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Differential equations on convex sets

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Introduction

Recent developments in the theory of semi-groups of nonlinear transformations in Banach or Hilbert spaces have sharply brought into focus the fact that these theories must be developed for semi-groups on convex sets in order to achieve their full scope. Motivated by the results of [**1**], [**6**] and [**10**], the purpose of this note is to establish existence of solutions of a Cauchy problem of the form

$$\frac{du}{dt}=g(u, t), \qquad u(0)=x, \tag{1}$$

where the function g is only defined on a set of the form $C\times[0, a]$ for some convex set C in a Banach space. The methods used are not new (see, e.g., [**3**], [**8**]), but the main result seems to have gone unnoticed and serves to clarify some of the theory of semi-groups of nonlinear transformations and the related theory of accretive mappings in Banach spaces.

Simple (but basic) existence theorems for (1) are established in Section 1. Section 2 contains applications of these results to the theory of nonlinear pseudo-contractive and accretive operators. For aesthetic reasons, applications to the semi-group theory (where one must deal with "multi-valued" mappings) are not given here.

§ 1. Existence and Uniqueness

The main topic of this section is existence. Uniqueness is established only in simple cases of interest. Let X be a real Banach space and C be a closed convex subset of X. We begin by establishing a local existence theorem of some generality. Denote by $B_r(x)$ the closed ball of radius r in X centered at x. Consider the set

$$K=(B_r(x)\cap C)\times[0, a]. \tag{1.1}$$

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The Cauchy problem to be treated has the form

$$\frac{du}{dt} = \lambda(u, t)(f(u, t)-u), \qquad u(0) = x \in C, \tag{1.2}$$

where λ is a continuous positive-valued function on K and $f: K \to C$.

In order to describe appropriate assumptions, we need a few definitions. First, a function mapping a subset of a Banach space Z into a Banach space W is called *demicontinuous* if it is continuous from the (relative) strong topology on its domain into the weak topology of W. Next, let X^* be the dual space of X. Set

$$\langle x, y\rangle = \inf\{\xi^*(x) : \xi^* \in X^* \quad \text{and} \quad \xi^*(y) = \|y\|^2 = \|\xi^*\|^2\}. \tag{1.3}$$

In (1.3), "inf" can be replaced by "min." The following properties of $\langle\, ,\rangle$ will be used freely in the sequel: (a) $\langle \alpha y + x, y\rangle = \alpha\|y\|^2 + \langle x, y\rangle$, (b) $\langle \beta x, \gamma y\rangle = \gamma\beta\langle x, y\rangle$ for $\gamma\beta \geqq 0$, and (c) $\langle z+x, y\rangle \leqq \|z\|\|y\| + \langle x, y\rangle$. The first result is:

THEOREM I. *Let λ be continuous and f demicontinuous on K. Assume there are constants c_2, $c_1 > 0$, L, M and a function $g: [0, \infty) \to [0, \infty]$ satisfying*

$$g \text{ is monotone increasing, } \lim_{s \downarrow 0} g(s) = 0, \tag{1.4}$$

such that

$$c_2 \geqq \lambda(w, h) \geqq c_1 > 0 \qquad \text{for } (w, h) \in K \tag{1.5}$$

$$\|f(w, h)\| \leqq M \qquad \text{for } (w, h) \in K \tag{1.6}$$

and

$$\langle f(z, \alpha) - f(y, \beta), u-v\rangle \leqq g(|\alpha-\beta| + \|z-u\| + \|y-v\|) + L\|u-v\|^2 \tag{1.7}$$

whenever (z, α), $(y, \beta) \in K$, $u, v \in B_r(x) \cap C$. Then (1.2) *has a strongly continuous, weakly continuously differentiable solution u on $[0, b]$ where*

$$b = c_2^{-1} \min\left(\frac{r}{\|x\|+M}, ac_1\right).$$

REMARK. Whenever we speak of a solution of an equation of the form (1), we will mean a strongly continuous, weakly continuously differentiable function u on some interval $[0, b]$ (or $[0, b)$). The nature of (1.7) is dictated by the desire to cover the cases (i) f is Lipschitz continuous, (ii) f is pseudo-contractive and locally uniformly continuous, (iii) f is pseudo-contractive, demicontinuous, and X^* is uniformly convex, under one assumption. See Section 2, especially Lemma 2.6.

