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Hecke operators in cohomology of groups

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Given a group G, with a subgroup Γ, one can always formulate the so-called Hecke rings whose elements are certain double cosets, called Hecke operators as introduced by Shimura in [4]. The study of the action of Hecke operators on the cohomology groups $H^k(\Gamma, \rho)$ with a linear representation ρ of G, defined by Kuga in [2], appears to be important in the number theory of automorphic forms, in the formulation of various "trace formulas", when the groups were Lie groups with discrete subgroups Γ, where the cohomology groups $H^k(\Gamma, \rho)$ were treated analytically and expressed as spaces of harmonic forms associated with the representation ρ.

In this paper, we shall deal purely algebraically with the Hecke operators on the cohomology groups $H^k(\Gamma, A)$ of arbitrary subgroups Γ of any abstract group G over a G-module A. The action of Hecke operators on $H^k(\Gamma, A)$, formulated by Kuga in [2] when G is a Lie group, turns out to be a sort of transfer map in the cohomology of groups.

In Section I, we described the Hecke rings $\mathfrak{R}(G, \Delta, \Gamma)$, and in Section II we obtained a representation of the Hecke rings $\mathfrak{R}(G, \Delta, \Gamma)$ over the cohomology groups $H^k(\Gamma, A)$ with an explicit formula. In the last section, we computed the effect of Hecke operators on $H^k(\Gamma, A)$ for a cyclic group Γ of $SL(2, \boldsymbol{Z}/p\boldsymbol{Z})$.

I. Hecke rings

1. Let G be a group. Two subgroups Γ and Γ' of G are said to be commensurable, denoted by $\Gamma \approx \Gamma'$, if the intersection of Γ and Γ' is of finite index with respect to both Γ and Γ'; in notation, $\Gamma \approx \Gamma' \Leftrightarrow [\Gamma : \Gamma \cap \Gamma'] < \infty$ and $[\Gamma' : \Gamma \cap \Gamma'] < \infty$. Then the commensurability is an equivalence relation and is invariant under conjugation, namely, $\Gamma \approx \Gamma'$ if and only if $\alpha^{-1}\Gamma\alpha = \Gamma^{\alpha} \approx \Gamma'^{\alpha}$. Let $\tilde{\Gamma}$ be the set of all elements α of G with $\Gamma^{\alpha} \approx \Gamma$.

PROPOSITION 1.1. *$\tilde{\Gamma}$ is a subgroup of G.*

PROOF. Given α and β in $\tilde{\Gamma}$, we have $\Gamma^{\alpha\beta} = (\alpha^{-1}\Gamma\alpha)^{\beta} \approx \Gamma^{\beta} \approx \Gamma$ and so $\alpha\beta$ belongs to $\tilde{\Gamma}$. By substituting α^{-1} for β, $\Gamma = (\alpha^{-1}\Gamma\alpha)^{\alpha^{-1}} \approx \Gamma^{\alpha^{-1}}$ implies $\alpha^{-1} \in \tilde{\Gamma}$.

We shall utilize some of the conventional notations: $\boldsymbol{Z}$ for the set of integers, $N(\Gamma)$ for the normalizer of Γ in G, $\#(S)$ for the cardinality of a set S and $\#(G)$ or $|G|$ for the order of a group G, in particular, $\#(\Gamma\backslash G)$ or $|\Gamma\backslash G|$ for the cardinality of the collection of all right cosets of Γ in G.

Given x and y of G, $\Gamma x\Gamma=\Gamma y\Gamma$ if and only if $x\gamma y^{-1}\in\Gamma$ for some γ in Γ, which, in turn, gives rise to an equivalence relation on the collection of right cosets of Γ in G, namely, Γx and Γy belong to a same double coset if and only if $x=\gamma_1 y\gamma_2$ for some γ_1 and γ_2 of Γ. Hence we can abuse the notation by writing $(\Gamma\backslash G)/\Gamma=\Gamma^{\backslash G/}\Gamma$, and call it the double coset decomposition. By specializing $\tilde{\Gamma}$ for G, we can choose a transversal Ω, so that $\Gamma^{\backslash\tilde{\Gamma}/}\Gamma=\{(\Gamma\omega\Gamma)\mid\omega\in\Omega\}=\bigcup_{\omega\in\Omega}(\Gamma\omega\Gamma)$, the disjoint union of elements $(\Gamma\omega\Gamma)$ of double cosets, and set-theoretically, $\tilde{\Gamma}=\bigcup_{\omega\in\Omega}\Gamma\omega\Gamma$, the disjoint union of sets $\Gamma\omega\Gamma$, that is, the set of all elements of the form $\gamma_1\omega\gamma_2$ for γ_1, γ_2 in Γ and $\omega\in\Omega$.

PROPOSITION 1.2. *With the notations as above, we have*

(i) $\tilde{\Gamma}=\{\alpha\mid\alpha\in G$ *and* $\#(\Gamma\backslash\Gamma\alpha\Gamma)<\infty\}=\cup\{\Gamma x\Gamma\mid\#((\Gamma^x\cap\Gamma)\backslash\Gamma)<\infty,\ x\in G\}$

(ii) $G\supset\tilde{\Gamma}\supset N(\Gamma)\supset\Gamma$

(iii) *If Γ is a normal subgroup of G, or either $|\Gamma|<\infty$ or $[G:\Gamma]<\infty$, then $G=\tilde{\Gamma}$.*

PROOF. $\alpha\in\tilde{\Gamma}\Leftrightarrow\Gamma^\alpha\approx\Gamma\Leftrightarrow\#(\Gamma^\alpha\cap\Gamma\backslash\Gamma)<\infty\Leftrightarrow\#(\Gamma\backslash\Gamma\alpha\Gamma)<\infty$ because $\#(\Gamma^\alpha\cap\Gamma\backslash\Gamma)=\#(\Gamma\backslash\Gamma\alpha\Gamma)$. This implies (i), and (ii) and (iii) are immediate from Proposition 1.1.

