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Noninvariant hypersurfaces of almost contact manifolds

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§ 1. Introduction.

A hypersurface of an almost contact manifold does not in general possess an almost complex structure as is seen by the example of S^4 in R^5 or in S^5. When considered as a unit sphere in R^6, S^5 carries a contact metric structure with respect to which S^4 cannot be imbedded as an invariant hypersurface. In fact, Theorem 5 says that it is impossible to imbed a manifold as an invariant hypersurface of a contact space. This situation is in marked contrast with the well-known fact that a hypersurface (real codimension 1) of an almost complex manifold admits an almost contact structure. However, this hypersurface is clearly not invariant, since the real codimension is 1, for, otherwise it admits an almost complex structure.

We are thereby led to consider noninvariant hypersurfaces of almost contact manifolds M. These again admit almost complex structures, but, in addition, there is a distinguished 1-form α induced by the contact form of M. This situation is examined in detail when the ambient space is affinely cosymplectic.

The metric case is especially interesting. Indeed, if M is quasi-Sasakian (e.g., a normal contact or cosymplectic space) and P is a noninvariant hypersurface, then P carries a symplectic, in fact, a Kaehlerian structure, with Kaehler metric γ.

§ 2. Hypersurfaces of almost contact manifolds.

Let $M(\phi, \xi, \eta)$ be a $(2n+1)$-dimensional almost contact manifold whose structure is defined by a linear transformation field ϕ acting in each tangent space M_m of M, $m \in M$, a vector field ξ on M and a *contact form* η such that

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$$\eta(\xi)=1\,, \qquad \phi\xi=0\,,$$
(2.1)
$$\eta\circ\phi=0\,, \qquad \phi^2=-I_{M_m}+\xi\otimes\eta\,,$$

where I_{M_m} is the identity of M_m.

An almost contact manifold $M(\phi, \xi, \eta)$ is called *normal* if the almost complex structure J' on $M\times R$ given by

$$J'\Big(x, f\frac{d}{dt}\Big)=\Big(\phi x-f\xi,\ \eta(x)\frac{d}{dt}\Big)\,,$$

where f is a C^∞ real-valued function and x is a vector field on M, gives rise to a complex structure on $M\times R$. In this case, the tensor field $[\phi, \phi]+d\eta\otimes\xi$ of type (1,2) vanishes where

$$[\phi, \phi](x, y)=[\phi x, \phi y]-\phi[\phi x, y]-\phi[x, \phi y]+\phi^2[x, y]\,.$$

Consider a $2n$-dimensional manifold P imbedded in M with imbedding map

$$i:\ P\longrightarrow M\,,$$

and assume that for each $m\in P$ the vector $\xi_{i(m)}$ does not belong to the tangent hyperplane of the hypersurface. We therefore have

(2.2) $$\phi i_*X=i_*JX+\alpha(X)\xi\,,$$

(2.3) $$\phi\xi=0\,,$$

where J and α are tensor fields of type (1,1) and (0,1), respectively, on P, and i_* is the differential of i. If $\alpha\neq 0$, then the submanifold $i(P)$ is called a *noninvariant hypersurface* of M. On the other hand, if the 1-form α vanishes, $i(P)$ is called an *invariant hypersurface* of M. A hypersurface may, of course, be neither invariant nor noninvariant. In the remainder of this section, unless specified otherwise $i(P)$ will be a noninvariant hypersurface of the almost contact manifold M. We shall occasionally refer to P as the hypersurface.

Applying ϕ to the relation (2.2) and then using (2.1), (2.2) and (2.2) again, we obtain

$$-i_*X+\eta(i_*X)\xi=i_*(J^2X)+\alpha(JX)\xi\,,$$

from which

(2.4) $$J^2=-I_{P_m}$$

and

(2.5) $$C\alpha=i^*\eta\,,$$

where I_{P_m} is the identity of P_m, $m\in P$, i^* is the dual map of i_* and $C\alpha$ is the 1-form on P defined by $C\alpha(X)=\alpha(JX)$.

Thus, the hypersurface P admits an almost complex structure J and a

1-form α whose vanishing means that the tangent hyperplane of the hypersurface is invariant by ϕ, that is, the hyperplane defined by $\eta(i(m))=0$ is the tangent hyperplane $i(P)_{i(m)}$.