PROOF OF THEOREM I. We begin by rewriting (1.2) as the system

$$(1.8)\qquad \begin{cases} \text{(a)} \quad \dfrac{du}{d\tau} = (f(u,t)-u)\,, & u(0)=x \\ \text{(b)} \quad \dfrac{dt}{d\tau} = (\lambda(u,t))^{-1}\,, & t(0)=0 \end{cases}$$

and then rewrite (1.8) in integral form

$$(1.9)\qquad \begin{cases} u(\tau) = xe^{-\tau} + \displaystyle\int_0^\tau e^{(\alpha-\tau)} f(u(\alpha), t(\alpha))d\alpha \\ t(\tau) = \displaystyle\int_0^\tau \frac{1}{\lambda(u(\alpha), t(\alpha))} d\alpha\,. \end{cases}$$

The systems (1.8) and (1.9) are equivalent for strongly continuous, weakly continuously differentiable functions. Moreover, (1.8) and (1.2) are equivalent. In particular, if $u(\tau)$, $t(\tau)$ is a solution pair of (1.8), then $t=t(\tau)$ may be inverted to obtain $\tau=\tau(t)$ and $u(\tau(t))$ is a solution of (1.2).

The main tool of our proof is the following simple lemma.

LEMMA 1.10. *Set* $T=\text{Min}\left(\frac{r}{\|x\|+M}, ac_1\right)$. *Let* $w:[0,\gamma]\to B_r(x)\cap C$ *and* $h:[0,\gamma]\to[0,a]$, *where* $\gamma\leqq T$, *be continuous. Then*

$$u(\tau) = xe^{-\tau} + \int_0^\tau e^{(\alpha-\tau)} f(w(\alpha), h(\alpha))d\alpha$$

$$t(\tau) = \int_0^\tau \frac{1}{\lambda(w(\alpha), h(\alpha))} d\alpha$$

are defined on $[0,\gamma]$, $u:[0,\gamma]\to B_r(x)\cap C$, $t:[0,\gamma]\to[0,a]$. *Moreover,*

$$\|u'(\tau)\| \leqq \|x\|+M+MT$$

$$|t'(\tau)| \leqq 1/c_1$$

for $0\leqq\tau\leqq\gamma$.

PROOF. The assertions concerning t follow at once from (1.5) and $\gamma\leqq T\leqq ac_1$. Concerning u, we first observe

$$f(w(\alpha), h(\alpha)) \in C$$

implies

$$\frac{\displaystyle\int_0^\tau e^{(\alpha-\tau)} f(w(\alpha), h(\alpha))d\alpha}{\displaystyle\int_0^\tau e^{(\alpha-\tau)} d\tau}$$

lies in the weak convex closure of C, which is C. Now $\int_0^\tau e^{(\alpha-\tau)}d\alpha = 1-e^{-\tau}$, so

$$u(\tau) = xe^{-\tau} + (1-e^{-\tau}) \frac{\displaystyle\int_0^\tau e^{(\alpha-\tau)} f(w(\alpha), h(\alpha))d\alpha}{1-\exp(-\tau)}$$

is a convex combination of elements of C and thus lies in C. Moreover, since $(w(\alpha), h(\alpha)) \in K$ for $0 \leqq \alpha \leqq \gamma$, we have

$$\|u(\tau)-x\| \leqq (1-e^{-\tau})\|x\| + \int_0^\tau |e^{(\alpha-\tau)}| \, \|f(w(\alpha), h(\alpha))\| d\alpha$$
$$\leqq (1-e^{-\tau})(\|x\|+M),$$

by (1.6), so

$$\|u(\tau)-x\| \leqq (1-e^{-\tau})(\|x\|+M) \leqq \tau(\|x\|+M)$$
$$\leqq T(\|x\|+M) \leqq r.$$

The simple but important fact that $u(\tau) \in C$ seems to have been missed by other authors, and is the main novelty of our proof. The estimate on $\|u'(\tau)\|$ is trivial.

To complete the proof of Theorem I we set up the following scheme: Let n be a positive integer and define functions u_n, t_n on $[-T/n, T]$ by

(1.11) (a) $$\begin{cases} t_n(\tau) = 0 & \text{for } -1/n \leqq \tau \leqq 0 \\ u_n(\tau) = x & \text{for } -1/n \leqq \tau \leqq 0 \end{cases}$$

(b) $$\begin{cases} t_n(\tau) = \int_0^\tau \lambda(u_n(\alpha-\varepsilon_n), t_n(\alpha-\varepsilon_n))^{-1} d\alpha & 0 \leqq \tau \leqq T \\ u_n(\tau) = xe^{-\tau} + \int_0^\tau e^{(\alpha-\tau)} f(u_n(\alpha-\varepsilon_n), t_n(\alpha-\varepsilon_n)) d\alpha & 0 \leqq \tau \leqq T \end{cases}$$

with $\varepsilon_n = T/n$. The implications of Lemma 1.10 here are that u_n, t_n are defined on $[0, T]$ by (1.11) (b) and take values in $B_r(x) \cap C$ and $[0, a]$, respectively. Moreover,