2. Let $\boldsymbol{Z}[\tilde{\Gamma},\Gamma]$ be the free $\boldsymbol{Z}$-module over the set $\Gamma^{\backslash\tilde{\Gamma}/}\Gamma$ of all distinct double cosets of Γ in $\tilde{\Gamma}$. Now we shall introduce a multiplication $\circ$ on $\boldsymbol{Z}[\tilde{\Gamma},\Gamma]$ as follows: Let $(\Gamma\alpha\Gamma)$ and $(\Gamma\beta\Gamma)$ be elements of $\boldsymbol{Z}[\tilde{\Gamma},\Gamma]$ with right coset decompositions $\Gamma\alpha\Gamma=\cup\Gamma\alpha_i$ and $\Gamma\beta\Gamma=\cup\Gamma\beta_j$ where the disjoint unions $\cup$ are taken over $i=1,2,\cdots,a$ and $j=1,2,\cdots,b$. Then define

$$(\Gamma\alpha\Gamma)\circ(\Gamma\beta\Gamma)=\sum_{(\Gamma\gamma\Gamma)}m(\Gamma\alpha\Gamma,\Gamma\beta\Gamma;\Gamma\gamma\Gamma)(\Gamma\gamma\Gamma),$$

where α, β and γ are in the prefixed transversal Ω, and $m(\Gamma\alpha\Gamma,\Gamma\beta\Gamma;\Gamma\gamma\Gamma)=\#\{(i,j)\mid\alpha_i\beta_j\in\Gamma\gamma\}$.

LEMMA 1.1. *The multiplication $\circ$, defined above, on $\boldsymbol{Z}[\tilde{\Gamma},\Gamma]$ is well-defined.*

PROOF. The multiplication is, indeed, independent of the choice of coset representations, because

$$\begin{aligned}\#\{(i,j)\mid\alpha_i\beta_j\in\Gamma\gamma\}&=\#\{(i,j)\mid\overline{\alpha_i\gamma_j^{-1}}\gamma_j\beta_j\in\Gamma\gamma\}\\&=\#\{(i,j)\mid\alpha_i\gamma_j\beta_j\in\Gamma\gamma\},\end{aligned}$$

where $\gamma\in\Omega$, γ_j's are in Γ and $\overline{\alpha_i\gamma_j^{-1}}$ denotes the coset representative of the right Γ coset to which $\alpha_i\gamma_j^{-1}$ belongs, i.e., $\alpha_i\gamma_j^{-1}\in\Gamma\cdot\overline{\alpha_i\gamma_j^{-1}}$.

The sum is a finite sum since $\#\{(i,j)\mid 1\leqq i\leqq a,\ 1\leqq j\leqq b\}$ is finite and

$\{\Gamma\gamma\Gamma \mid \gamma \in \Omega\}$ is a disjoint set in $\tilde{\Gamma}$.

By extending this operation bilinearly, we obtain an associative ring $\mathcal{R}(G, \tilde{\Gamma}, \Gamma)$ with the identity $(\Gamma) = (\Gamma \cdot 1 \cdot \Gamma)$ associated with $G \supset \tilde{\Gamma} \supset \Gamma$ over the $\boldsymbol{Z}$-module $\boldsymbol{Z}[\tilde{\Gamma}, \Gamma]$; for proof see [4]. By taking a semi-group Δ with $\tilde{\Gamma} \supset \Delta \supset \Gamma$, we obtain an associative ring $\mathcal{R}(G, \Delta, \Gamma)$ with the same construction as $\mathcal{R}(G, \tilde{\Gamma}, \Gamma)$, called a Hecke ring, associated with G, Δ, and Γ. ([2]).

COROLLARY 1.1. *The structure constants $m(\Gamma\alpha\Gamma, \Gamma\beta\Gamma; \Gamma\gamma\Gamma)$ of a Hecke ring $\mathcal{R}(G, \tilde{\Gamma}, \Gamma)$ are always non-negative integers, and are equal to $\sharp\{\Gamma\backslash\{\Gamma\alpha^{-1}\Gamma \cdot \gamma \cap \Gamma\beta\Gamma\}\}$.*

PROOF. The first statement is obvious from Lemma 1.1. For the second statement, consider the σ-ring $\mathfrak{L}$, generated by the set of all Γ right cosets, as subsets of $\tilde{\Gamma}$. Introduce a natural right Γ-invariant measure μ by

$$\mu(E) = \sharp\{\Gamma\backslash E\} \qquad \text{for } E \in \mathfrak{L}.$$

Then for a characteristic function $\chi_{\Gamma\alpha\Gamma}$ with $\Gamma\alpha\Gamma \subset \tilde{\Gamma}$, we have

$$\int_{\tilde{\Gamma}} |\chi_{\Gamma\alpha\Gamma}| \, d\mu = \mu(\Gamma\alpha\Gamma),$$

and the convolution $*$, with respect to the measure space $(\tilde{\Gamma}, \mathfrak{L}, \mu)$, of the characteristic functions $\chi_{\Gamma\alpha\Gamma}$ and $\chi_{\Gamma\beta\Gamma}$ evaluated at $\gamma' \in \Gamma\gamma\Gamma$, shows that

$$\mu(\Gamma\alpha^{-1}\Gamma\gamma' \cap \Gamma\beta\Gamma) = m(\Gamma\alpha\Gamma, \Gamma\beta\Gamma; \Gamma\gamma\Gamma)$$

without depending on the choice of γ' in $\Gamma\gamma\Gamma$. For details, see [1].

For $\Gamma = 1 (\in G)$, we have $G = \tilde{\Gamma}$ and the Hecke ring $\mathcal{R}(G, G, 1)$ is exactly the integral group ring $\boldsymbol{Z}(G)$ of G. If Γ is normal, or either $|\Gamma| < \infty$ or $[G : \Gamma] < \infty$, then $G = \tilde{\Gamma}$. For a Hecke ring $\mathcal{R}(G, G, \Gamma)$ with $|\Gamma| < \infty$, or other examples, see [1]. The above corollary indicates that any Hecke ring may be realized as a convolution algebra with respect to an invariant measure, naturally generalized from the counting measure on G with respect to a subgroup Γ, which is, in turn, used for defining integral group rings.