We introduce a symmetric affine connection $\tilde{\nabla}$ on M and define an affine connection ∇ on P with respect to the affine normal ξ by

$$\tilde{\nabla}_{i_*X} i_*Y = i_*\nabla_X Y + h(X, Y)\xi\,, \tag{2.6}$$

where h is a symmetric tensor field of type $(0, 2)$ on P called the *second fundamental form* of P with respect to ξ.

Suppose that the almost contact structure on M is normal. Then, since $[J', J']=0$, the torsion tensor field S on M of type $(1, 2)$ given by

$$S(x, y)=[\phi x, \phi y]-\phi[\phi x, y]-\phi[x, \phi y]+\phi^2[x, y]+d\eta(x, y)\xi \tag{2.7}$$

vanishes. Putting $y=\xi$ in (2.7), we obtain

$$L_\xi\phi=0\,, \qquad L_\xi\eta=0\,, \tag{2.8}$$

where L_ξ is the Lie derivative operator with respect to ξ. The tensor field S is also given by

$$\begin{aligned} S(x, y)=&\tilde{\nabla}_{\phi x}(\phi y)-\tilde{\nabla}_{\phi y}(\phi x)-\phi(\tilde{\nabla}_{\phi x}y-\tilde{\nabla}_y(\phi x)) \\ &-\phi(\tilde{\nabla}_x(\phi y)-\tilde{\nabla}_{\phi y}x)+\phi^2(\tilde{\nabla}_x y-\tilde{\nabla}_y x) \\ &+(\tilde{\nabla}_x\eta(y)-\tilde{\nabla}_y\eta(x)-\eta([x, y]))\xi\,, \end{aligned}$$

or

$$\begin{aligned} S(x, y)=&(\tilde{\nabla}_{\phi x}\phi)y-(\tilde{\nabla}_{\phi y}\phi)x+\phi(\tilde{\nabla}_y\phi)x-\phi(\tilde{\nabla}_x\phi)y \\ &+[(\tilde{\nabla}_x\eta)y-(\tilde{\nabla}_y\eta)x]\xi\,. \end{aligned} \tag{2.9}$$

Thus, by (2.2) and (2.6)

$$\begin{aligned} S(i_*X, i_*Y)=&(\tilde{\nabla}_{i_*JX+\alpha(X)\xi}\phi)i_*Y-(\tilde{\nabla}_{i_*JY+\alpha(Y)\xi}\phi)i_*X \\ &+\phi((\tilde{\nabla}_{i_*Y}\phi)i_*X-(\tilde{\nabla}_{i_*X}\phi)i_*Y)+((\tilde{\nabla}_{i_*X}\eta)i_*Y-(\tilde{\nabla}_{i_*Y}\eta)i_*X)\xi \\ =&(\tilde{\nabla}_{i_*JX}\phi+\alpha(X)\tilde{\nabla}_\xi\phi)i_*Y-(\tilde{\nabla}_{i_*JY}\phi+\alpha(Y)\tilde{\nabla}_\xi\phi)i_*X \\ &+\phi\{(\tilde{\nabla}_{i_*Y}\phi)i_*X-(\tilde{\nabla}_{i_*X}\phi)i_*Y)+((\tilde{\nabla}_{i_*X}\eta)i_*Y-(\tilde{\nabla}_{i_*Y}\eta)i_*X\}\xi \\ =&\tilde{\nabla}_{i_*JX}(\phi i_*Y)-\phi\tilde{\nabla}_{i_*JX}(i_*Y)+\alpha(X)(\tilde{\nabla}_\xi\phi)i_*Y-\tilde{\nabla}_{i_*JY}(\phi i_*X) \\ &+\phi\tilde{\nabla}_{i_*JY}(i_*X)-\alpha(Y)(\tilde{\nabla}_\xi\phi)i_*X \\ &+\phi(\tilde{\nabla}_{i_*Y}(\phi i_*X)-\phi\tilde{\nabla}_{i_*Y}(i_*X)-\tilde{\nabla}_{i_*X}(\phi i_*Y)+\phi\tilde{\nabla}_{i_*X}(i_*Y)) \\ &+\{\tilde{\nabla}_{i_*X}(\eta(i_*Y))-\eta(\tilde{\nabla}_{i_*X}(i_*Y))-\tilde{\nabla}_{i_*Y}(\eta(i_*X))+\eta(\tilde{\nabla}_{i_*Y}(i_*X))\}\xi \\ =&\tilde{\nabla}_{i_*JX}(i_*JY+\alpha(Y)\xi)-\phi(i_*\nabla_{JX}Y+h(JX, Y)\xi)+\alpha(X)(\tilde{\nabla}_\xi\phi)i_*Y \\ &-\tilde{\nabla}_{i_*JY}(i_*JX+\alpha(X)\xi)+\phi(i_*\nabla_{JY}X+h(JY, X)\xi)-\alpha(Y)(\tilde{\nabla}_\xi\phi)i_*X \\ &+\phi\{\tilde{\nabla}_{i_*Y}(i_*JX+\alpha(X)\xi)-\phi(i_*\nabla_Y X+h(Y, X)\xi) \end{aligned}$$