(1.12) $$\|u_n'(\tau)\| \leqq R, \qquad \|t_n'(\tau)\| \leqq R$$

for $R = \max(\|x\|+M+MT, c_1^{-1})$. The sequence $\{t_n\}_1^\infty$ is uniformly bounded and equicontinuous on $[0, T]$. Thus it has a uniformly convergent subsequence $\{t_{n(j)}\}$. Define $w_{jk}(\tau) = \|e^\tau(u_{n(j)}(\tau) - u_{n(k)}(\tau))\|^2$. By [8, Lemma 1.3] (which we use hereafter without comment), the Lipschitz continuity of w_{jk}, and (1.7), we have

(1.13) $$\frac{d}{d\tau} w_{jk}(\tau) = 2e^{2\tau} \langle f(u_{n(j)}(\tau-\varepsilon_{n(j)}), t_{n(j)}(\tau-\varepsilon_{n(j)})) - f(u_{n(k)}(\tau-\varepsilon_{n(k)}), t_{n(k)}(\tau-\varepsilon_{n(k)})), u_{n(j)}(\tau)-u_{n(k)}(\tau) \rangle$$
$$\leqq 2e^{2\tau} g(r_{jk}(\tau)) + 2Lw_{jk}(\tau)$$

for almost all τ, $0 \leqq \tau \leqq T$, where

$$r_{jk}(\tau) = |t_{n(j)}(\tau-\varepsilon_{n(j)}) - t_{n(k)}(\tau-\varepsilon_{n(k)})|$$
$$+ \|u_{n(j)}(\tau-\varepsilon_{n(j)}) - u_{n(j)}(\tau)\|$$
$$+ \|u_{n(k)}(\tau-\varepsilon_{n(k)}) - u_{n(k)}(\tau)\|.$$

In view of (1.12) and the uniform convergence of $\{t_{n(j)}\}$,

$$\lim_{j,k\to\infty}\sup_{0\leqq\tau\leqq T} r_{jk}(\tau)=0\,,$$

so (1.13) and (1.4) give

$$\frac{d}{d\tau}w_{jk}\leqq\beta_{jk}+2Lw_{jk}\,,\qquad \text{a. e. in } \tau\,, \tag{1.14}$$

where $\lim_{j,k\to\infty}\beta_{jk}=0$. However, $w_{jk}(0)=0$ and (1.14) imply

$$w_{jk}(\tau)\leqq\Big(\frac{e^{2L\tau}-1}{L}\Big)\beta_{jk}\leqq\Big(\frac{e^{2LT}-1}{L}\Big)\beta_{jk}\,. \tag{1.15}$$

Thus $\lim_{j,k\to\infty}w_{jk}(\tau)=0$ uniformly on $0\leqq\tau\leqq T$, and $\lim_{j\to\infty}u_{n(j)}(\tau)=u(\tau)$ exists uniformly. If $t(\tau)=\lim_{j\to\infty}{}' t_{n(j)}(\tau)$, then clearly $u(\tau)$, $t(\tau)$ are solutions of (1.9) on $[0, T]$. Finally, (1.5) implies

$$t(T)=\int_0^T\frac{1}{\lambda(u(\tau),\, t(\tau))}d\tau\geqq T/c_2$$

so $t=t(\tau)$ can be inverted to find $\tau=\tau(t)$ for $0\leqq t\leqq T/c_2$ at least. The proof is complete.

With regard to uniqueness, we content ourselves with noting two simple facts.

THEOREM 1.16. *Let the assumptions of Theorem I be satisfied with the additional condition that* $\lambda(u, t)=k$ *for some positive constant* k. *Then solutions of* (1.2) *are unique.*

PROOF. The proof is trivial. Let u, v be solutions of (1.2). Then, by (1.7),

$$\frac{d}{dt}\|u-v\|^2=2\langle u'(t)-v'(t),\, u(t)-v(t)\rangle$$
$$\leqq-2k\|u(t)-v(t)\|^2+2Lk\|u(t)-v(t)\|^2,\quad \text{a. e. in } t\,,$$

so $u(0)-v(0)=0$ implies $u(t)-v(t)=0$ for $t\geqq0$.

REMARK. The condition $f: K\to C$ is not required for Theorem 1.16.

