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Double ruled surfaces and their canonical systems*)

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Generally we shall follow the definitions and notations in Weil [4] and we shall consider projective varieties exclusively. Thus *varieties* are projective varieties, and surfaces and curves are (projective) varieties of dimension two and one respectively. To state our results, we first recall and introduce several definitions. k denotes, once and for always, an algebraically closed subfield of the field of complex numbers.

DEFINITION. (i) A variety U is a *rational variety over* k if and only if U is birationally equivalent over k to a projective space $\boldsymbol{P}_n$. (ii) A surface S is a *ruled surface over* k *with the base* B if and only if S is birationally equivalent over k to the product of the projective line $\boldsymbol{P}_1$ and a curve B defined over k. (iii) A surface S is a *double ruled surface over* k *with the base* B if and only if there is a rational mapping defined over k of degree two of S to a ruled surface $\boldsymbol{P}_1 \times B$ over k. S is a *double plane over* k if and only if there is a rational mapping defined over k of degree two of S to a rational surface over k (or the projective plane). (iv) We say that $\pi: S \rightarrow B$ is a *pencil over* k *of curves* or S has a pencil over k of curves if and only if S is a non-singular surface defined over k, B a non-singular curve defined over k, π a morphism defined over k, and a generic fibre $F_b = pr_S[\Gamma_\pi \cdot (S \times b)]$, $b \in B$, is irreducible (a curve defined over $k(b)$).

The purpose of this note is to find the image S_K of a double ruled surface S over k under the rational mapping induced by the canonical system. It turns out that, if S_K is of dimension two, then it is a ruled surface *over* k (Theorem 1); in particular we see that, if S is a double plane over k, then the image S_K is a *rational variety over* k (Corollary 2 to Theorem 1). These results remind us of a well-understood property of the canonical system of hyperelliptic curves. On the way to reach Theorem 1, the following results are proven and used. Proposition 2 generalizes, in some sense, Lüroth's Theorem to the effect that if a surface is the image of a rational mapping

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defined over k of a ruled surface over k then it is a ruled surface *over k* again. In Proposition 3 it is proven that double ruled surfaces carry with them pencils of rational curves, or of elliptic curves, or of hyperelliptic curves.

NOTATIONS. If V is a variety defined over a field k', then $R_{k'}(V)$ denotes the function-field over k' of V, and $L(D)=L_{V/k'}(D)$ denotes the vector-space over k' of all functions $u \in R_{k'}(V)$ with $\operatorname{div}(u)+D \geqq 0$, D being a divisor rational over k' on V. If b is a point (a variety of dimension 0), then $k'(b)$ denotes the field generated over k' by the coordinates of an affine representative of b. If $\pi: U \rightarrow V$ is a rational mapping, then Γ_π denotes the graph of π and $\pi^*(v)$ denotes the cycle $pr_U[\Gamma_\pi \cdot (U \times v)]$ on U, v being a point on V ([**4**, p. 222]).

§1. Fibered surfaces and ruled surfaces.

PROPOSITION 1. *Let $h: S \rightarrow V$ be a rational mapping defined over k of a non-singular surface S to a variety V of dimension 1 or 2. Then, applying to S a finite sequence of dilatations $\sigma=\sigma_n \cdots \sigma_1$, we have a surface $S^*=\sigma(S)$ such that the rational mapping $h \circ \sigma^{-1}: S^* \rightarrow V$ is a morphism. (S^* and $h \circ \sigma^{-1}$ are also defined over k.)*

This generalizes Theorem 1 in Šafarevič [**3**, p. 14]. Recall that, as the fundamental locus of h is of codimension 2, the number of fundamental points of h is finite (Weil [**4**, p. 201, Corollary 2 to Theorem 9]). The proof given in [**3**] does not make full use of the fact that the mapping is birational, but it uses only the fact that the number of fundamental points of a rational mapping of a surface is finite. Therefore he has actually proven the above proposition.

We shall often use a theorem of Bertini to the effect

(1) *Let S, B be non-singular surface and curve defined over k, and $\pi: S \rightarrow B$ be a morphism over k of S onto B. If a generic fibre $F_b=\pi^*(b)$, $b \in B$, is irreducible (a curve defined over $k(b)$), then it is non-singular.* (Akizuki [**1**]).

