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On injective modules

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In [3] C. Faith and E. A. Walker gave a characterization of a left artinian ring in terms of module theories. That is, a ring B is left artinian if and only if every injective left B-module is a direct sum of injective hulls of simple left B-modules. Under the assumption that a ring B is commutative, P. Vámos investigated in [9] some conditions for B to be locally artinian. One part of this paper is concerned with these results, that is, we give some conditions for a commutative ring R such that there exists a finitely generated injective R-module. The details are the following: Let R be a commutative ring with the noetherian total quotient ring. Then we have the followings: (1) There is a torsion-free and finitely generated injective R-module if and only if there exists a maximal ideal $\mathfrak{M}$ in R such that $R_{\mathfrak{M}}$ is an artinian local ring (Theorem 1). (2) There is a cyclic injective R-module if and only if there exists a maximal ideal $\mathfrak{M}$ in R such that $R_{\mathfrak{M}}$ is a self-injective artinian local ring (Theorem 3).

The other part of this paper is concerned with the ring property of an injective hull of a commutative ring. Let B be a ring and let $E_B(B)$ be an injective hull of B. Then we call $E_B(B)$ a B-*algebra* only when $E_B(B)$ (identifying B with its canonical image in $E_B(B)$) has a B-algebra structure and the multiplication between an element of B and an element of $E_B(B)$ as a B-algebra coincides with the multiplication as a B-module. In [7] B. L. Osofsky gave an example of a non commutative ring B whose injective hull is not a B-algebra. Even when a ring is commutative, such a ring exists (Theorem 4). Now we give here a necessary and sufficient condition for a commutative ring of special type such that its injective hull is an R-algebra. The result is the following: Let R be a commutative ring whose total quotient ring is artinian. Then an injective hull of R can be made into an R-algebra if and only if the total quotient ring of R is a self-injective ring (Theorem 6).

In this paper we assume always that a ring is commutative and has a unit element and a module is unitary. Let R be a ring. We denote an injective hull of an R-module M by $E_R(M)$, the set of all the regular elements ($=$non zero-divisors) in R by $S(R)$, and the total quotient ring of R by $Q(R)$.

If it is clear from context we sometimes denote by S (resp. Q) instead of $S(R)$ (resp. $Q(R)$). Terminologies and notations are due to [**1**] and [**5**].

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§1. Existence of an injective module of finite type.

First we observe the properties of an injective hull of a simple module over a noetherian local ring.

LEMMA 1. *Let R be a noetherian local ring with the maximal ideal $\mathfrak{M}$. Then the following conditions are equivalent.*

(1) *R is an artinian local ring with $\mathfrak{M}^n=0$ for some $n>0$.*

(2) *$A_n=E_R(R/\mathfrak{M})$, where $A_i=\{x\in E_R(R/\mathfrak{M})\mid \mathfrak{M}^i x=0\}$.*

(3) *$E_R(R/\mathfrak{M})$ is a finitely generated R-module.*

PROOF. The equivalence of (1) and (3) follows from Theorem 5 of [**8**]. (1)→(2): This is trivial. (2)→(1): Assume that $A_n=E_R(R/\mathfrak{M})$ and $A_n\neq A_{n-1}$. Now, if $\mathfrak{M}^n\neq 0$, then there is an R-homomorphism f of Rx $(0\neq x\in\mathfrak{M}^n)$ into $E_R(R/\mathfrak{M})$ defined by $f(x)=y$, where $y\in A_1=\{z\in E_R(R/\mathfrak{M})\mid\mathfrak{M}z=0\}$. Since $E_R(R/\mathfrak{M})$ is injective, there is an R-homomorphism g of R into $E_R(R/\mathfrak{M})$ such that the restriction of g to Rx is f. Set $g(1)=a$. Then $y=f(x)=g(x)=xa$ and $x\in\mathfrak{M}^n$, and so $xa=0$. This is impossible. Thus $\mathfrak{M}^n=0$.

When R is a noetherian ring, in [**6**] E. Matlis showed that $E_R(R/\mathfrak{P})\cong E_{R_\mathfrak{P}}(R_\mathfrak{P}/\mathfrak{P}R_\mathfrak{P})$ as an R-module and as an $R_\mathfrak{P}$-module for any prime ideal $\mathfrak{P}$ of R. We can now omit the condition that R is noetherian.