COROLLARY 1.17. *Let the assumptions of Theorem I be satisfied and* f *be independent of* t. *If* λ *has the property that*

$$\frac{dt}{d\tau}=(\lambda(u(\tau),\, t))^{-1}\,,\qquad t(0)=0$$

has a unique solution for each continuous $u:[0, T]\to C\cap B_r(x)$, (*e.g., if* λ *is independent of* t) *then the solution of* (1.2) *is* (*locally*) *unique.*

PROOF. As remarked earlier, (1.2) and (1.8) are locally equivalent. By Theorem 1.16, (1.8) (a) has a unique solution if f is independent of t. Then t is obtained uniquely (by assumption) from (1.8) (b). The proof is complete.

Hereafter the discussion is restricted to the autonomous case for clarity. The results developed below are of especial interest for applications.

It is convenient to restate the regularity condition (1.7) in the autonomous case.

DEFINITION 1.18. Let $\Omega \subseteqq X$. $R(\Omega)$ denotes the set of mappings $f: \Omega \to X$ with the following properties: (i) f is demicontinuous, (ii) f is bounded on Ω, and (iii) there is a constant L and a function $g: [0, \infty) \to [0, \infty]$ satisfying (1.4) such that

$$\langle f(z)-f(y), u-v\rangle \leqq g(\|z-u\|+\|y-v\|)+L\|u-v\|^2$$

for all $z, y, u, v \in \Omega$. $R_l(\Omega)$ denotes the set of functions $f: \Omega \to X$ such that each $x \in \Omega$ has a neighborhood $B_r(x)$ for which $f \in R(\Omega \cap B_r(x))$.

THEOREM 1.19. *Let C be a closed convex subset of X and $f \in R_l(C)$. If $f(\partial C) \subseteqq C$, where ∂C is the boundary of C, then for each $x \in C$ there is a positive number a such that*

$$\frac{du}{dt} = f(u)-u, \qquad u(0)=x \tag{1.20}$$

has a solution on $[0, a]$.

PROOF. Assume first that $\partial C = C$. In this case, if $x \in C$ there is a neighborhood $B_r(x)$ such that $f \in R(B_r(x) \cap C)$ and Theorem I applies directly with $\lambda = 1$, $K = (B_r(x) \cap C) \times [0, 1]$. Thus the only nontrivial case is if C has interior, because then we do not require $f(C) \subseteqq C$. Assume $x \in \operatorname{int} C$. Then there is a neighborhood $B_r(x)$ such that $B_r(x) \subset C$ and $f \in R(B_r(x))$. In this situation we may again apply Theorem I directly (by taking X as the set C of Theorem I). The difficult case is if $x \in \partial C$. Again, there is a neighborhood $B_r(x)$ such that $f \in R(C \cap B_r(x))$, and hence there are constants M, L such that

$$\|f(z)\| \leqq M \qquad \text{for } z \in C \cap B_r(x) \tag{1.21}$$

and

$$\langle f(w)-f(z), w-z\rangle \leqq L\|w-z\|^2 \qquad \text{for } w, z \in C \cap B_r(x)\,. \tag{1.22}$$

We will approximate the problem (1.20) by approximating both x and f as follows: Choose $x_0 \in \operatorname{int} C$, $0 < \gamma < 1$, and consider

$$\frac{du}{dt} = \gamma f(u)+(1-\gamma)x_0-u\,, \qquad u(0)=y \tag{1.23}$$

for $y \in \operatorname{int}(C \cap B_r(x))$. As in the case $x \in \operatorname{int} C$ above, (1.23) has a solution $u(t, \gamma, y)$ on some interval $[0, a]$ for $y \in \operatorname{int}(C \cap B_r(x))$. The estimate (1.21) implies