$$
\begin{aligned}
&\qquad -\tilde{\nabla}_{i_*X}(i_*JY+\alpha(Y)\xi)+\phi(i_*\nabla_X Y+h(X,Y)\xi)\}\\
&\quad +\{\tilde{\nabla}_{i_*X}((\alpha\circ J)Y)-\eta(i_*\nabla_X Y+h(X,Y)\xi)\\
&\qquad -\tilde{\nabla}_{i_*Y}((\alpha\circ J)X)+\eta(i_*\nabla_Y X+h(Y,X)\xi)\}\xi\\
&= i_*\nabla_{JX}(JY)+h(JX,JY)\xi+\nabla_{JX}(\alpha(Y))\xi+\alpha(Y)\tilde{\nabla}_{i_*JX}\xi\\
&\qquad -i_*J\nabla_{JX}Y-\alpha(\nabla_{JX}Y)\xi+\alpha(X)(\tilde{\nabla}_\xi\phi)i_*Y\\
&\quad -i_*\nabla_{JY}(JX)-h(JY,JX)\xi-\nabla_{JY}(\alpha(X))\xi-\alpha(X)\tilde{\nabla}_{i_*JY}\xi\\
&\qquad +i_*J\nabla_{JY}X+\alpha(\nabla_{JY}X)\xi-\alpha(Y)(\tilde{\nabla}_\xi\phi)i_*X\\
&\quad +\phi(i_*\nabla_Y(JX)+h(Y,JX)\xi)+\alpha(X)\phi(\tilde{\nabla}_{i_*Y}\xi)+i_*\nabla_Y X-\eta(i_*\nabla_Y X)\xi\\
&\quad -\phi(i_*\nabla_X(JY)+h(X,JY)\xi)-\alpha(Y)\phi(\tilde{\nabla}_{i_*X}\xi)-i_*\nabla_X Y+\eta(i_*\nabla_X Y)\xi\\
&\quad +\{\nabla_X((\alpha\circ J)Y)-\eta(i_*\nabla_X Y)-\nabla_Y((\alpha\circ J)X)+\eta(i_*\nabla_Y X)\}\xi\\
&= i_*\nabla_{JX}(JY)+\nabla_{JX}(\alpha(Y))\xi+\alpha(Y)\tilde{\nabla}_{\phi i_*X-\alpha(X)\xi}\xi\\
&\qquad -i_*J\nabla_{JX}Y-\alpha(\nabla_{JX}Y)\xi+\alpha(X)(\tilde{\nabla}_\xi\phi)i_*Y\\
&\quad -i_*\nabla_{JY}(JX)-\nabla_{JY}(\alpha(X))\xi-\alpha(X)\tilde{\nabla}_{\phi i_*Y-\alpha(Y)\xi}\xi\\
&\qquad +i_*J\nabla_{JY}X+\alpha(\nabla_{JY}X)\xi-\alpha(Y)(\tilde{\nabla}_\xi\phi)i_*X\\
&\quad +i_*J\nabla_Y(JX)+\alpha(\nabla_Y(JX))\xi+\alpha(X)\phi\tilde{\nabla}_{i_*Y}\xi+i_*\nabla_Y X\\
&\quad -i_*J\nabla_X(JY)-\alpha(\nabla_X(JY))\xi-\alpha(Y)\phi\tilde{\nabla}_{i_*X}\xi-i_*\nabla_X Y\\
&\qquad +\{\nabla_X((\alpha\circ J)Y)-\nabla_Y((\alpha\circ J)X)\}\xi\\
&= i_*\{\nabla_{JX}(JY)-\nabla_{JY}(JX)-J(\nabla_{JX}Y-\nabla_Y(JX))\\
&\qquad -J(\nabla_X(JY)-\nabla_{JY}X)-(\nabla_X Y-\nabla_Y X)\}\\
&\quad +(\tilde{\nabla}_\xi\phi+\phi\tilde{\nabla}\xi-(\tilde{\nabla}\xi)\phi)(\alpha(X)i_*Y-\alpha(Y)i_*X)\\
&\quad +\{\nabla_{JX}(\alpha(Y))-\nabla_{JY}(\alpha(X))-\alpha(\nabla_{JX}Y)+\alpha(\nabla_{JY}X)\\
&\qquad -\alpha(\nabla_X(JY))+\alpha(\nabla_Y(JX))+\nabla_X((\alpha\circ J)Y)-\nabla_Y((\alpha\circ J)X)\}\xi\\
&= i_*\{[JX,JY]-J[JX,Y]-J[X,JY]-[X,Y]\}\\
&\qquad +L_\xi\phi\{\alpha(X)i_*Y-\alpha(Y)i_*X\}\\
&\quad +\{(\nabla_{JX}\alpha)Y-(\nabla_{JY}\alpha)X+(\nabla_X\alpha)(JY)-\nabla_Y\alpha(JX)\}\xi,
\end{aligned}
$$