II. Hecke operators on $H^k(\Gamma, A)$

Let G be a group, Γ a subgroup of G and A a unitary left $\boldsymbol{Z}[G]$-module where $\boldsymbol{Z}[G]$ is the integral group ring of G. Let $\{Y_k, \partial_k, \varepsilon\}$ be a free and acyclic $\boldsymbol{Z}[G]$-complex, augmented by $\varepsilon : Y_0 \to Z^+$ for non-negative integers k with $\partial_0 = \varepsilon$. Hereafter, for the sake of convenience, we will call this complex an f. a. a. G-complex. The k-th cohomology group $H^k(G, A)$ of G with coefficients in A is uniquely defined and independent of the choice of f. a. a. G-complexes, because of the existence of chain transformations $\{\varphi_k : Y_k \to Y'_k\}$ between any two f.a.a. G-complexes $\{Y_k, \partial_k, \varepsilon\}$ and $\{Y'_k, \partial'_k, \varepsilon\}$ with the property that any such two are homotopic. Therefore the $\boldsymbol{Z}$-module End $(H^k(G, A))$ contains a submodule which is isomorphic to a submodule of End $(\mathrm{Hom}_G(Y_k, A))$, consisting of those elements which commute with the boundary operators ∂_k for $k \geqq 0$ without depending on the choice of f. a. a. G-complexes.

1. Let $\{Y_k, \partial_k, \varepsilon\}$ be an f.a.a. G-complex and Y_k a free G-module with a basis $\{b\}$. Let G be decomposed into a union of right cosets of Γ with a complete system $\Lambda = \{\lambda\}$ of representatives λ, namely, $G = \bigcup_{\lambda \in \Lambda} \Gamma\lambda$. Then Y_k is also a free $\boldsymbol{Z}[\Gamma]$-module with the corresponding basis $\{\lambda b\}$ and so $\{Y_k, \partial_k, \varepsilon\}$ becomes an f.a.a. Γ-complex. Therefore any f.a.a. G-complex might just as well be used for defining the cohomology groups $H^k(\Gamma, A)$.

For a given element $\Gamma\alpha\Gamma$ of the Hecke ring $\mathscr{R}(G, \Delta, \Gamma)$ with a coset decomposition $\Gamma\alpha\Gamma = \bigcup_{i=1}^{n} \Gamma\alpha_i$ we shall define the action of $\Gamma\alpha\Gamma$ on $H^k(\Gamma, A)$, denoted by $(H^k(\Gamma, A)|S_{\Gamma\alpha\Gamma})$, as follows:

Let $\{Y_k, \partial_k, \varepsilon\}$ be an f.a.a. G-complex.

Given a k-th cochain f of $\mathrm{Hom}_\Gamma(Y_k, A)$,

$$(f|S_{\Gamma\alpha\Gamma}) = \sum_{i=1}^{n} \alpha_i^{-1} \circ (f \circ \alpha_i)\,.$$

As a preparation needed in the sequel, we will observe a mapping τ of Γ into itself. Given $\alpha \in \Delta \subset \tilde{\Gamma}$, suppose the double coset $\Gamma\alpha\Gamma$ has a coset decomposition $\Gamma\alpha\Gamma = \bigcup_{i=1}^{n} \Gamma\alpha_i$ with a complete system $\{\alpha_i\}_{i=1}^{n}$ of representatives α_i. Then for any element γ of Γ the set $\{\alpha_i\gamma\}_{i=1}^{n}$ is also a complete system of representatives of the very same coset decomposition of $\Gamma\alpha\Gamma$ modulo Γ, and we have between the two systems $\{\alpha_i\}$ and $\{\alpha_i\gamma\}$ the following relation:

$$\alpha_i\gamma = \tau_i(\gamma) \cdot \alpha_{i^\gamma} \qquad \text{for } 1 \leqq i \leqq n$$

with $\tau_i(\gamma) \in \Gamma$ and $\alpha_{i^\gamma} \in \{\alpha_i\}$, where $\alpha_{i^\gamma} = \overline{\alpha_i\gamma}$ in our earlier notation.

Then it is easy to see that $(1^\gamma, 2^\gamma, \cdots, n^\gamma)$ is a permutation of $(1, 2, \cdots, n)$ with $i^{(\gamma\gamma')} = (i^\gamma)^{\gamma'}$ and $\tau_i(\gamma\gamma') = \tau_i(\gamma) \cdot \tau_{i^\gamma}(\gamma')$.

PROPOSITION 2.1. *With the notations as above, the operator $S_{\Gamma\alpha\Gamma}$ is a $\boldsymbol{Z}[\Gamma]$-homomorphism and independent of the choice of representatives of coset decomposition of $\Gamma\alpha\Gamma$ modulo Γ.*

PROOF. Let $\Gamma\alpha\Gamma$ be decomposed of $\bigcup_{i=1}^{n} \Gamma\alpha_i$. Given $f \in \mathrm{Hom}\,(Y_k, A)$ and $\gamma \in \Gamma$, observe, for $x \in Y_k$,

$$(f|S_{\Gamma\alpha\Gamma})(\gamma x) = \sum_{i=1}^{n} \alpha_i^{-1} f(\alpha_i\gamma x)$$

$$= \gamma \sum_{i=1}^{n} \gamma^{-1}\alpha_i^{-1} f(\tau_i(\gamma)\alpha_{i^\gamma} \cdot x)$$

$$=\gamma\sum_{i=1}^{n}\gamma^{-1}\alpha_i^{-1}\tau_i(\gamma)\cdot f(\alpha_{i\gamma}\cdot x)$$

$$=\gamma\sum_{i=1}^{n}\alpha_{i\gamma}^{-1}f(\alpha_{i\gamma}x)=\gamma\sum_{i=1}^{n}\alpha_i f(\alpha_i x)$$

since $\gamma^{-1}\alpha_i^{-1}\tau_i(\gamma)=\alpha_{i\gamma}^{-1}$ and $\{\alpha_i\}_{i=1}^n=\{\alpha_{i\gamma}\}_{i=1}^n$.