$$\|\gamma f(z)+(1-\gamma)x_0-z\| \leqq M+\|x_0\|+\|x\|+r = M_1 \tag{1.24}$$

for $z \in B_r(x) \cap C$. It follows that if $y \in \operatorname{int}(C \cap B_{r/4}(x))$ and $u(t, \gamma, y)$ is defined on $[0, a]$, $a \leqq r/(4M_1)$, then $u(t, \gamma, y) \in C \cap B_{r/2}(x)$ for $0 \leqq t \leqq a$. Moreover, it follows that if $a < r/(4M_1)$, $y \in \operatorname{int}(C \cap B_{r/4}(x))$, $u(t, \gamma, y)$ is defined on $[0, a]$ and $u(a, \gamma, y) \in \operatorname{int}(C \cap B_r(x))$, then $u(t, \gamma, y)$ may be continued to a solution on $[0, a+\delta]$ for some $\delta > 0$. Hence, if we can show $u(a, \gamma, y) \in \operatorname{int}(C \cap B_r(x))$ whenever $y \in \operatorname{int}(C \cap B_{r/4}(x))$, $a \leqq r/(4M_1)$, and $u(t, \gamma, y)$ is defined on $[0, a]$, it follows in a standard way that $u(t, \gamma, y)$ may be defined on $[0, r/(4M_1)]$. Assume $u(a, \gamma, y) \notin \operatorname{int}(C \cap B_r(x))$, where $u(t, \gamma, y)$ is defined on $[0, a]$, $a \leqq r/(4M_1)$, and $y \in \operatorname{int}(C \cap B_{r/4}(x))$. Then $u(a, \gamma, y) \in \partial(C \cap B_r(x))$ and $u(a, \gamma, y) \in C \cap B_{r/2}(x)$ imply $u(a, \gamma, y) \in \partial C$. Set $z = u(a, \gamma, y)$. Since z does not meet the open convex set $\operatorname{int} C$, there is an element $\xi^* \in X^*$ such that

$$\xi^*(w) < \xi^*(z) \qquad \text{for } w \in \operatorname{int} C. \tag{1.25}$$

Clearly, (1.25) implies

$$\xi^*(w) \leqq \xi^*(z) \qquad \text{for } w \in C. \tag{1.26}$$

Set $v(t) = u(t, \gamma, y)$. We have, for $0 \leqq t < a$,

$$\begin{aligned} z = v(a) &= v(t) + \int_t^a (\gamma f(v(s)) + (1-\gamma)x_0 - v(s))ds \\ &= v(t) + (a-t)[\gamma f(z) + (1-\gamma)x_0 - z] + (a-t)\varepsilon(t), \end{aligned} \tag{1.27}$$

where

$$\varepsilon(t) = (a-t)^{-1} \int_a^t (\gamma(f(v(s)) - f(z)) + (z - v(s)))ds$$

has the property

$$w\text{-}\lim_{t \uparrow a} \varepsilon(t) = 0, \tag{1.28}$$

since v is continuous, f is demicontinuous, and $v(a) = z$. Now choose $w = v(t)$ in (1.26), replace z according to (1.27), divide by $(a-t) > 0$, let t increase to a, and use (1.28) to find $0 \leqq \xi^*(\gamma f(z) + (1-\gamma)x_0 - z)$ or

$$\xi^*(z) \leqq \xi^*(\gamma f(z) + (1-\gamma)x_0). \tag{1.29}$$

However, $z \in \partial C$ implies $f(z) \in C$ (by assumption), so $x_0 \in \operatorname{int} C$, $0 < \gamma < 1$ imply $(\gamma f(z) + (1-\gamma)x_0) \in \operatorname{int} C$. Thus (1.29) and (1.25) are contradictory, and $z \in \partial C$ is not possible. We conclude that if $0 < \gamma < 1$, $y \in \operatorname{int}(C \cap B_{r/4}(x))$, then $u(t, \gamma, y)$ is defined for $0 \leqq t \leqq r/(4M_1)$ and takes values in $\operatorname{int}(C) \cap B_{r/2}(x)$. We next show that if $\{(\gamma_n, y_n)\}$ is a sequence such that $\gamma_n \nearrow 1$, $y_n \in \operatorname{int} C$ and $y_n \to x$, then $u(t) = \lim_{n\to\infty} u(t, \gamma_n, y_n)$ exists uniformly in t on $[0, r/(4M_1)]$, so that $u(t)$ is a solution of (1.20) on $[0, r/(4M_1)]$. By virtue of (1.21), (1.22) and (1.23) one has, for almost all t,

$$\frac{d}{dt}\|u(t,\gamma,w)-u(t,\eta,y)\|^2=-2\|u(t,\gamma,w)-u(t,\eta,y)\|^2$$
$$+2\langle\gamma(f(u(t,\gamma,w))-f(u(t,\eta,y)))+(\gamma-\eta)f(u(t,\eta,y))$$
$$+(\eta-\gamma)x_0,\ u(t,\gamma,w)-u(t,\eta,y)\rangle$$
$$\leqq 2(\gamma L-1)\|u(t,\gamma,w)-u(t,\eta,y)\|^2$$
$$+|\gamma-\eta|(M+\|x_0\|)\|u(t,\gamma,w)-u(t,\eta,y)\|,$$

and it follows that

$$\|u(t,\gamma,w)-u(t,\eta,y)\|\leqq e^{(\gamma L-1)t}\|u(0,\gamma,w)-u(0,\eta,y)\|$$
$$+|\gamma-\eta|(M+\|x_0\|)\frac{(e^{(\gamma L-1)t}-1)}{(\gamma L-1)}$$
$$=e^{(\gamma L-1)t}\|w-y\|+|\gamma-\eta|(M+\|x_0\|)\frac{(e^{(\gamma L-1)t}-1)}{(\gamma L-1)}.$$

Thus $u(t,\gamma,y)$ is Lipschitz continuous on $[0, r/(4M_1)]\times(0,1)\times \operatorname{int}(C\cap B_{r/4}(x))$, and the proof is complete.

