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On complex hypersurfaces of the complex projective space II

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§ 1. Introduction

Let $P_{n+1}(C)$ be the complex projective space of complex dimension $n+1$ with the Fubini-Study metric of constant holomorphic sectional curvature 1 and let M be a complex hypersurface of $P_{n+1}(C)$ with the induced Kaehler structure. The purpose of this paper is to prove the following theorem.

THEOREM. *Let M be a complete complex hypersurface of $P_{n+1}(C)$. If $n \geqq 2$ and if every sectional curvature of M is greater than* 1/8, *then M is a complex hyperplane $P_n(C)$.*

Postponing the proof of the theorem to the following section, we shall list here results in the same direction. For the sake of simplicity, we shall adopt the following notations: for example,

$K > \delta$: every sectional curvature of M is greater than δ,

$H > \delta$: every holomorphic sectional curvature of M is greater than δ.

A. *If M is complete and if $K \geqq \frac{1}{4}$, then $M = P_n(C)$ provided $n \geqq 2$.*

In a recent paper ([**5**]), K. Nomizu proved (A) in case of $n \geqq 3$. But (A) is an immediate consequence of the following well known results ([**1**], [**2**], [**7**]):

(a) $H \leqq 1$ for a complex hypersurface of $P_{n+1}(C)$.

(b) If $H \geqq 0$, then a maximum curvature is holomorphic.

(c) If $n \geqq 2$ and if $\delta \leqq K \leqq 1$, then $\frac{\delta(8\delta+1)}{1-\delta} \leqq H$.

The assumption of (A), together with (a) and (b), implies $\frac{1}{4} \leqq K \leqq 1$ and hence (a) and (c) imply $H=1$ so that $M = P_n(C)$.

In [**6**] we proved

B. *If M is complete and if $H > \frac{1}{2}$, then $M = P_n(C)$.*

Let $z_0, z_1, \cdots, z_{n+1}$ be a homogeneous coordinate system of $P_{n+1}(C)$ and let $Q_n(C) = \{(z_0, \cdots, z_{n+1}) \in P_{n+1}(C) \mid \sum z_i^2 = 0\}$. Then it is known that $\frac{1}{2} \leqq H \leqq 1$

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and $0 \leqq K \leqq 1$ for $Q_n(C)$ $(n \geqq 2)$ and $H = K = \frac{1}{2}$ for $Q_1(C)$. Hence (B) can not be improved.

Combining (B) and (c), we have an improvement of (A).

C. *If M is complete and if $K > \frac{\sqrt{73}-3}{32}$, then $M = P_n(C)$ provided $n \geqq 2$.*

Clearly our theorem is an improvement of (C). We have the following conjecture.

D. *If M is complete and if $K > 0$, then $M = P_n(C)$ provided $n \geqq 2$.*

§ 2. Proof of theorem

Let M be a complete complex hypersurface of $P_{n+1}(C)$ with the induced metric $g = 2\sum g_{\alpha\bar{\beta}} dz^\alpha d\bar{z}^\beta$ and the fundamental 2-form $\Phi = \frac{2}{\sqrt{-1}} \sum g_{\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta$. Let $S = 2\sum R_{\alpha\bar{\beta}} dz^\alpha d\bar{z}^\beta$ be the Ricci tensor of M.

PROPOSITION 1. *Let M be a complete complex hypersurface of $P_{n+1}(C)$. If $S - \frac{n}{2} g$ is positive definite, then $M = P_n(C)$.*

PROOF. Since $S - \frac{n}{2} g$ is positive definite, a theorem of Myers ([4]) implies M is compact. Hence, by a well known theorem of Chow, M is algebraic.

The first Chern class $c_1(M)$ of M is represented by the closed 2-form

$$\gamma = \frac{1}{2\pi\sqrt{-1}} \sum R_{\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta .$$

We denote $[\Phi]$ and $[\gamma]$ the cohomology classes represented by Φ and γ, respectively, so that $c_1(M) = [\gamma]$.