In order to show the independence of $S_{\Gamma\alpha\Gamma}$ from the systems of representatives, let $\{\gamma(\alpha_i)\cdot\alpha_i\}_{i=1}^n$ be another complete system of representatives of coset decomposition of $\Gamma\alpha\Gamma$ modulo Γ with $\gamma(\alpha_i)\in\Gamma$. Given $f\in\mathrm{Hom}_\Gamma(Y_k, A)$ and $x\in Y_k$ consider the action of $S_{\Gamma\alpha\Gamma}$ with respect to the system $\{\gamma(\alpha_i)\cdot\alpha_i\}_{i=1}^n$, namely,

$$(f|S_{\Gamma\alpha\Gamma})(x)=\sum_{i=1}^{n}\alpha_i^{-1}\cdot\gamma(\alpha_i)^{-1}f(\gamma(\alpha_i)\cdot\alpha_i\cdot x)$$

$$=\sum_{i=1}^{n}\alpha_i^{-1}f(\alpha_i x)\,.$$

The Γ-homomorphism $S_{\Gamma\alpha\Gamma}$, well-defined on $\mathrm{Hom}_\Gamma^*(Y_k, A)$ will induce a homomorphism on $H^k(\Gamma, A)$, again denoted by $S_{\Gamma\alpha\Gamma}$ by the following

PROPOSITION 2.2. *With the notations as above, we have*

$$\delta_k S_{\Gamma\alpha\Gamma}=S_{\Gamma\alpha\Gamma}\delta_k$$

where δ_k is the k-th coboundary operator for $k\geqq 0$ defined by $(f|\delta_k)=\delta_k f=f\partial_{k+1}$. In fact, we have, for $x\in Y_{k+1}$,

$$((f|\delta_k)|S_{\Gamma\alpha\Gamma})(x)=\sum_{i=1}^{n}\alpha_i^{-1}(f|\delta_k)\cdot\alpha_i x$$

$$=\sum_{i=1}^{n}\alpha_i^{-1}(f\cdot\partial_{k+1})(\alpha_i x)$$

$$=\sum_{i=1}^{n}\alpha_i^{-1}f(\partial_{k+1}(\alpha_i x))=\sum_{i=1}^{n}\alpha_i^{-1}f\cdot\alpha_i(\partial_{k+1}x)$$

$$=(f|S_{\Gamma\alpha\Gamma})(\partial_{k+1}x)=((f|S_{\Gamma\alpha\Gamma})|\delta_k)(x)\,.$$

We have established that the operators $S_{\Gamma\alpha\Gamma}$, associated to $\Gamma\alpha\Gamma$ of $\mathfrak{R}(G, \varDelta, \Gamma)$ are defined on the cohomology groups $H^k(\Gamma, A)$ of Γ over A, which are particularly derived from an f.a.a. G-complex $\{Y_k, \partial_k, \varepsilon\}$, which is, in fact, an f.a.a. Γ-complex. However, since $H^k(\Gamma, A)$ are independent of the choice of f.a.a. Γ-complexes as mentioned earlier, $S_{\Gamma\alpha\Gamma}$ are well-defined on $H^k(\Gamma, A)$, only depending on Γ and A. We call $S_{\Gamma\alpha\Gamma}$ Hecke operators on $H^k(\Gamma, A)$. Now we shall establish our main property that $H^k(\Gamma, A)$ is a unitary right $\mathfrak{R}(G, \varDelta, \Gamma)$-module for $k\geqq 0$. For that purpose, we shall extend the definition of Hecke operators linearly on the module structure of $\mathfrak{R}(G, \varDelta, \Gamma)$ by the formula $S_{(\sum_\omega n(\omega)\cdot\Gamma\omega\Gamma)}=\sum_\omega n(\omega)\cdot S_{\Gamma\omega\Gamma}$, with $n(\omega)\in\boldsymbol{Z}$, and $\sum n(\omega)\cdot\Gamma\omega\Gamma\in\mathfrak{R}(G, \varDelta, \Gamma)$.

PROPOSITION 2.3. *With the notations as above, the mapping S is a representation of the Hecke rings $\mathcal{R}(G, \Delta, \Gamma)$ over $H^k(\Gamma, A)$ for each $k \geqq 0$.*

PROOF. Let $\Gamma\alpha\Gamma$ and $\Gamma\beta\Gamma$ be elements of $\mathcal{R}(G, \Delta, \Gamma)$ with right coset decompositions $\Gamma\alpha\Gamma=\bigcup_{i=1}^{a}\Gamma\alpha_i$ and $\Gamma\beta\Gamma=\bigcup_{j=1}^{b}\Gamma\beta_j$, and $\Gamma\alpha\Gamma\circ\Gamma\beta\Gamma=\sum m(\Gamma\alpha\Gamma, \Gamma\beta\Gamma; \Gamma\gamma\Gamma)\ \Gamma\gamma\Gamma$ where the sum runs through the finite set $\{\Gamma\gamma\Gamma \mid \gamma\in\Omega'\}$ for a subset Ω' of Ω, determined by $\Gamma\alpha\Gamma$ and $\Gamma\beta\Gamma$ in their product, namely, all $\Gamma\gamma\Gamma\subset\Gamma\alpha\Gamma\beta\Gamma$. Let $\Gamma\gamma\Gamma$ be decomposed in $\Gamma\gamma\Gamma=\bigcup_{k=1}^{c}\Gamma\gamma_k$ for each γ in Ω'. It follows from the Corollary to Lemma 1.1

$$m(\Gamma\alpha\Gamma, \Gamma\beta\Gamma; \Gamma\gamma\Gamma)=\#\{(i, j) \mid \Gamma\alpha_i\beta_j=\Gamma\gamma\}$$
$$=\#\{(i, j) \mid \Gamma\alpha_i\beta_j=\Gamma\gamma_k\} \qquad \text{for each } \gamma_k,\ 1\leqq k\leqq c.$$