Also we shall frequently use

(2) *If S is a surface defined over k, then there is a non-singular surface which is birationally equivalent over k to S.* (Zariski [**5**])

We first generalize Lüroth's Theorem to the effect

PROPOSITION 2. *Let R be the function-field over k of a ruled surface over k. Let R' be an intermediary field between R and k over which R is of finite degree:*

$$R \geqq R', \quad [R: R'] < \infty,$$

then R' is also the function-field of a ruled surface over k.

PROOF. Let S and S' be non-singular models of R/k and R'/k respectively.

The inclusion $R' \leqq R$ induces a rational mapping $f: S \rightarrow S'$ over k. In view of Proposition 1, we may assume that f is a morphism. Let q and q' be the irregularities of S and S' respectively. Since the bigenus P_2 of S is 0, the bigenus P_2' of S' is 0. It follows from this, by a theorem of Castelnuovo-Zariski, that if $q'=0$ then S' is a rational (ruled) surface over k. (Cf. Zariski [**6**, p. 303].) Assume that $q' \geqq 1$. Let $\pi: S \rightarrow A$, $\pi': S' \rightarrow A'$ be the Albanese mappings of S, S', and call B, B' the images of S, S' by π and π' respectively. π and π' are morphisms as is well known. Since k is algebraically closed, A, π, B, A', π', B' are defined over k. We have $q \geqq q' \geqq 1$. Since the geometric genus of S is 0 and so the geometric genus of S' is 0, B and B' are non-singular irreducible curves (Šafarevič [**3**, p. 54, Theorem 3]). There exists a homomorphism $\lambda: A \rightarrow A'$ and a constant $c' \in B'$ such that $\pi' \circ f = \lambda \circ \pi + c'$. Replacing π' by $\pi' - c'$, we may assume that $c'=0$. Thus we have a commutative diagram

$$\begin{array}{ccc} S & \xrightarrow{f} & S' \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi'} \\ B & \xrightarrow{\lambda} & B' \end{array} \; .$$

Let z be a generic point of S over k. Then $b=\pi(z)$, $z'=f(z)$, $b'=\pi'(z')=\lambda(b)$ are generic points of B, S', B' respectively. The locus of z over $k(b)$ is the fibre $F_b=\pi^*(b)$ of π, which is a non-singular curve defined over $k(b)$. (Cf. Šafarevič [**3**, p. 55, Theorem 4] and (1).) We also see that the genus of F_b is zero. Similarly, the locus of z' over $k(b')$ is the fibre $F_{b'}=\pi'^*(b')$ of π', which is a non-singular curve defined over $k(b')$. Clearly, if $P \in F_b$, then $f(P) \in F_{b'}$. Since $k(z')$ is a regular extension of $k(b')$ and since $k(b)$ is algebraic over $k(b')$, $k(z')$ and $k(b)$ are linearly disjoint over $k(b')$. Hence z' is a generic point of $F_{b'}$ over $k(b)$, too. Thus $z \leadsto z'=f(z)$ induces a rational mapping $F_b \rightarrow F_{b'}$ defined over $k(b)$. Since F_b is of zero genus, the genus of $F_{b'}$ is 0. It follows, from this and Noether's Theorem ([**3**, p. 53, Theorem 2]), that S' is a ruled surface over k. Proposition 2 is thereby proved.

§ 2. Surfaces with pencils of hyperelliptic curves.

Let F be a non-singular curve defined over a field k'. F is hyperelliptic over k' if and only if there is a divisor $\mathfrak{a}$ on F rational over k' such that $\deg(\mathfrak{a})=2$ and $\dim L_{F/k'}(\mathfrak{a}) \geqq 2$. It is known that, if a curve F is hyperelliptic over k', then the canonical system on F induces a rational mapping $F \rightarrow C$ to a curve of zero genus (Chevalley [**2**, p. 74, Theorem 9]).