Let h be the generator of $H^2(P_{n+1}(C), Z)$ corresponding to the divisor class of a hyperplane $P_n(C)$. Then the first Chern class $c_1(P_{n+1}(C))$ of $P_{n+1}(C)$ is given by

$$c_1(P_{n+1}(C)) = (n+2)h .$$

Let $j: M \to P_{n+1}(C)$ be the imbedding and let $\tilde{h}$ be the image of h under the homomorphism $j^*: H^2(P_{n+1}(C), Z) \to H^2(M, Z)$. Let d be the degree of the algebraic manifold M. Then we have

$$c_1(M) = (n-d+2)\tilde{h} . \tag{1}$$

Let Ψ be the fundamental 2-form of $P_{n+1}(C)$ so that

$$c_1(P_{n+1}(C)) = \frac{n+2}{8\pi} [\Psi] .$$

These, together with the fact that $\Phi = j^*\Psi$, imply

$$[\Phi] = 8\pi\tilde{h} . \tag{2}$$

Since $S-\frac{n}{2}g$ is positive definite, $c_1(M)-\frac{n}{8\pi}[\Phi]$ is positive definite. This, together with (1) and (2), implies that $\frac{n-d+2}{8\pi}[\Phi]-\frac{n}{8\pi}[\Phi]$ is positive definite. Hence we have $d<2$, that is, $d=1$. (Q. E. D.)

Let A be the tensor field of type (1, 1) associated to the second fundamental form of the imbedding. Let J be the complex structure of M and let $e_1, \cdots, e_n, Je_1, \cdots, Je_n$ be an orthonormal basis of $T_x(M)$ with respect to which the matrix of A is of the form

$$\begin{pmatrix} \lambda_1 & & & & & \\ & \ddots & & & 0 & \\ & & \lambda_n & & & \\ & & & -\lambda_1 & & \\ & 0 & & & \ddots & \\ & & & & & -\lambda_n \end{pmatrix}.$$

The eigenvalues $\lambda_1, \cdots, \lambda_n, -\lambda_1, \cdots, -\lambda_n$ of A are called the principal curvature.

PROPOSITION 2. *Let M be a complex hypersurface of $P_{n+1}(C)$. If every principal curvature lies in the interval $\left(-\frac{1}{2}, \frac{1}{2}\right)$, then $S-\frac{n}{2}g$ is positive definite.*

PROOF. Let R be the curvature tensor of M. Then the equation of Gauss is

$$\begin{aligned} g(R(X, Y)Z, W) = & g(AX, W)g(AY, Z)-g(AX, Z)g(AY, W) \\ & +g(JAX, W)g(JAY, Z)-g(JAX, Z)g(JAY, W) \\ & +\frac{1}{4}\{g(X, W)g(Y, Z)-g(X, Z)g(Y, W) \\ & +g(JX, W)g(JY, Z)-g(JX, Z)g(JY, W) \\ & +2g(X, JY)g(JZ, W)\}. \end{aligned}$$

It follows immediately that

$$S(X, Y)=\frac{n+1}{2}g(X, Y)-2g(AX, AY).$$

Let $X=\sum X^\alpha e_\alpha+\sum X^{\alpha^*}Je_\alpha$. Then we have

$$S(X, X)=\frac{n+1}{2}g(X, X)-2\sum \lambda_\alpha^2(X^\alpha X^\alpha+X^{\alpha^*}X^{\alpha^*}).$$

Since $\lambda_\alpha^2<\frac{1}{4}$ for $\alpha=1, \cdots, n$, we have $S(X, X)>\frac{n}{2}g(X, X)$, that is, $S-\frac{n}{2}g$ is positive definite. (Q. E. D.)

Let $K(X, Y)$ be the sectional curvature of M determined by two vectors X and Y. Then we have, for $\alpha \neq \beta$,

$$K(e_\alpha+e_\beta, Je_\alpha-Je_\beta)=\frac{1}{4}-\frac{\lambda_\alpha^2+\lambda_\beta^2}{2}.$$

Since every sectional curvature is greater than 1/8, we have $\lambda_\alpha^2+\lambda_\beta^2<\frac{1}{4}$ so that $\lambda_\alpha^2<\frac{1}{4}$ for $\alpha=1, \cdots, n$. This, together with Propositions 1 and 2, implies $M=P_n(\boldsymbol{C})$.

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