Therefore for $f\in C^k$, the k-th cochain group, we have

$$(f \mid S_{\Gamma\alpha\Gamma\circ\Gamma\beta\Gamma})=\sum_{\gamma\in\Omega'} m(\Gamma\alpha\Gamma, \Gamma\beta\Gamma; \Gamma\gamma\Gamma)\Big(\sum_{k=1}^{c}\gamma_k^{-1}\circ f\circ\gamma_k\Big)$$
$$=\sum_{\overline{\alpha_p\beta_q}}\#\{(i, j) \mid \Gamma\alpha_i\beta_j=\Gamma\overline{\alpha_p\beta_q}\}\cdot(\overline{\alpha_p\beta_q}^{-1}\circ f\circ\overline{\alpha_p\beta_q})$$
$$=\sum_{i,j}(\alpha_i\beta_j)^{-1}\circ f\circ(\alpha_i\beta_j)=((f \mid S_{\Gamma\alpha\Gamma}) \mid S_{\Gamma\beta\Gamma})$$

where the second sum runs through the set $\bigcup_{\gamma\in\Omega'}\{\overline{\alpha_p\beta_q}=\gamma_k,\ 1\leqq k\leqq c\}$ (k and c depend on γ), because for a pair (p, q) with $1\leqq p\leqq a$ and $1\leqq q\leqq b$, there exists a $\gamma\in\Omega'$ and some k such that $\Gamma\overline{\alpha_p\beta_q}=\Gamma\gamma_k$, and vice versa, by the fact that for every $\gamma\in\Omega'$, $\Gamma\gamma\Gamma\subset\Gamma\alpha\Gamma\beta\Gamma=\Gamma\alpha\Gamma\{\bigcup_j\Gamma\beta_j\}=\bigcup_{i,j}\Gamma\alpha_i\beta_j$ with $1\leqq i\leqq a$ and $1\leqq j\leqq b$.

2. An explicit formula for Hecke operators. In practice, we will find it convenient to have an explicit and computable formula for Hecke operators $S_{\Gamma\alpha\Gamma}$. For this purpose we will utilize a specific Γ-complex, namely, the standard homogeneous f.a.a. Γ-complex $\{X_k, \partial_k, \varepsilon\}$, defined as follows:

For $k\geqq 0$, X_k is the free $\boldsymbol{Z}[\Gamma]$-module, generated by the set $\Gamma\times\Gamma\times\cdots\times\Gamma$ of all $(k+1)$-tuples of elements of Γ and the Γ-homomorphism ∂_k is defined homogeneously by

$$\partial_k(\gamma_0, \gamma_1, \cdots, \gamma_k)=\sum_{i=0}^{k}(-1)^i(\gamma_0, \gamma_1, \cdots, \hat{\gamma}_i, \cdots, \gamma_k)$$

for $k>0$ and for $k=0$ we set ∂_k to be the augmentation $\varepsilon: X_0\rightarrow\boldsymbol{Z}^+$.

PROPOSITION 2.4. *With the notations as above, the Hecke operator, associated to $\Gamma\alpha\Gamma$ of $\mathcal{R}(G, \Delta, \Gamma)$ on $H^k(\Gamma, A)$ with respect to the standard homogeneous f.a.a. Γ-complex $\{X_k, \partial_k, \varepsilon\}$, denoted by $T_{\Gamma\alpha\Gamma}$, is expressible as follows: Given $f\in \mathrm{Hom}_\Gamma(X_k, A)$ and γ_i's in Γ,*

$$(f|T_{\Gamma\alpha\Gamma})(\gamma_0, \gamma_1, \cdots, \gamma_k) = \sum_{i=1} \alpha_i^{-1} f(\tau_i(\gamma_0), \tau_i(\gamma_1), \cdots, \tau_i(\gamma_k)),$$

provided that $\Gamma\alpha\Gamma = \bigcup_{i=1}^{a} \Gamma\alpha_i$.

PROOF. Let $\{Y_k, \partial_k, \varepsilon\}$ be the standard homogeneous f.a.a. G-complex, which is also an f.a.a. Γ-complex through a (but fixed) coset decomposition $G = \bigcup_{\lambda\in\Lambda} \Gamma\lambda$ with a complete system $\Lambda = \{\lambda\}$ of representatives. Then a set of mappings φ_k of Y_k into X_k, defined by

$$\varphi_k : (g_0, g_1, \cdots, g_k) \longrightarrow (\gamma_0, \gamma_1, \cdots, \gamma_k)$$

with $g_i = \gamma_i\lambda$ $(\gamma_i \in \Gamma)$ is a chain transformation of $\{Y_k, \partial_k, \varepsilon\}$ to $\{X_k, \partial_k, \varepsilon\}$, and the set of inclusion mappings θ_k of X_k into Y_k is a chain transformation of $\{X_k, \partial_k, \varepsilon\}$ to $\{Y_k, \partial_k, \varepsilon\}$. Now we have the following commutative diagram:

$$\begin{array}{ccc}
\mathrm{Hom}_\Gamma(Y_k, A) & \xrightarrow{\quad S_{\Gamma\alpha\Gamma} \quad} & \mathrm{Hom}_\Gamma(Y_k, A) \\
\uparrow \varphi_k^* & & \downarrow \theta_k^* \\
\mathrm{Hom}_\Gamma(X_k, A) & \xrightarrow[\quad T_{\Gamma\alpha\Gamma} \quad]{} & \mathrm{Hom}_\Gamma(X_k, A)
\end{array}$$

where $\{\varphi_k^*\}$ and $\{\theta_k^*\}$ are the induced homomorphisms by $\{\varphi_k\}$ and $\{\theta_k\}$ respectively. In other words, $T_{\Gamma\alpha\Gamma} = \varphi_k^* \cdot S_{\Gamma\alpha\Gamma} \cdot \theta_k^*$, that is, explicitly, for $f \in \mathrm{Hom}_\Gamma(X_k, A)$ γ_i's in Γ and $\Gamma\alpha\Gamma = \bigcup_{i=1}^{n} \Gamma\alpha_i$ with $\alpha_i \in \Lambda$,