REMARK. The above proof, in the case $\operatorname{int} C\neq\emptyset$, required only the case $C=X$ of Theorem I, which is (more or less) known. This proof employed a crucial observation of H. Brezis, which allowed a strengthening of Theorem 1.19 over an earlier version, and for which the author is indebted.

Under various conditions, the local solutions of Theorem 1.19 may be extended to global solutions. The most common and useful cases are covered in the next result.

THEOREM II. *Let C be a closed convex subset of X and $f\in R_l(C)$. Let f satisfy $f(\partial C)\subseteqq C$ and either*

(i) *there is a constant K such that*

$$\langle f(x)-f(y),\ x-y\rangle\leqq K\|x-y\|^2 \qquad \text{for } x,y\in C,$$

or

(ii) *There is a positive-valued continuous function $\psi:[0,\infty)\to[0,\infty)$ such that $\|f(x)\|\leqq\psi(\|x\|)$ and $\int_0^\infty(\psi(s))^{-1}ds$ diverges.*

Then

$$\frac{du}{dt}=f(u)-u, \qquad u(0)=x \tag{1.30}$$

has a unique solution on $[0,\infty)$ for each $x\in C$.

PROOF FOR CASE (i). By Zorn's Lemma, there is a maximal solution u of (1.30) on an interval $[0,b)$, or $[0,b]$ with $b<\infty$, which is not a proper restriction of a solution on another interval with left end point 0. The interval must have the form $[0,b)$, for if u is on $[0,b]$ then Theorem 1.19 allows one to solve

$$\frac{dv}{dt}=f(v)-v\,,\qquad v(0)=u(b)$$

on an interval $[0, a]$, $a>0$, and defining

$$u_e(t)=\begin{cases} u(t) & t\in[0, b] \\ v(t-b) & t\in[b, b+a] \end{cases}$$

extends u to a solution on $[0, b+a]$. If u is defined on $[0, b)$ only, $b<\infty$, we will show $\lim_{t\uparrow b} u(t)$ exists, which trivially allows extension of u to a solution on $[0, b]$, contradicting maximality. Hence $b=\infty$. Consider numbers t, τ, $\delta\geqq 0$ such that $t+\tau+\delta<b$. We have, for almost all t,

$$\begin{aligned}\frac{d}{dt}\|u(t+\tau+\delta)-u(t+\tau)\|^2 &= -2\|u(t+\tau+\delta)-u(t+\tau)\|^2 \\ &\quad +2\langle f(u(t+\tau+\delta))-f(u(t+\tau)),\, u(t+\tau+\delta)-u(t+\tau)\rangle \\ &\leqq 2(K-1)\|u(t+\tau+\delta)-u(t+\tau)\|^2\,,\end{aligned}$$

and hence

$$\|u(t+\tau+\delta)-u(t+\tau)\|\leqq e^{(K-1)t}\|u(\tau+\delta)-u(\tau)\|\,. \tag{1.31}$$

Now u is continuously weakly differentiable, so u is strongly differentiable almost everywhere. Let τ be a point of strong differentiability of u. Then dividing (1.31) by $\delta>0$ and letting $\delta\downarrow 0$ we have

$$\|u'(t+\tau)\|\leqq e^{(K-1)t}\|u'(\tau)\|\,. \tag{1.32}$$

Hence $\max(e^{(K-1)b}, 1)\limsup_{\tau\downarrow 0}\|u'(\tau)\|$ is a Lipschitz constant for u on $[0, b)$ and $\lim_{t\uparrow b} u(t)$ exists. The proof is complete.

The proof for case (ii) of Theorem II is similar, only differing in the proof of the existence of $\lim_{t\uparrow b} u(t)$ if u is defined on $[0, b)$. In this case one obtains a bound on $\|u\|$ first using

$$\begin{aligned}\frac{d}{dt}\|e^t u\| &\leqq \left\|\frac{d}{dt}(e^t u)\right\| \\ &= \|e^t f(u)\|\leqq e^t\phi(e^{-t}e^t\|u\|)\,,\end{aligned}$$

so

$$\frac{d}{dt}v\leqq e^t\phi(e^{-t}v)$$

where $v=\|e^t u\|$, and then a bound on $\frac{du}{dt}$ follows at once. See, e.g., [7, pp. 26-30].