LEMMA 2. *Let R be a ring and $\mathfrak{P}$ be a prime ideal of R. Then $E_R(R/\mathfrak{P})\cong E_{R_\mathfrak{P}}(R_\mathfrak{P}/\mathfrak{P}R_\mathfrak{P})$ as an R $(R_\mathfrak{P})$-module.*

PROOF. We first prove that $E_R(R/\mathfrak{P})$ can be regarded as an $R_\mathfrak{P}$-module. For any $r\in R-\mathfrak{P}$, we define an R-homomorphism $T_r\colon E_R(R/\mathfrak{P})\to E_R(R/\mathfrak{P})$ by $T_r(x)=rx$ for all x in $E_R(R/\mathfrak{P})$. Then T_r is a monomorphism. In fact, if $rx=0$ $(0\neq x\in E_R(R/\mathfrak{P}))$, then $Rx\cap R/\mathfrak{P}\neq 0$ because $E_R(R/\mathfrak{P})$ is an essential extension of $R/\mathfrak{P}$, and $r\,(Rx\cap R/\mathfrak{P})=0$. Since $\mathfrak{P}$ is a prime ideal of R, this is impossible. Thus $\operatorname{Ker}(T_r)=0$.

As $E_R(R/\mathfrak{P})$ is an indecomposable injective R-module by Theorem 2.4 of [**6**] and as T_r is a monomorphism, T_r is an automorphism. For any $r\in R-\mathfrak{P}$ and for any $x\in E_R(R/\mathfrak{P})$, there is only one element y in $E_R(R/\mathfrak{P})$ such that $ry=x$. Therefore $E_R(R/\mathfrak{P})$ can be regarded as an $R_\mathfrak{P}$-module. Next we show that $E_R(R/\mathfrak{P})$ is injective as an $R_\mathfrak{P}$-module. Let $\mathfrak{A}$ be any ideal of $R_\mathfrak{P}$ and let $f\in\operatorname{Hom}_R(\mathfrak{A}, E_R(R/\mathfrak{P}))$. Then f can be regarded as an R-homomorphism. As $E_R(R/\mathfrak{P})$ is injective as an R-module, there is an R-homomorphism g of $R_\mathfrak{P}$ into $E_R(R/\mathfrak{P})$ such that the restriction of g to $\mathfrak{A}$ is f. g can be regarded as an $R_{\mathfrak{P}}$-homomorphism which is an extension of f because for any $r \in R-\mathfrak{P}$, T_r is an automorphism. Thus $E_R(R/\mathfrak{P})$ is injective as an $R_{\mathfrak{P}}$-module. Consider an injective hull $E_{R_{\mathfrak{P}}}(R_{\mathfrak{P}}/\mathfrak{P}R_{\mathfrak{P}})$ in $E_R(R/\mathfrak{P})$ of $R_{\mathfrak{P}}/\mathfrak{P}R_{\mathfrak{P}}$, as an $R_{\mathfrak{P}}$-module. Then since $E_R(R/\mathfrak{P})$ is an indecomposable injective R-module and since any $R_{\mathfrak{P}}$-module can be regarded as an R-module, we have $E_R(R/\mathfrak{P})=E_{R_{\mathfrak{P}}}(R_{\mathfrak{P}}/\mathfrak{P}R_{\mathfrak{P}})$. The proof is completed.

Let R be a ring and let M be an R-module. Then M is said to be *torsion-free* in case for any element x in M and for any element s in $S(R)$, $sx=0$ implies $x=0$, and M is said to be *torsion* in case for any x in M there is an element s in $S(R)$ such that $sx=0$.

It is easily seen that for any R-module M, $t(M)=\{x \in M \mid sx=0$ for some s in $S(R)\}$ is a unique maximal submodule of M which is torsion (this submodule of M is called *the torsion submodule* of M and is denoted by $t(M)$), and that M is torsion-free if and only if $t(M)=0$. For any R-module M, M is called *a divisible R-module* if $sM=M$ for all s in $S(R)$.

Let M be an R-module. Then, if M is injective, M is divisible. It is easily seen that if M is a torsion-free and divisible R-module, then M can be regarded as a $Q(R)$-module.