that is,

$$
\begin{aligned}
S(i_*X, i_*Y) &= i_*[J,J](X,Y)+L_\xi\phi\{\alpha(X)i_*Y-\alpha(Y)i_*X\}\\
&\quad +\{d\alpha(JX,Y)+d\alpha(X,JY)\}\xi.
\end{aligned}
\tag{2.10}
$$

Hence, we have

THEOREM 1. *A noninvariant hypersurface of a normal almost contact manifold $M(\phi,\xi,\eta)$ is a complex manifold carrying a 1-form $\alpha=C^{-1}i^*\eta$ whose differential has bidegree (1, 1) with respect to the complex structure J.*

COROLLARY ([2], [4]). *An invariant hypersurface of an almost contact manifold is an almost complex manifold. If the almost contact structure is normal, then the almost complex structure is integrable.*

The above computation also yields

THEOREM 2. *Let ξ be an infinitesimal automorphism of the almost contact structure $M(\phi, \xi, \eta)$. If, for every noninvariant hypersurface,* (a) *the induced almost complex structure J is integrable and* (b) *the differential of the induced 1-form $C^{-1}i^*\eta$ is of bidegree* (1, 1) *with respect to J, then $M(\phi, \xi, \eta)$ is normal.*

§ 3. Hypersurfaces of affinely cosymplectic and normal contact manifolds.

Let M be an almost contact manifold with a symmetric connection $\tilde{\nabla}$ and denote by ∇ the induced connection on the noninvariant hypersurface P [see formula (2.6)]. If we put

$$(\nabla_X i_*)Y = \tilde{\nabla}_{i_*X} i_* Y - i_* \nabla_X Y ,$$

the equation of Gauss and Weingarten are

$$(\nabla_X i_*)Y = h(X, Y)\xi , \qquad h(X, Y) = h(Y, X) , \tag{3.1}$$

and

$$\tilde{\nabla}_{i_*X}\xi = -i_* HX + \omega(X)\xi , \tag{3.2}$$

where h and H are the second fundamental tensors (of type (0, 2) and (1, 1), respectively) of P with respect to ξ, and ω is a 1-form on P defining the connection on the affine normal bundle.

Covariant differentiation of (2.2), in which X is replaced by Y, yields after applying (3.1), (3.2) and then (2.2) again

$$\begin{aligned}(\tilde{\nabla}_{i_*X}\phi)i_*Y = {} & [h(X, JY) + (\nabla_X \alpha)(Y) + \alpha(Y) \cdot \omega(X)]\xi \\ & + i_*[(\nabla_X J)Y - \alpha(Y)HX] .\end{aligned} \tag{3.3}$$