The local uniqueness of solutions of (1.30) follows from Theorem 1.16 and $f\in R_l(C)$, and this extends to global uniqueness in the usual way.

§ 2. Applications to Accretive Operators

Let $\Omega \subseteqq X$. A function $f: \Omega \to X$ is called *pseudo-contractive* in [3] if

$$\|(1+r)(u-v)-r(f(u)-f(v))\| \geqq \|u-v\| \qquad \text{for } r \geqq 0,\ u, v \in \Omega . \tag{2.1}$$

It is known that (2.1) is equivalent to

$$\langle f(u)-f(v), u-v\rangle \leqq \|u-v\|^2 \qquad \text{for } u, v \in \Omega , \tag{2.2}$$

see [6]. In turn, (2.2) is usually described by saying the map $u \to u-f(u)$ is *accretive*, i.e. f is pseudo-contractive if and only if $I-f$ is accretive. In [4] F. Browder proved the following theorems:

THEOREM 2 OF [4]. *Let X be a uniformly convex Banach space, B a closed ball in X, G an open set in X containing B. Let f be a pseudo-contractive mapping of G into X such that f maps the boundary of B into B. Suppose also that f is demicontinuous and either* (a) *f is uniformly continuous in the strong topology on bounded subsets of X, or* (b) *X^* is uniformly convex. Then f has a fixed point in B.*

THEOREM 3 OF [4]. *Let X be a uniformly convex Banach space, C a closed bounded convex subset of X, G an open subset of X which contains C and such that C has positive distance from $X-G$. Suppose f is a nonexpansive mapping of G into X which carries the boundary of C into C. Then f has a fixed point in C.*

The following theorem generalizes these results by eliminating the open set G containing C. Moreover, we remove the restriction that B is a ball in the first result above, and allow C to be unbounded. See [5] for other generalizations.

THEOREM 2.3. *Let C be a closed convex subset of the uniformly convex space X. Let $f \in R_l(C)$ satisfy*

$$\langle f(u)-f(v), u-v\rangle \leqq \|u-v\|^2 \qquad \textit{for } u, v \in C \tag{2.4}$$

and $f(\partial C) \subseteqq C$. If

$$I-f \textit{ is unbounded on each unbounded subset of } C, \tag{2.5}$$

then f has a fixed point in C.

Before proving this theorem, let us show how Theorems 2 and 3 above are subsumed.

LEMMA 2.6. *Let $h: \Omega \to X$ satisfy $\langle h(u)-h(v), u-v\rangle \leqq K\|u-v\|^2$ for $u, v \in \Omega$ and some fixed number K. If h is bounded on Ω and either* (a) *h is uniformly continuous on Ω, or* (b) *X^* is uniformly convex, then there is a number L and a function $g: [0, \infty) \to [0, \infty]$ satisfying $\lim_{r \downarrow 0} g(r) = 0$ such that:*

$$\langle h(z)-h(y),\, u-v\rangle \leqq g(\|z-u\|+\|y-v\|)+L\|u-v\|^2 .$$

PROOF. We prove the lemma under the condition (a). The proof for (b) is similar, and uses the fact that if X^* is uniformly convex, then the map $x\to\xi^*\in X^*$, where $\xi^*(x)=\|x\|^2=\|\xi^*\|^2$, is uniformly continuous on bounded sets. We have

$$\begin{aligned}\langle h(z)-h(y),\, u-v\rangle &= \langle h(z)-h(u)+h(v)-h(y)+h(u)-h(v),\, u-v\rangle \\ &\leqq \|h(z)-h(u)\|\,\|u-v\|+\|h(v)-h(y)\|\,\|u-v\|+K\|u-v\|^2 \\ &\leqq \frac{1}{2}(\|h(z)-h(u)\|^2+\|h(v)-h(y)\|^2)+(K+1)\|u-v\|^2 .\end{aligned}$$

Setting $g(r)=\sup\{\|h(z)-h(u)\|^2 : \|u-z\|\leqq r \text{ and } u, z\in\Omega\}$, the result follows.

Thus, if f satisfies the conditions of Theorem 2 or Theorem 3, then $f\in R_l(C)$. Moreover, if C is bounded, then (2.5) is vacuously satisfied.