$$\begin{aligned}
&(f|\varphi_k^* S_{\Gamma\alpha\Gamma}\theta_k^*)(\gamma_0, \gamma_1, \cdots, \gamma_k) \\
&= \sum_{i=1}^{n} \alpha_i^{-1}(f|\varphi_k^*)(\alpha_i\gamma_0, \alpha_i\gamma_1, \cdots, \alpha_i\gamma_k) \\
&= \sum_{i=1}^{n} \alpha_i^{-1}(f|\varphi_k^*)(\tau_i(\gamma_0)\alpha_{i\gamma_0}, \tau_i(\gamma_1)\alpha_{i\gamma_1}, \cdots, \tau_i(\gamma_k)\alpha_{i\gamma_k}) \\
&= \sum_{i=1}^{n} \alpha_i^{-1} f(\tau_i(\gamma_0), \tau_i(\gamma_1), \cdots, \tau_i(\gamma_k))
\end{aligned}$$

since $\alpha_{i\gamma_j}$ are all in the system $\{\alpha_i\}_{i=1}^{n}$.

REMARK. The results in this section can be obtained also by the method, utilized in [5], whose argument runs somewhat longer.

III. Hecke operators on $H^k(\Gamma, A)$ of a cyclic group Γ

In this section we would like to give an explicit description of the action of Hecke operators on specific cohomology groups.

Let G denote $SL(2, \boldsymbol{Z}/p\boldsymbol{Z})$ for a prime number p, Γ the cyclic subgroup $\langle T\rangle$, generated by $T=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$ of order p, and A a two-dimensional vector space over $\boldsymbol{Z}/p\boldsymbol{Z}$. By letting G operate on A as linear transformations from the left, A becomes a left unitary $\boldsymbol{Z}[G]$-module.

We note that $G=\tilde{\Gamma}\supset\Gamma$.

LEMMA 3.1. *Let α be $\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ of $SL(2, \boldsymbol{Z}/p\boldsymbol{Z})$. Then $c\neq 0$ if and only if $\alpha^{-1}\Gamma\alpha\cap\Gamma=e$, the identity matrix.*

PROOF. Suppose $\alpha^{-1}\Gamma\alpha\cap\Gamma\neq e$. Then there exist $\begin{pmatrix}1 & m\\ 0 & 1\end{pmatrix}$ and $\begin{pmatrix}1 & n\\ 0 & 1\end{pmatrix}$ with integers $1<m, n<p$ such that $\begin{pmatrix}1 & m\\ 0 & 1\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix}=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}1 & n\\ 0 & 1\end{pmatrix}$, from which it follows that c must be 0.

Conversely, if α is of the form $\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ with $c=0$, then we can find T^n of Γ which belongs to $\alpha^{-1}\Gamma\alpha$, provided that n is one of those which satisfy the equation $md=an$ for some non-zero integer m.

COROLLARY 3.1. *If α in $SL(2, \boldsymbol{Z}/p\boldsymbol{Z})$ is of the form $\begin{pmatrix}a & b\\ 0 & a^{-1}\end{pmatrix}$, then we have $\Gamma\alpha\Gamma=\Gamma\alpha$.*

PROOF. For $\alpha=\begin{pmatrix}a & b\\ 0 & a^{-1}\end{pmatrix}\in G$, $\alpha^{-1}\Gamma\alpha\cap\Gamma=\Gamma$ or equivalently $\Gamma\alpha=\alpha\Gamma$, since $\gamma\alpha=\alpha\gamma'$ for γ, γ' in Γ, $md=an$ is solvable for any n. Hence $\Gamma\Gamma\alpha=\Gamma\alpha=\Gamma\alpha\Gamma$.

We recall a few notations. Let $\Gamma\alpha\Gamma$ be decomposed into a disjoint union of right cosets $\Gamma\alpha\Gamma=\bigcup\Gamma\alpha_i$. Then $\alpha_i\gamma=\tau_i(\gamma)\cdot\overline{\alpha_i\gamma}$ where $\tau_i(\gamma)\in\Gamma$ and $\overline{\alpha_i\gamma}$ is the representative of the coset to which $\alpha_i\gamma$ belongs with respect to a pre-chosen right transversal $\{\alpha_i\}$ for Γ in $\Gamma\alpha\Gamma$, i.e., $\overline{\alpha_i\gamma}\in\{\alpha_i\}$. As in the proof of Proposition 1.2, it follows that $\Gamma\alpha\Gamma=\bigcup\Gamma\alpha\gamma_i$ where $\{\gamma_i\}$ is a right transversal for the coset decomposition $(\Gamma^\alpha\cap\Gamma)\backslash\Gamma$. In fact, $\Gamma\alpha\gamma=\Gamma\alpha\gamma'$ if and only if $\gamma'\gamma^{-1}\in\Gamma^\alpha\cap\Gamma$.

PROPOSITION 3.1. *Let $\alpha\in G$ be of the form $\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ with $c\neq 0$ and $\gamma\in\Gamma$. Then $\tau_i(\gamma)=e$ for all i, with respect to a coset decomposition $\Gamma\alpha\Gamma=\bigcup\Gamma\alpha\gamma_i$, described above. If $\alpha=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ with $c=0$, we have $\Gamma\alpha\Gamma=\Gamma\alpha$ and $\tau_1(\gamma)=\gamma^{a^2}$.*

PROOF. For the first case, it follows from Lemma 3.1 that for every γ of Γ, $\alpha\gamma$ is a coset representative and $\overline{\alpha\gamma}=\alpha\gamma$, yielding $\tau_i(\gamma)=e$. For the second case, we have $\alpha\gamma=\tau_1(\gamma)\alpha$ for $\gamma\in\Gamma$, since $\Gamma\alpha\Gamma=\Gamma\alpha$. Hence $\tau_1(\gamma)$ is of the form $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ with $xd=an$ for $\gamma=\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ and so we have $x=a^2n$.