THEOREM 1. *Let R be a ring with the noetherian total quotient ring. Then there is a torsion-free and finitely generated injective R-module if and only if there exists a maximal ideal $\mathfrak{M}$ in R such that $R_{\mathfrak{M}}$ is an artinian local ring.*

PROOF. If $R_{\mathfrak{M}}$ is an artinian local ring for some maximal ideal $\mathfrak{M}$ in R, then by Theorem 5 of [8] $E_R(R/\mathfrak{M})$ is a finitely generated R-module. On the other hand, as $\mathfrak{M}R_{\mathfrak{M}}$ is nilpotent, $\mathfrak{M}$ is not a regular ideal of R, and so $S(R) \subseteq R-\mathfrak{M}$. Thus $E_R(R/\mathfrak{M})$ is torsion-free because $E_R(R/\mathfrak{M}) \cong E_{R_{\mathfrak{M}}}(R_{\mathfrak{M}}/\mathfrak{M}R_{\mathfrak{M}})$ by Lemma 2.

Assume that there is a torsion-free and finitely generated injective R-module M. Then M can be regarded as a $Q(R)$-module and it is injective as a $Q(R)$-module by Lemma 2.1 of [5]. Since $Q=Q(R)$ is a noetherian ring, using Proposition 3.1 of [6], M can be expressed as follows:

$$M=\Sigma \oplus E_Q(Q/\mathfrak{P}_i'),$$

where $\mathfrak{P}_i'$ is a prime ideal of Q for all i. Thus $E_Q(Q/\mathfrak{P}_i')$ is finitely generated as an R-module, a fortiori as a Q-module. Furthermore, $\mathfrak{P}_i=\mathfrak{P}_i' \cap R$ is not regular, and so the quotient field of $Q/\mathfrak{P}_i'$ is equal to that of $R/\mathfrak{P}_i$. On the other hand, by Theorem 3.4 of [6],

$$A_1=\{x \in E_Q(Q/\mathfrak{P}_i') \mid x\mathfrak{P}_i'=0\}$$

is isomorphic to the quotient field of $Q/\mathfrak{P}_i'$ as a vector space. Moreover A_1 is finitely generated as a Q-module because Q is noetherian and A_1 is a submodule of a finitely generated R-module $E_Q(Q/\mathfrak{P}_i')$. Thus the quotient field of

$Q/\mathfrak{P}'_i$ is integral over $Q/\mathfrak{P}'_i$, and hence $\mathfrak{P}'_i$ is a maximal ideal of Q. From the facts that $A_n/A_{n-1}(A_j=\{x\in E_Q(Q/\mathfrak{P}'_i)\mid x(\mathfrak{P}'_i)^j=0\}$, $A_n=E_Q(Q/\mathfrak{P}'_i)$, and $A_n\neq A_{n-1}$) is a finite dimensional vector space over $Q/\mathfrak{P}'_i$ and is a finitely generated R-module, and that the quotient field of $R/\mathfrak{P}_i$ is equal to $Q/\mathfrak{P}'_i$, we have that the quotient field of $R/\mathfrak{P}_i$ is integral over $R/\mathfrak{P}_i$. Hence $\mathfrak{P}_i$ is a maximal ideal of R.

Next, let us prove that $R_{\mathfrak{P}_i}$ is artinian. Since $E_Q(Q/\mathfrak{P}'_i)=E_{Q_{\mathfrak{P}'_i}}(Q_{\mathfrak{P}'_i}/\mathfrak{P}'_iQ_{\mathfrak{P}'_i})$ by Lemma 2, $Q_{\mathfrak{P}'_i}$ is an artinian local ring by Lemma 1. Furthermore, $Q_{\mathfrak{P}'_i}\supseteqq R_{\mathfrak{P}_i}$ and $S(R)\subseteqq R-\mathfrak{P}$, and so, we have $R_{\mathfrak{P}_i}=Q_{\mathfrak{P}'_i}$.

Next let us consider a condition for a ring R that the torsion submodule of any finitely generated injective R-module is zero.

LEMMA 3. *Let M be a cyclic module over a ring R. Then the torsion submodule of M is isomorphic to $(\mathfrak{A}Q\cap R)/\mathfrak{A}$, where $M=R/\mathfrak{A}$.*

A torsion R-module is *of bounded order* in case it is annihilated by some element in $S(R)$.

