(t)(x)y$ for $x\in D$ and $y\in X$.*

(b) *For each* $x\in D$, $y\in X$, $dT(t)(x)y$ *is strongly continuous in* $t$.

PROOF. (a) For the proof of (a) it suffices to show that $T_m(t)$ are analytic and locally uniformly bounded in $D$. For if so, then Corollary 5.3 (b) and Theorem A would yield that $T(t)$ is analytic in $D$ (hence $T(t)\in F(D)$) and $dT_m(t)(x)y=\delta T_m(t)(x;y)\to\delta T(t)(x;y)=dT(t)(x)y$ strongly as $m\to\infty$ for $x\in D$ and $y\in X$. Now, fix an $\eta>0$. Then (4.3) and Lemma 7.1 (a) imply that $(I-hU_\eta)^{-1}$ maps $D$ into itself and is analytic in $D$ (and hence $\in F(D)$). Thus $(I-hU_\eta)^{-n}\in F(D)$ for $h>0$ and $n$, by the chain rule of $F$-differentiability. On the other hand, (4.5) implies that for any $T>0$, the contractions $\{(I-hU_\eta)^{-n};\ h,\ \eta>0,\ nh\in[0,T]\}$ are locally uniformly bounded in $D$. Hence, by Corollary 5.3 (a) and Theorem A we see that $T_m(t)$ are analytic and locally uniformly bounded in $D$.

(b) Fix any $x\in D$ and $y\in X$ with $\|y\|=1$. Let $\rho(x;y)$ be the supremum of all numbers $\rho$ such that $|\xi|\leqq\rho$ implies that $x+\xi y\in D$, and $C$ be any circle $|\xi|=\rho'<\rho(x;y)$. Then by the Cauchy's integral formula [**19**; p. 111],

$$dT(t)(x)y=\frac{1}{2\pi i}\int_C\xi^{-2}T(t)(x+\xi y)d\xi\,,\qquad \text{for } t\geqq 0\,.$$

Now, let $0\leqq t,\ t'\leqq T$, then we have that

$$dT(t')(x)y-dT(t)(x)y=\frac{1}{2\pi i}\int_C\xi^{-2}\{T(t')(x+\xi y)-T(t)(x+\xi y)\}d\xi\,.$$

Since the integrand is uniformly bounded with respect to $\xi\in C$ and $t,\ t'\in[0,T]$, and since the convergence $\lim_{t'\to t}T(t')(x+\xi y)=T(t)(x+\xi y)$ holds uniformly for $\xi\in C$, it follows that $\lim_{t'\to t}dT(t')(x)y=dT(t)(x)y$. Q. E. D.

PROOF OF THEOREM 7.1. Fix an $m$ and let $t\geqq 0$ and $x\in D$. Since $T_m(t)\in F(D)$ by Lemma 7.2 (a), we have that

$$\begin{aligned}h^{-1}\{T_m(t+h)x-T_m(t)x\}&=h^{-1}\{T_m(t)[x+(T_m(h)x-x)]-T_m(t)x\}\\&=dT_m(t)(x)[h^{-1}(T_m(h)x-x)]+h^{-1}o(\|T_m(h)x-x\|)\quad\text{as } h\to+0\,.\end{aligned}$$

Since $D\subset D(U_m)$, passing to the limit as $h\to+0$, $(d/dt)T_m(t)x=U_mT_m(t)x=dT_m(t)(x)U_mx$. The derivative $(d/dt)T_m(t)x$ is strongly continuous in $t$, and so, we have that

$$T_m(t)x-x=\int_0^t dT_m(s)(x)U_mx\,ds\,,\qquad\text{for } t\geqq 0 \text{ and } x\in D\,. \tag{7.1}$$

On the other hand, $\|T_m(t)(x+y)-T_m(t)x-dT_m(t)(x)y\|=o(\|y\|)$ as $\|y\|\to 0$, and hence $\|dT_m(t)(x)(y/\|y\|)\|\leqq\|T_m(t)(x+y)-T_m(t)x\|/\|y\|+o(\|y\|)/\|y\|\leqq 1+o(\|y\|)/\|y\|$. This means that $\|dT_m(t)(x)\|\leqq 1$ for $t\geqq 0$ and $x\in D$. Thus, the set $\{dT_m(s)(x)U_mx;\ s\geqq 0,\ m\}$ is bounded for $x\in D\cap D(A)$. Hence, Lemmas 7.1 (b) and 7.2 (a) imply that

(7.2) $$\lim_{m\to\infty} dT_m(s)(x)U_m x = dT(s)(x)Ax .$$

By Lemma 7.2 (b), $dT(s)(x)Ax$ is strongly continuous in $s \geqq 0$. Applying the dominated convergence theorem to (7.1), we have

$$T(t)x - x = \int_0^t dT(s)(x)Ax\, ds \qquad \text{for } t \geqq 0 \text{ and } x \in D \cap D(A) .$$

Finally, we show that $dT(t)(x)Ax = AT(t)x$ for $t \geqq 0$. By (7.2), $U_m T_m(t)x = AJ_{\eta_m} T_m(t)x \to dT(t)(x)Ax$ strongly for each $t \geqq 0$. Since $\lim_{m\to\infty} J_{\eta_m} T_m(t)x = T(t)x$ and $A$ is closed, it follows that $T(t)x \in D(A)$ and $dT(t)(x)Ax = AT(t)x$. Q.E.D.

Waseda University

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