Case I: *M is affinely cosymplectic.* An almost contact manifold $M(\phi, \xi, \eta)$ with a symmetric affine connection $\tilde{\nabla}$ is said to be *affinely cosymplectic* (see [5]) if

$$\tilde{\nabla}\phi = 0 , \qquad \tilde{\nabla}\eta = 0 . \tag{3.4}$$

An affinely cosymplectic manifold is clearly normal as is seen from (2.9). From (2.1) one easily sees that the relations (3.4) imply that

$$\tilde{\nabla}\xi = 0 \tag{3.5}$$

also. Hence, by (3.2), $HX = 0$ and $\omega(X) = 0$. Moreover, $\tilde{\nabla}\phi$ being zero, we also have

$$\nabla J = 0$$

and

$$(\nabla_X \alpha)(Y) = -h(X, JY) .$$

Case II: *M is affinely Sasakian.* An almost contact manifold $M(\phi, \xi, \eta)$ with a symmetric affine connection $\tilde{\nabla}$ is said to be *affinely Sasakian* if it is normal and

$$\phi = \tilde{\nabla} \xi .$$

If $M(\phi, \xi, \eta)$ is affinely Sasakian, then by (2.2) and (3.2), we have

$$i_* JX + \alpha(X)\xi = -i_* HX + \omega(X)\xi ,$$

that is

$$J = -H \tag{3.6}$$

and

$$\alpha = \omega . \tag{3.7}$$

If M is Sasakian (see § 4), that is, if M is a normal contact manifold with a compatible metric g, then M is affinely Sasakian with respect to the Riemannian connection, so the relation (3.6) cannot hold; for, H is symmetric and J is skew symmetric with respect to g. Thus, there are no noninvariant hypersurfaces of a Sasakian manifold.

If for every vector field X on P, $HX = 0$, then $\tilde{\nabla}_{i_* X}\xi$ and ξ are proportional by (3.2). Hence the affine normals are parallel along the hypersurface. In this case, P is said to be *totally flat.*

PROPOSITION 1. *Let P be a noninvariant hypersurface of an affinely cosymplectic manifold. Then P is totally flat and*

$$\nabla J = 0 ,$$
$$(\nabla_X \alpha)(Y) = -h(X, JY) ,$$
$$\omega = 0 .$$

COROLLARY. *Let P be an invariant hypersurface of an affinely cosymplectic manifold. Then*

$$\nabla J = 0 ,$$
$$h = 0 ,$$
$$\omega = 0 .$$

PROPOSITION 2. *Let P be a noninvariant hypersurface of an affinely Sasakian manifold. Then*

$$J = -H$$

and

$$\alpha = \omega .$$

PROPOSITION 3. *There are no noninvariant hypersurfaces of a Sasakian manifold.*

If the affine normal ξ of $i(P)$ is torse-forming (see [7]), P is said to be *affinely umbilical*. In this case, $H=\lambda I$ for some function λ on $i(P)$. If the ambient space is affinely cosymplectic, λ vanishes. On the other hand, if the ambient space is affinely Sasakian, then, by Proposition 2, the hypersurface cannot be totally umbilical.

PROPOSITION 4. *A noninvariant hypersurface of an affinely Sasakian space cannot be affinely umbilical.*

§4. Hypersurfaces of almost contact metric spaces.

An almost contact manifold $M(\phi, \xi, \eta)$ admits a Riemannian metric g such that

$$\begin{aligned} &g(\phi X, Y)=-g(X, \phi Y), \\ &g(X, \xi)=\eta(X), \end{aligned} \tag{4.1}$$

and in this case we denote the manifold with structure (ϕ, ξ, η) by $M(\phi, \eta, g)$.

Let $P(J, \alpha, G)$ be a noninvariant hypersurface of $M(\phi, \eta, g)$ where G is the induced metric on P, that is $G=i^*g$. By (4.1),

$$g(\phi i_*X, i_*Y)=-g(i_*X, \phi i_*Y),$$

so by (2.2)

$$g(i_*JX, i_*Y)+\alpha(X)\eta(i_*Y)=-g(i_*X, i_*JY)-\alpha(Y)\eta(i_*X).$$

The induced metric G on $P(J, \alpha)$ is given by

$$G(X, Y)=g(i_*X, i_*Y).$$

Hence by (2.5)

$$G(JX, Y)+\alpha(X)C\alpha(Y)=-G(X, JY)-\alpha(Y)C\alpha(X),$$

that is

$$(G+\alpha\otimes\alpha)(JX, Y)=-(G+\alpha\otimes\alpha)(X, JY).$$

PROPOSITION 5. *The noninvariant hypersurface $P(J, \alpha, G)$ of the almost contact manifold $M(\phi, \eta, g)$ admits an hermitian metric*