LEMMA 4. *Let $x=(x_i)_{i=1,2,\cdots,n}$ be a finite set of zero-divisors in a ring R. Then the torsion submodule of R/Rx_i is of bounded order if and only if the set $F_x=\{(Rx:Rs)\mid s\in S(R)\}$ satisfies the maximal condition, where $Rx=\Sigma Rx_i$.*

PROPOSITION 1. *Let R be a ring with the noetherian total quotient ring and assume that for any finite set $x=(x_i)_{i=1,2,\cdots,n}$ of zero-divisors in R, the set $F_x=\{(Rx:Rs)\mid s\in S(R)\}$ satisfies the maximal condition. Then the torsion submodule of any finitely generated R-module is of bounded order.*

PROOF. Let $M=R/\mathfrak{A}$, where $\mathfrak{A}$ is an ideal of R. If $\mathfrak{A}$ is a regular ideal of R, then $t(M)$ is of bounded order. Assume that $\mathfrak{A}$ is not regular. Then $\mathfrak{A}Q$ is a proper ideal of Q and it is finitely generated since Q is noetherian. We may assume that $\mathfrak{A}Q=Qy_1+Qy_2+\cdots+Qy_n$ any $y_i\in\mathfrak{A}$ for all i. Now, by Lemma 3 $(\mathfrak{A}Q\cap R)/\mathfrak{A}$ is the torsion submodule of $R/\mathfrak{A}$. By the assumption and by Lemma 4, $(\mathfrak{A}Q\cap R)/\Sigma Ry_i$ is of bounded order.

But $(\mathfrak{A}Q\cap R)/\mathfrak{A}$ is a homomorphic image of $(\mathfrak{A}Q\cap R)/\Sigma Ry_i$, and so $(\mathfrak{A}Q\cap R)/\mathfrak{A}$ is of bounded order. Therefore the torsion submodule of any cyclic R-module is of bounded order. Let M be a finitely generated R-module with the generators $z_1, z_2, \cdots, z_t$. If $t=1$, then the proposition is true. Assume that for any R-module generated by at most $t-1$ elements the proposition is true. Consider the following exact sequence:

$$0\longrightarrow Rz_1\longrightarrow M\longrightarrow M/Rz_1\longrightarrow 0.$$

Then we have the exact sequence: $0\rightarrow t(Rz_1)\rightarrow t(M)\rightarrow t(M/Rz_1)$ with $t(Rz_1)$ and $t(M/Rz_1)$ being of bounded order by the induction hypothesis. Thus $t(M)$ is of bounded order.

THEOREM 2. *Let R be a ring with the noetherian total quotient ring and* *assume that for any finite set* $x=(x_i)_{i=1,2,\cdots,n}$ *of zero-divisors in* R, *the set* $F_x=\{(Rx:Rs)\mid s\in S(R)\}$ *satisfies the maximal condition. Then there is a finitely generated injective* R-*module if and only if there exists a maximal ideal* $\mathfrak{M}$ *of* R *such that* $R_{\mathfrak{M}}$ *is artinian local.*

PROOF. By Proposition 1, the torsion submodule of any finitely generated R-module is of bounded order. Furthermore, the torsion submodule of any divisible R-module is also divisible. Hence the torsion submodule of any finitely generated injective R-module is zero, and so every finitely generated injective R-module is always torsion-free. From this we have the result by Theorem 1.

If R is a noetherian ring, then for every finitely generated R-module M, $t(M)$ is of bounded order. Thus we have the following corollary.

COROLLARY 1. *Let* R *be a noetherian ring. Then there is a finitely generated injective* R-*module if and only if there is a maximal ideal* $\mathfrak{M}$ *in* R *such that* $R_{\mathfrak{M}}$ *is artinian local.*

If R is an integral domain, then the torsion submodule of any finitely generated R-module is of bounded order. For any proper prime ideal $\mathfrak{P}$ of R, $R_{\mathfrak{P}}$ is not artinian. Hence we have the following corollary.

COROLLARY 2. *Let* R *be an integral domain. Then if there is a finitely generated injective* R-*module,* R *is a field.*

The author does not know whether there is a ring R such that there is a finitely generated injective R-module which is not torsion-free and not cyclic.

§ 2. Existence of a cyclic injective module.

In this section, we investigate the conditions for a ring over which there is a cyclic injective module.

LEMMA 5. *Let* R *be a ring and let* $\mathfrak{A}$ *be an ideal of* R. *Suppose that* $R/\mathfrak{A}$ *is divisible as an* R-*module. Then* $R/\mathfrak{A}$ *is a torsion-free* R-*module.*

PROOF. If $\mathfrak{A}$ is a regular ideal of R, then $R/\mathfrak{A}$ is divisible and a torsion R-module of bounded order, and so $R/\mathfrak{A}=0$. If $\mathfrak{A}$ is not a regular ideal of R, then $S(R)\subset R-\mathfrak{A}$ and hence, for any s in $S(R)$ the class $\bar{s}$ in $R/\mathfrak{A}$ containing s is not zero in $R/\mathfrak{A}$. By the assumption $\bar{s}(R/\mathfrak{A})=R/\mathfrak{A}$ for any s in $S(R)$. Thus $\bar{s}$ is invertible in $R/\mathfrak{A}$. Hence $R/\mathfrak{A}$ is torsion-free as an R-module.