PROPOSITION 3. *If a surface S is a double ruled surface over k, then S is birationally equivalent over k to either a (rational or irrational) ruled surface* *over k, or an elliptic surface, or a surface with a pencil over k of hyperelliptic curves.*

PROOF. By the assumption of Proposition 3, there is a rational mapping over k, $f: S \to \boldsymbol{P}_1 \times B$ such that $[R_k(S): R_k(\boldsymbol{P}_1 \times B)] = \deg f = 2$, where B is a non-singular curve defined over k. Let $\pi_0: \boldsymbol{P}_1 \times B \to B$ be the projection. Its generic fibre $C_b = \pi_0^*(b) = \boldsymbol{P}_1 \times b$ is a curve defined over $k(b)$ and of genus 0. Replacing S by its non-singular model and using Proposition 1, we may assume that S is non-singular and f is a morphism. Hence $\pi = \pi_0 \circ f: S \to B$ is a morphism. Call z a generic point of S over k, and put $(t, b) = f(z) \in \boldsymbol{P}_1 \times B$. We have $b = \pi(z) = \pi_0(t, b)$, and the generic fibre $F_b = \pi^*(b) = f^*(t, b)$ is the locus of z over $k(b)$ and a prime rational cycle over $k(b)$ on S.

CASE 1 where $k(b)$ is algebraically closed in $k(z)$: Then $k(z)$ is a regular extension of $k(b)$, and F_b is a curve defined over $k(b)$. It follows from (1) that F_b is non-singular. If F_b is of genus 0, then S is a ruled surface over k by Noether's Theorem. If F_b is of genus 1, then S is, *by definition*, an elliptic surface, (F_b may not have a rational point over $k(b)$.) If the genus of F_b is $\geqq 2$, then it follows from the isomorphisms

$$R_{k(b)}(F_b) \cong k(b)(z) = k(z) \cong R_k(S)$$

$$F_{k(b)}(C_b) \cong k(b)(t, b) = k(t, b) \cong R_k(\boldsymbol{P}_1 \times B)$$

that $[R_{k(b)}(F_b): R_{k(b)}(C_b)] = 2$ and that F_b is a hyperelliptic curve defined over $k(b)$.

CASE 2 where $k(b)$ is not algebraically closed in $k(z)$: Then call $k(b')$ the algebraic closure of $k(b)$ in $k(z)$. Since $(t, b) = f(z)$ is a generic point of $\boldsymbol{P}_1 \times B$ over k, $k(t)$ and $k(b)$ are linearly disjoint over k, and $k(t, b)$ is a regular extension of $k(b)$. Hence $k(t, b)$ and $k(b')$ are linearly disjoint over $k(b)$, and we have $2 = [k(z): k(t, b)] \geqq [k(t, b'): k(t, b)] = [k(b'): k(b)] > 1$. (Cf. Weil [**4**, p. 5, Proposition 6].) It follows from this that $k(z) = k(t, b')$. Since $k(b')$ is algebraically closed in $k(z)$, $k(t, b') = k(z)$ is a regular extension of $k(b')$. We may assume that b' is the coordinates of a generic point of a non-singular curve B' over k. The inclusion $k(b') \leqq k(z)$ defines a rational mapping $\pi': S \to B'$. In view of Proposition 1, we may assume that π' is a morphism. The fibre $G_{b'} = \pi'^*(b')$ of π' is the locus of z over $k(b')$. It follows, from $R_{k(b')}(G'_{b'}) \cong k(b', z) = k(t, b') \cong R_{k(b')}(\boldsymbol{P}_1)$, that $G_{b'}$ is of genus 0 and that S is a ruled surface over k. Proposition 3 is thereby proved.