$$G^*=G+\alpha\otimes\alpha. \tag{4.2}$$

We show that a Kaehlerian metric can be defined on P. To this end, put

$$\Omega^*(X, Y)=G^*(JX, Y) \tag{4.3}$$

and

$$\Phi(x, y)=g(\phi x, y), \tag{4.4}$$

where x and y are vector fields on M. The 2-forms Ω^* and Φ are known as the *fundamental forms* of $P(J, G^*)$ and $M(\phi, \eta, g)$, respectively. Then,

$$\begin{aligned}\Phi(i_*X, i_*Y) &= g(\phi i_*X, i_*Y)\\ &= g(i_*JX+\alpha(X)\xi, i_*Y)\\ &= g(i_*JX, i_*Y)+\alpha(X)g(\xi, i_*Y)\\ &= G(JX, Y)+\alpha(X)\eta(i_*Y)\\ &= G(JX, Y)+\alpha(X)C\alpha(Y)\\ &= G^*(JX, Y)-C\alpha(X)\alpha(Y)+\alpha(X)C\alpha(Y)\,,\end{aligned}$$

that is,

$$i^*\Phi = \Omega^* - C\alpha\wedge\alpha\,. \tag{4.5}$$

Since the 2-form Φ is of maximal rank and i is a regular map, the tensor $\gamma = G - C\alpha\otimes C\alpha$ defines a positive definite Riemannian metric, and it is easily checked that it is hermitian with respect to J. In fact, if Φ is closed, γ is an almost Kaehler metric and $\Omega^*-C\alpha\wedge\alpha$ is the fundamental 2-form of the almost Kaehler manifold $P(J, \gamma)$. If the structure on M is normal, $P(J, \gamma)$ is Kaehlerian.

A $(2n+1)$-dimensional manifold M carrying a 1-form η with the property

$$\eta\wedge(d\eta)^n \neq 0$$

is said to have a *contact structure*, and in this case, M is called a *contact manifold.* It is well-known that on a contact manifold there exists an almost contact metric structure (ϕ, η, g) with contact form η defining the contact structure and

$$g(\phi x, y) = d\eta(x, y)\,.$$

A normal contact metric manifold is also called a *Sasakian manifold.*

An almost contact metric manifold is *almost cosymplectic* if its fundamental and contact forms are both closed. If, in addition, the structure is normal, it is called *cosymplectic*, and in this case, the contact form has vanishing covariant derivative. Examples are provided by $N\times R$ or $N\times S^1$ where N is an (almost) Kaehler manifold. For complete simply connected cosymplectic spaces by the de Rham decomposition theorem, one sees that the only examples are products with one factor Kaehlerian [3].

An almost contact metric structure is said to be *quasi-Sasakian* if it is normal and its fundamental form is closed (see [1]). Thus, Sasakian and cosymplectic manifolds are quasi-Sasakian.

THEOREM 3. *Let $M(\phi, \eta, g)$ be a quasi-Sasakian manifold and $P(J, \alpha, G)$ a noninvariant hypersurface of M with metric G induced by g. Then, $P(J, \alpha, \gamma)$ is Kaehlerian.*

COROLLARY ([2]). *Let $M(\phi, \eta, g)$ be a quasi-Sasakian manifold and $P(J, G)$ an invariant hypersurface of M with metric G induced by g. Then, $P(J, G)$ is a Kaehler manifold.*

We show that the hermitian metric G^* of Proposition 5 is also a Kaehler metric provided the ambient space is cosymplectic and $\alpha=0$ is completely integrable.

Let $M(\phi, \eta, g)$ be a cosymplectic manifold. Then, since Φ is closed,

$$(4.6)\qquad (\tilde{\nabla}_{i_*X}\Phi)(i_*Y, i_*Z)+(\tilde{\nabla}_{i_*Y}\Phi)(i_*Z, i_*X)+(\tilde{\nabla}_{i_*Z}\Phi)(i_*X, i_*Y)=0.$$

On the other hand, from (4.5)

$$(4.7)\qquad \nabla_X(i^*\Phi)=\nabla_X\Omega^*-\nabla_X C\alpha\wedge\alpha-C\alpha\wedge\nabla_X\alpha.$$