Let R denote the integral group ring $\boldsymbol{Z}[\Gamma]$ of $\Gamma=\langle T\rangle$ with $T=\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ in $G=SL(2, \boldsymbol{Z}/p\boldsymbol{Z})$, and $N=1+T+T^2+\cdots+T^{p-1}$, $D=T-1$ in R, which operate on A. The special f.a.a. Γ-complex:

$$\xrightarrow{N} R \xrightarrow{D} R \xrightarrow{N} R \xrightarrow{D} \cdots \xrightarrow{D} R \xrightarrow{N} R \xrightarrow{D} R \xrightarrow{\varepsilon} \boldsymbol{Z} \longrightarrow 0$$

is a free resolution, customarily denoted by W, from which we obtain the isomorphism I^* of $H_W^{2n}(\Gamma, A)$ onto $A^\Gamma=\{a\in A \mid Ta=a\}=H^{2n}(\Gamma, A)$ and $H_W^{2n+1}(\Gamma, A)$ onto $A/DA=H^{2n+1}(\Gamma, A)$, $(n\geqq 0)$, induced from the cochain isomorphism I: $\mathrm{Hom}_\Gamma(R, A)\cong A$ by $I(\varphi)=\varphi(e)$.

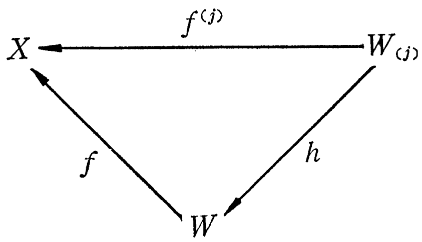
Let X be, as before, the standard homogeneous f.a.a. Γ-complex $\{X_k, \partial_k, \varepsilon\}$ with X_k being the R-module on $(k+1)$-copies $\Gamma\times\Gamma\times\cdots\times\Gamma$ of Γ, and $W_{(j)}$ for the free resolution: for $D^{(j)}=T^{(j)}-1$ with positive integers j,

$$\xrightarrow{N} R \xrightarrow{D^{(j)}} R \xrightarrow{N} R \xrightarrow{D^{(j)}} \cdots \xrightarrow{D^{(j)}} R \xrightarrow{N} R \xrightarrow{D^{(j)}} R \xrightarrow{\varepsilon} \boldsymbol{Z} \longrightarrow 0 .$$

Then among these three f.a.a. Γ-complexes, we have the following useful functorial chain transformations.

PROPOSITION 3.2. *With the notations as above, we have the commutative diagram:*

(A)



where the chain transformations f, $f^{(j)}$ and h are defined as follows: for $f=\{f_k\}$, with $e=I$, the unit matrix,

$$f_0=\mathit{identity}$$

$$f_1(e)=(e, T)$$

$$f_{2n}(e)=\sum_{\gamma_1,\gamma_2,\cdots,\gamma_n\in\Gamma}(e, \gamma_1, T\gamma_1, \gamma_2, T\gamma_2, \cdots, \gamma_n, T\gamma_n)$$

and

$$f_{2n+1}(e)=\sum_{\gamma_1,\gamma_2,\cdots,\gamma_n\in\Gamma}(e, T, \gamma_1, T\gamma_1, \gamma_2, T\gamma_2, \cdots, \gamma_n, T\gamma_n)$$

for all natural numbers n, and for $f^{(j)}=\{f_k^{(j)}\}$,

$$f_0^{(j)} = identity$$

$$f_1^{(j)}(e) = (e, T^{(j)})$$

$$f_{2n}^{(j)}(e) = \sum_{\gamma_1, \gamma_2, \cdots, \gamma_n \in \Gamma} (e, \gamma_1, T^j\gamma_1, \gamma_2, T^j\gamma_2, \cdots, \gamma_n, T^j\gamma_n)$$

and

$$f_{2n+1}^{(j)}(e) = \sum_{\gamma_1, \gamma_2, \cdots, \gamma_n \in \Gamma} (e, T^j, \gamma_1, T^j\gamma_1, \gamma_2, T^j\gamma_2, \cdots, \gamma_n, T^j\gamma_n)$$

for all natural numbers n, *and for* $h = \{h_k\}$,

$$h_{2n} = j^n \quad \text{and} \quad h_{2n+1} = j^n \cdot (1 + T + T^2 + \cdots + T^{j-1})$$

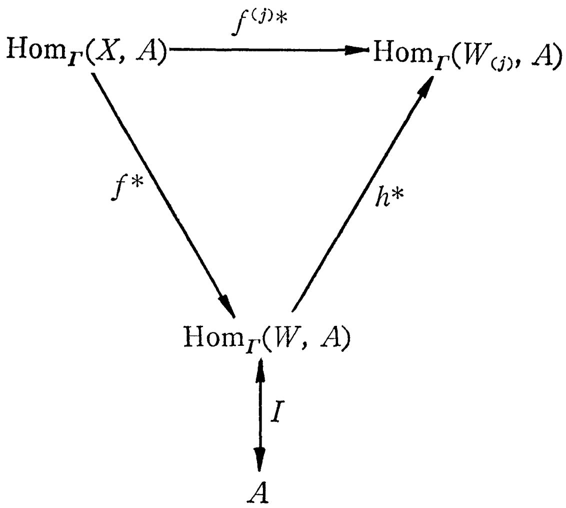
for all natural numbers n *with the mapping*

$$h_k : R \to R \quad \text{by} \quad h_k(y) = h_k \cdot y, \quad \text{the module product.}$$

PROOF. A straightforward checking.

From the diagram (A), the functor Hom yields the following commutative diagram:

(B)



Before obtaining the effect of actions of Hecke operators on $H^k(\Gamma, A)$, we notice the following

LEMMA 3.2. *Let* $\mathrm{Hom}_\Gamma(W, A)$ *be the following cochain complex, derived from the free resolution* W:

$$\longleftarrow \mathrm{Hom}_\Gamma(R, A) \xleftarrow{N^*} \mathrm{Hom}_\Gamma(R, A) \xleftarrow{D^*}$$

$$\cdots \xleftarrow{D^*} \mathrm{Hom}_\Gamma(R, A) \xleftarrow{N^*} \mathrm{Hom}_\Gamma(R, A) \xleftarrow{D^*} \mathrm{Hom}_\Gamma(R, A) \longleftarrow 0$$

Then in odd dimensions $2n+1$, *for every* $\varphi \in \mathrm{Hom}_\Gamma(R, A)$, φ *is cohomologous to* $\varphi \cdot \gamma$ *in* $\mathrm{Hom}_\Gamma(R, A)$ *for* $\gamma \in \Gamma$.