PROOF OF THEOREM 2.3. Under the assumptions of Theorem 2.3, we may apply Theorem II to obtain a solution u of

$$\frac{du}{dt}=f(u)-u\,, \qquad u(0)=x\in C$$

on $[0, \infty)$. Define $S(t): C\to C$ by $S(t)x=u(t)$. One easily checks that $S(t)S(\tau)=S(t+\tau)$, $S(0)x=x$ and $\|S(t)x-S(t)y\|\leqq\|x-y\|$. Moreover, $\left\|\frac{d}{dt}S(t)x\right\|$ is monotone decreasing in t. It follows that $S(t)x$ is bounded in t for fixed x (since $\|S(t_n)x\|\to\infty$ implies $\|f(S(t_n)x)-S(t_n)x\|=\left\|\frac{d}{dt}S(t_n)x\right\|\to\infty$ by assumption). Under these conditions on S, Browder [3] proved there is a point $x_0\in C$ such that $S(t)x_0=x_0$ for $t\geqq 0$. Hence

$$0=\frac{d}{dt}S(t)x_0=f(S(t)x_0)-S(t)x_0=f(x_0)-x_0\,,$$

and x_0 is a fixed point of f.

REMARK. Theorem 2.3 above also contains Theorems 5 and 6 of [4] as simple special cases.

An interesting result of a similar nature is:

THEOREM 2.7. *Let C be a closed convex subset of X and $f\in R_l(C)$. Let ω, ε be real numbers such that*

$$\varepsilon>0\,, \qquad \varepsilon\omega<1 \tag{2.8}$$

and

$$\langle f(x)-f(y),\, x-y\rangle \leqq (\omega+1)\|x-y\|^2 \qquad \text{for } x, y\in C. \tag{2.9}$$

If $f(\partial C)\subseteq C$, then for each $v\in C$ there is an element $x\in C$ such that

$$(1+\varepsilon)x-\varepsilon f(x)=x-\varepsilon(f(x)-x)=v\,.$$

PROOF. Fix $v\in C$ and consider the function

$$\bar{f}(x)=\frac{\varepsilon}{1+\varepsilon}f(x)+\frac{1}{1+\varepsilon}v\,. \tag{2.10}$$

Clearly $\bar{f}\in R_l(C)$ and $\bar{f}$ satisfies $f(\partial C)\subseteqq C$. Also

$$\langle \bar{f}(x)-\bar{f}(y),\ x-y\rangle \leqq \frac{\varepsilon}{1+\varepsilon}(\omega+1)\|x-y\|^2\,, \tag{2.11}$$

and thus

$$\langle (\bar{f}(x)-x)-(\bar{f}(y)-y),\ x-y\rangle \leqq -\kappa\|x-y\|^2\,, \tag{2.12}$$

where $\kappa=(1-\varepsilon(1+\varepsilon)^{-1}(\omega+1))$ is positive by (2.8). Theorem II now supplies a solution u on $[0,\infty)$ of

$$\frac{du}{dt}=\bar{f}(u)-u\,,\qquad u(0)=z$$

for each $z\in C$. Define $S(t):C\to C$ by $S(t)z=u(t)$ as before. Again $S(0)$ is the identity and $S(t)S(\tau)=S(t+\tau)$. This time, however, (2.12) yields

$$\frac{d}{dt}\|S(t)x-S(t)y\|^2\leqq -2\kappa\|S(t)x-S(t)y\|^2\,,$$

so

$$\|S(t)x-S(t)y\|\leqq e^{-\kappa t}\|x-y\|\,. \tag{2.13}$$

Hence $S(t)$, $t>0$, is a strict contraction mapping of C into C and has a unique fixed point. Define $x(t)\in C$ for $t>0$ by $S(t)x(t)=x(t)$. Since $(S(t/n))^n=S(t)$, $x(t/n)=x(t)$, and $x(mt)=x(t)$ for positive integers m, n and $t>0$. Thus $x(m/n)$ is independent of m, n, the fixed point of $S(t)$ is the same for all rational t, and, by continuity, for all t. Let $S(t)x_0=x_0$ for $t>0$. As before, $\bar{f}(x_0)=x_0$, which reduces at once to $(1+\varepsilon)x_0-\varepsilon f(x_0)=v$. The proof is complete.

REMARK. $\|S(t)x-x_0\|=\|S(t)x-S(t)x_0\|\leqq e^{-\kappa t}\|x-x_0\|$ implies $\lim_{t\to\infty}S(t)x=x_0$ for all $x\in C$.

For some of the implications of Theorem 2.7, see [2].

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