Substituting (4.7) in (4.6), we get

$$\begin{aligned}
&(\nabla_X\Omega^*)(Y, Z)+(\nabla_Y\Omega^*)(Z, X)+(\nabla_Z\Omega^*)(X, Y)\\
&\qquad -\{(\nabla_X C\alpha)Y\cdot\alpha(Z)+C\alpha(Y)(\nabla_X\alpha)Z\\
&\qquad -(\nabla_X C\alpha)Z\cdot\alpha(Y)-C\alpha(Z)(\nabla_X\alpha)Y\\
&\qquad +(\nabla_Y C\alpha)Z\cdot\alpha(X)+C\alpha(Z)(\nabla_Y\alpha)X\\
&\qquad -(\nabla_Y C\alpha)X\cdot\alpha(Z)-C\alpha(X)(\nabla_Y\alpha)Z\\
&\qquad +(\nabla_Z C\alpha)X\cdot\alpha(Y)+C\alpha(X)(\nabla_Z\alpha)Y\\
&\qquad -(\nabla_Z C\alpha)Y\cdot\alpha(X)-C\alpha(Y)(\nabla_Z\alpha)X\}\\
&=0.
\end{aligned}$$

But, since η is closed,

$$(\nabla_X C\alpha)Y=(\nabla_Y C\alpha)X,$$

so that

$$\begin{aligned}
&d\Omega^*(X, Y, Z)+C\alpha(X)d\alpha(Y, Z)\\
&\qquad +C\alpha(Y)d\alpha(Z, X)+C\alpha(Z)d\alpha(X, Y)=0.
\end{aligned}$$

By Theorem 1, since M is normal, J is integrable and $d\alpha$ is of bidegree (1, 1). Hence,

$$\begin{aligned}
&d\Omega^*(X, Y, Z)+C\alpha(X)d\alpha(JY, JZ)\\
&\qquad +C\alpha(Y)d\alpha(JZ, JX)+C\alpha(Z)d\alpha(JX, JY)=0.
\end{aligned}$$

If $\alpha=0$ is completely integrable, then $d\alpha=\alpha\wedge\beta$ for some 1-form β. Thus, in particular

$$\begin{aligned}
C\alpha(X)d\alpha(JY, JZ)&=C\alpha(X)\cdot\alpha\wedge\beta(JY, JZ)\\
&=C\alpha(X)\{C\alpha(Y)C\beta(Z)-C\alpha(Z)C\beta(Y)\}.
\end{aligned}$$

We conclude that Ω^* is closed and the metric $G+\alpha\otimes\alpha$ is a Kaehler metric.

THEOREM 4. *If $M(\phi, \eta, g)$ is a cosymplectic manifold, then the hypersurface*

$P(J, \alpha, G+\alpha\otimes\alpha)$ is Kaehlerian if and only if $\alpha=0$ is completely integrable.

From (4.7), we obtain

COROLLARY 4.1. *An invariant hypersurface of an (almost) cosymplectic manifold is an (almost) Kaehler manifold.*

An application of Proposition 3 also yields

COROLLARY 4.2. *Let $M(\phi, \eta, g)$ be a cosymplectic manifold and $P(J, \alpha, G)$ a totally flat hypersurface of M. Then, the manifold $P(J, \alpha)$ with metric $G+\alpha\otimes\alpha$ is Kaehlerian.*

THEOREM 5. *There does not exist an invariant hypersurface of a contact manifold.*

PROOF. Let $i(P)$ be an invariant hypersurface of the contact manifold M. Since the fundamental form Φ of M is derived, namely, $\Phi=d\eta$, and since $i(P)$ is invariant, $i^*\eta$ vanishes. Hence

$$i^*\Phi=i^*d\eta=di^*\eta=0\ .$$

Moreover, $i(P)$ being invariant, we have

$$\phi i_*X=i_*JX\,,$$

from which

$$0=i^*\Phi=\Omega\,,$$

which is impossible.

Theorem 5 also follows from a theorem of Sasaki: The highest dimension of integrable submanifolds of the contact distribution of a contact manifold of dimension $2n+1$ is n. ([6], Theorem 17.3).

COROLLARY. *S^4 cannot be imbedded as an invariant hypersurface of R^5 or S^5 regarded as a (normal) contact manifold.*

This is also a consequence of the corollary to Theorem 3 since the second Betti number of S^4 vanishes.

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