PROOF. Observe $\varphi - (\varphi \circ T) \in D^*(\mathrm{Hom}_\Gamma(R, A))$.

For any α of $G=SL(2, \boldsymbol{Z}/p\boldsymbol{Z})$, the Hecke operator $S_{\Gamma\alpha\Gamma}$ on $H^k(\Gamma, A)$ was explicitly defined in Proposition 2.4 as follows: for $\varphi \in \mathrm{Hom}_\Gamma(X_k, A)$,

$$(\varphi | S_{\Gamma\alpha\Gamma})(\gamma_0, \gamma_1, \cdots, \gamma_k) = \sum \alpha_i^{-1} \cdot \varphi(\tau_i(\gamma_0), \tau_i(\gamma_1), \cdots, \tau_i(\gamma_k)).$$

LEMMA 3.3. *Let* α *be of the form* $\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}$ *in* $SL(2, \boldsymbol{Z}/p\boldsymbol{Z})$. *Then for* $[\xi] \in H^k(\Gamma, A)$

$$[\alpha\xi] = \begin{cases} [a \cdot \xi] & \textit{for } k=2n, \textit{ and} \\ [a^{-1} \cdot \xi] & \textit{for } k=2n, \quad \textit{for } n \geqq 0. \end{cases}$$

PROOF. Consider the cochain complex:

$$\xleftarrow{D} A \xleftarrow{N} A \xleftarrow{D} A \xleftarrow{N} A \xleftarrow{D} \cdots \xleftarrow{D} A \xleftarrow{N} A \xleftarrow{D} A \longleftarrow 0$$

with the operation N and D being the module product, from which we obtained

$$H^k(\Gamma, A) = \begin{cases} A^\Gamma & \text{for } k=2n \\ A/DA & \text{for } k=2n+1. \end{cases}$$

For $\xi \in \ker D = A^\Gamma$, $\alpha\xi = a\xi$, and for $\begin{pmatrix} x \\ y \end{pmatrix} \in A$, $\alpha\begin{pmatrix} x \\ y \end{pmatrix} = a^{-1}\begin{pmatrix} x \\ y \end{pmatrix}$ (mod DA).

PROPOSITION 3.3. *With the notations as above, for* $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$, *with* $c \neq 0$, *we have*

$$(H^k(\Gamma, A) | S_{\Gamma\alpha\Gamma}) = 0$$

for all non-negative integers k. *If* α *is of the form* $\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}$, *then on* $H^k(\Gamma, A)$,

$$S_{\Gamma\alpha\Gamma} = \begin{cases} a^{2n-1} & \textit{for } k=2n \\ a^{2n+3} & \textit{for } k=2n+1 \end{cases}$$

for $n \geqq 0$.

PROOF. For the first part of the proposition, we have $\tau_i(\gamma)=e$ for every $\gamma \in \Gamma$ from Proposition 3.1 with respect to a finite coset decomposition $\Gamma\alpha\Gamma = \bigcup \Gamma\alpha_i$.

From the diagrams (A) and (B), we obtain the following commutative diagram: with the induced isomorphisms f^*, $f^{(j)*}$, h^* and I^*, for each k,

$$\text{(C)}\quad \begin{array}{ccccccc} H^k_X(\Gamma, A) & \xleftrightarrow{\quad f_k^* \quad} & & & H^k_W(\Gamma, A) & \xleftrightarrow{\ I_k^*\ } & H^k(\Gamma, A) \\ \uparrow{\scriptstyle S_{\Gamma\alpha\Gamma}} & & & & & & \uparrow{\scriptstyle S_{\Gamma\alpha\Gamma}} \\ H^k_X(\Gamma, A) & \xleftrightarrow[f_k^{(j)*}]{} & H^k_{W_{(j)}}(\Gamma, A) & \xleftrightarrow[J_k^*]{} & H^k_W(\Gamma, A) & \xleftrightarrow[I_k^*]{} & H^k(\Gamma, A) \end{array}$$

where $J^* = (h^*)^{-1}$ with

$$J^*_{2n}([\varphi]) = [j^{-n}\cdot\varphi] \quad \text{and} \quad J^*_{2n+1}([\varphi]) = [j^{-n-1}\cdot\varphi],$$

by Lemma 3.2.

$\Gamma\alpha\Gamma = \bigcup_{\gamma\in\Gamma} \Gamma\alpha\gamma$, since $\Gamma^\alpha \cap \Gamma = e$. Hence $S_{\Gamma\alpha\Gamma}$ are zero operators.

For the rest of the proposition, we recall that $\Gamma\alpha\Gamma = \Gamma\alpha$ for $\alpha = \begin{pmatrix} a & b \\ 0 & a^{-1}\end{pmatrix}$, and $\tau(\gamma) = \gamma^{a^2}$ for $\gamma \in \Gamma$ from Proposition 3.1. Now, using Lemma 3.3, it is a matter of chasing the diagram (C):

Given $[\varphi] \in H^k_X[\Gamma, A]$ for $k = 2n$, letting $j = a^2$, we have

$$I_k^* f_k^* (\varphi \mid S_{\Gamma\alpha\Gamma}) = \alpha^{-1}\cdot j^n \cdot I_k^* J_k^* f_k^{(j)*}[\varphi] = a^{2n-1}\cdot I_k^* J_k^* f_k^{(j)*}[\varphi],$$

and for $k = 2n+1$,

$$I_k^* f_k^* (\varphi \mid S_{\Gamma\alpha\Gamma}) = \alpha^{-1}\cdot j^{n+1} I_k^* J_k^* f_k^{(j)*}[\varphi] = a^{2n+3}\cdot I_k^* J_k^* f_k^{(j)*}[\varphi].$$

q. e. d.

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