Conversely we have

PROPOSITION 4. *If $\pi: S \to B$ is a pencil over k of hyperelliptic curves, then S is a double ruled surface over k with the base B.*

PROOF. Take a generic point z of S over k. Then $b = \pi(z)$ is a generic point of B over k and z is a generic point of the hyperelliptic curve F_b over

$k(b)$. The canonical system of $F_b/k(b)$ induces a rational mapping $g: F_b \to C_b$ of F_b to a curve C_b which is defined over $k(b)$ and of genus 0, and $[R_{k(b)}(F_b): R_{k(b)}(C_b)]=2$ holds. $x=g(z)$ is a generic point of C_b over $k(b)$. Let S' be the locus of (x, b) over k. We have isomorphisms

$$R_{k(b)}(F_b) \cong k(b)(z) = k(z) \cong R_k(S)$$
$$R_{k(b)}(C_b) \cong k(b)(x) = k(x, b) \cong R_k(S'),$$

and $[R_k(S): R_k(S')]=2$. $(x, b) \to b$ defines a rational mapping $\pi_0': S' \to B$. The locus $\pi'^*(b)$ of (x, b) over $k(b)$ is the curve $C_b \times b$ of genus 0. Replacing S' by its non-singular model and using Proposition 1 and (1), we see that there is a pencil over k of curves $\pi_0: S_0 \to B$ such that the generic fibre $\pi_0^*(b)$ is a non-singular curve defined over $k(b)$ and of genus 0 and that $[R_k(S): R_k(S_0)]=2$. It follows from Noether's Theorem that S_0 is a ruled surface over k with the base B. Hence S is a double ruled surface over k with the base B. Proposition 4 is thereby proved.

REMARK. Clearly a ruled surface over k is a double ruled surface over k. However, if $\pi: S \to B$ is a pencil of elliptic curves over k, i.e., if a generic fibre $F_b=\pi^*(b)$ is a curve defined over $k(b)$ of genus 1, we do not know that S is a double ruled surface over k. (F_b may not have a rational point over $k(b)$!)

§ 3. The canonical systems of double ruled surfaces.

PROPOSITION 5. *Let $\pi: S \to B$ be a pencil of curves, and let F be an irreducible non-singular fibre. If F is not a component of a canonical divisor K on S, then the intersection cycle*

$$\mathfrak{k} = F \cdot K$$

is a canonical divisor of F.

PROOF. Let $F=\pi^*(b)$, $b \in B$, and let K be the divisor of a 2-form ω on S. Let k be an algebraically closed field of definition for S, B, π, ω and b. Let τ be an uniformizing parameter of b on B/k, and let $\operatorname{div}(\tau)=b+\sum n_i a_i$. It follows that $t=\tau \circ \pi$ is a uniformizing parameter of F on S/k and that $\operatorname{div}(t)=\pi^*(\operatorname{div}(\tau))=F+\sum n_i F_{a_i}$. It follows that $K+F-\operatorname{div}(t)=K-\sum n_i F_{a_i}$. The divisor of the Poincaré residue $\bar{\omega}$ of ω with respect to t is given by $\operatorname{div}(\bar{\omega})=F \cdot (K+F-\operatorname{div}(t))=F \cdot (K-\sum n_i F_{a_i})=F \cdot K$ (Zariski [7]). Proposition 5 is thereby proved.

Now we shall study the rational mapping induced by the canonical systems on double ruled surfaces. $\Phi_{mK}: S \to S_{mK}$ denotes the rational mapping induced by the pluri-canonical system $|mK|$ on a non-singular surface S. If the geometric genus p of a surface S is $0 \leqq p \leqq 2$, then the rational mapping Φ_K

is trivial, i.e., its image S_K is empty, a point, or the projective line.

PROPOSITION 6. (Cf. Šafarevič [3, p. 120, Lemma 5, 3) $\Rightarrow$ 1)].) *Let S be a non-singular surface defined over k of geometric genus $p \geqq 2$. If S has a pencil $\pi: S \to B$ over k of elliptic curves, i.e., a generic fibre $F_b = \pi^*(b)$ is a non-singular curve defined over $k(b)$ of genus 1, then Φ_{mK} is decomposed as $S \xrightarrow{\pi} B \to S_{mK}$, in particular S_{mK} is a curve.* ($m \geqq 1$).

PROOF. Let $\pi: S \to B$ be a pencil over k, of curves whose generic fibre F_b is irreducible. Let z be a generic point of S over k with $b = \pi(z)$. Then the fibre $F_b = \pi^*(b)$ is the locus of z over $k(b)$ and a non-singular curve defined over $k(b)$. Take a canonical divisor $K \geqq 0$, on S, rational over k. It is a matter of triviality to see that F_b is not defined over k. Hence the intersection cycle $\mathfrak{k}_b = F_b \cdot K$ is defined and a canonical divisor on F_b, rational over $k(b)$, by Proposition 5. Each function $u \in R_k(S)$ is defined along F_b and induces the function $\bar{u} \in R_{k(b)}(F_b)$. We see that $u \to \bar{u}$ induces an isomorphism

$$(3) \qquad R_k(S) \cong R_{k(b)}(F_b)$$

of fields, under which the subfield $R_k(B)$ goes to the constant field $k(b)$. We have, by [4, p. 251, Corollary to Theorem 3], $\operatorname{div}(\bar{u}) = (\operatorname{div}(u)) \cdot F_b$. If $\operatorname{div}(u) + mK \geqq 0$, then we have $\operatorname{div}(\bar{u}) + m\mathfrak{k}_b \geqq 0$. This shows that the mapping $u \to \bar{u}$ induces an injection

$$(4) \qquad L_k(mK) \longrightarrow L_{k(b)}(m\mathfrak{k}_b)\,.$$

Now assume that F_b is of genus 1. Then the canonical divisor $\mathfrak{k}_b$ is the null divisor and we have $L_{k(b)}(m\mathfrak{k}_b) = k(b)$. It follows from this and the isomorphism (3) that $L_k(mK) \leqq R_k(B)$ and that $R_k(S_{mK}) \leqq R_k(B)$. Proposition 6 is thereby proved.

THEOREM 1. *Let S be a non-singular surface defined over k of geometric genus $p \geqq 2$. If S has a pencil $\pi: S \to B$ over k of hyperelliptic curves (therefore, by Proposition 4, there is a rational mapping $f: S \to \boldsymbol{P}_1 \times B$ of degree 2 defined over k), then either* (a) *S_K is a ruled surface over k and Φ_K is decomposed as $S \xrightarrow{f} \boldsymbol{P}_1 \times B \to S_K$, or* (b) *$S_K$ is a curve and Φ_K is decomposed as $S \xrightarrow{\pi} B \to S_K$, or* (c) *$S_K$ is a rational curve over k and Φ_K is decomposed as $S \to \boldsymbol{P}_1 \times B \to S_K$.*

PROOF. We use the isomorphism (3) and the injection (4) in the first half of the proof of Proposition 6. By the assumption of Theorem 1, the generic fibre F_b of π is a hyperelliptic curve defined over $k(b)$. Hence the canonical system $|\mathfrak{k}_b|$ induces the rational mapping of F_b to a curve C_b of genus 0 defined over $k(b)$. The proof of Proposition 6 shows that the isomorphism (3) induces an isomorphism $R_k(\boldsymbol{P}_1 \times B) \cong R_{k(b)}(C_b)$. Hence it follows from $L_{k(b)}(\mathfrak{k}_b) \leqq R_{k(b)}(C_b)$ that $L_k(K) \leqq R_k(\boldsymbol{P}_1 \times B)$ and that

$$R_k(S_K) \leqq R_k(\boldsymbol{P}_1 \times B)\,.$$

(a) If S_K is a surface, then it is a ruled surface over k by Proposition 2.

(b) If $\dim S_K = \text{trans. deg. } k(L_k(K))/k = 1$ and $R_k(B)(L_k(K))$ is an algebraic extension of $R_k(B)$, then we have $R_k(B)(L_k(K)) = R_k(B)$ since $R_k(B)$ is algebraically closed in $R_k(\boldsymbol{P}_1 \times B)$. This implies that $R_k(S_K) = k(L_k(K)) \leqq R_k(B)$ and proves the assertion (b). (c) Finally assume that $\text{trans. deg. } k(L_k(K))/k = 1$ and $R_k(B)(L_k(K))$ is a transcendental extension of $R_k(B)$. Then $R_k(B)(L_k(K))$ is a field of algebraic functions of one variable over $R_k(B)$ of genus 0, since, under (3), $R_k(B)(L_k(K))/R_k(B)$ goes to a subfield of $R_{k(b)}(C_b)/k(b)$ whose genus is 0. It follows from

$$\text{trans. deg. } R_k(B)(L_k(K))/k = 2$$
$$= \text{trans. deg. } R_k(B)/k + \text{trans. deg. } k(L_k(K))/k$$

that $R_k(S_K)$ and $R_k(B)$ are linearly disjoint over k. (Cf. Weil [**4**, p. 18, Theorem 5].) This implies that $R_k(S_K) = k(L_k(K))$ over k has the same genus 0 as that of $R_k(B)(L_k(K))$ over $R_k(B)$ and proves our assertion (c). Theorem 1 is thereby proved.

The followings are immediate consequences of Proposition 3, Proposition 6, Theorem 1 and Lüroth's Theorem for curves.

COROLLARY 1. *If S is a non-singular double ruled surface over k, then the rational map Φ_K induced by the canonical system on S is not birational.*

COROLLARY 2. *If S is a non-singular double plane over k of geometric genus $p \geqq 2$, then the image S_K of the rational mapping induced by the canonical system on S is a rational variety over k of dimension* 1 *or* 2.

REMARK 1. Let S be a non-singular surface defined over k and K be a canonical divisor $\geqq 0$ on S rational over k. It is easy to see that the subfield $k(L_{S/k}(mK))$ of $R_k(S)$ is independent of the choice of models S and canonical divisors $K \geqq 0$. Hence our results in Theorem 1 and its Corollaries are properties of the function-fields and independent of the models S.

REMARK 2. An algebraic surface S is, by definition, *of general type* (or of fundamental type), if and only if, for some $m > 0$, $\dim L(mK) \geqq 2$ and S does not have a pencil of elliptic curves (Šafarevič [**3**, p. 120]). In view of the results in Theorem 1, we are inclined to consider surfaces of general type of geometric genus $p \geqq 2$ with a pencil of hyperelliptic curves as what correspond to hyperelliptic curves. However, differing from the case of dimension one, it will not be true in general that $R_k(S_K) = R_k(\boldsymbol{P}_1 \times B)$ in Theorem 1 even if S_K is a surface as we see it in Example 1 below.

EXAMPLE 1. Let F_i be a hyperelliptic curve defined over k, and $\mathfrak{k}_i \geqq 0$ be a canonical divisor on F_i rational over k, and let C_i be the image of F_i by the rational mapping induced by the canonical system $|\mathfrak{k}_i|$ $(1 \leqq i \leqq 2)$. C_1 and C_2 are curves of genus 0 defined over k. $S = F_1 \times F_2$ is a double ruled surface over k covering the ruled surface $C_1 \times F_2$. $K = \mathfrak{k}_1 \times F_2 + F_1 \times \mathfrak{k}_2$ is a canonical divisor on S. We see easily that $R_k(S_K) = k(L_{S/k}(K)) = R_k(C_1 \times F_2) \cap R_k(F_1 \times C_2) \subsetneqq R_k(C_1 \times F_2)$. This is an example of (a) in Theorem 1 with $R_k(S_K) \neq R_k(\boldsymbol{P}_1 \times B)$.

EXAMPLE 2. Let B be a non-hyperelliptic curve of genus $\geqq 3$, and E be an elliptic curve, both defined over k. Call $\mathfrak{k}$ a canonical divisor $\geqq 0$ on B rational over k. There is a degree 2 rational mapping $E \rightarrow C$ of E to a rational curve C since k is algebraically closed. $S = E \times B$ is a double ruled surface over k covering the ruled surface $C \times B$, and $K = E \times \mathfrak{k}$ is a canonical divisor on S. The linear system $|\mathfrak{k}|$ induces a birational mapping of B. It follows from this that $R_k(S_K) = R_k(B)$, which is an example of (b) in Theorem 1.

EXAMPLE 3. Let B be a hyperelliptic curve, and E be an elliptic curve, both defined over k. Call C the image of the rational mapping induced by the canonical system $|\mathfrak{k}|$ on B. $S = E \times B$ is a double ruled surface over k covering the ruled surface $E \times C$. We see easily that $R_k(S_K) = R_k(B) \cap R_k(E \times C) = R_k(C)$. This gives an example of (c) in Theorem 1.

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