THEOREM 3. *Let* R *be a ring with the noetherian total quotient ring. Then there is a cyclic injective* R-*module if and only if there exists a maximal ideal* $\mathfrak{M}$ *in* R *such that* $R_{\mathfrak{M}}$ *is a self-injective artinian local ring.*

PROOF. Assume that $R_{\mathfrak{M}}$ is a self-injective artinian local ring for some maximal ideal $\mathfrak{M}$ in R. Then $R_{\mathfrak{M}}$ is a finitely generated R-module. In fact, set $A_i=\{x\in R_{\mathfrak{M}}\mid \mathfrak{M}^iR_{\mathfrak{M}}x=0\}$. Then, by Theorem 1, $A_n=R_{\mathfrak{M}}$ for some positive integer n and A_i is a finitely generated $R_{\mathfrak{M}}$-module for $i=1, 2, \cdots, n$, and so A_i/A_{i-1} is finitely generated as an $R_{\mathfrak{M}}/\mathfrak{M}R_{\mathfrak{M}}$ $(=R/\mathfrak{M})$-module for $i=1, 2, \cdots, n$. Thus, $R_{\mathfrak{M}}$ is a finitely generated R-module.

Set $\mathfrak{A}=\{x\in R \mid xr=0$ for some r in $R-\mathfrak{M}\}$. Since $R_{\mathfrak{M}}$ is noetherian and it is finitely generated as an $R/\mathfrak{A}$-module, by Theorem 2 of [2], $R/\mathfrak{A}$ is a noetherian ring. Furthermore, for any $a\in\mathfrak{M}$, there is a positive integer m such that $a^m\in\mathfrak{A}$. In fact, since $R_{\mathfrak{M}}$ is artinian local, $\mathfrak{M}^m R_{\mathfrak{M}}=0$ for some positive integer m, and hence, there is an r in $R-\mathfrak{M}$ such that $ra^m=0$. Therefore $a^m\in\mathfrak{A}$. From the above remarks, $\mathfrak{M}/\mathfrak{A}$ is nilpotent, and so, $R/\mathfrak{A}$ is an artinian local ring because $\mathfrak{M}/\mathfrak{A}$ is a maximal ideal of $R/\mathfrak{A}$. Thus $R/\mathfrak{A}\cong R_{\mathfrak{M}}$, that is, $R_{\mathfrak{M}}$ is a cyclic R-module.

Hence $R_{\mathfrak{M}}$ is a cyclic injective R-module.

Conversely, assume that there is a cyclic injective R-module M and let $M=R/\mathfrak{A}$ ($\mathfrak{A}$: an ideal of R). Then $R/\mathfrak{A}$ can be regarded as a Q-module by Lemma 5 and the canonical map $f: R\to R/\mathfrak{A}$ is uniquely extended to a Q-homomorphism $h: Q\to R/\mathfrak{A}$. Set $\mathfrak{B}=\mathrm{Ker}(h)$. Then, by Lemma 2.1 of [5], $Q/\mathfrak{B}$ is injective as a Q-module. Moreover, $Q/\mathfrak{B}$ is a self-injective noetherian ring. In fact, for any $g\in\mathrm{Hom}_{Q/\mathfrak{B}}(\mathfrak{C}/\mathfrak{B}, Q/\mathfrak{B})$ ($\mathfrak{C}$: any ideal of Q containing $\mathfrak{B}$), there is a $Q/\mathfrak{B}$-homomorphism k which is an extension of g because g can be regarded as a Q-homomorphism of $\mathfrak{C}/\mathfrak{B}$ into $Q/\mathfrak{B}$ and $Q/\mathfrak{B}$ is injective as a Q-module.

Thus, by Lemma 2.8 of [5], we can express $Q/\mathfrak{B}$ as follows,

$$Q/\mathfrak{B}=Q_1\oplus Q_2\oplus\cdots\oplus Q_n \quad \text{(as a ring)},$$

where Q_i is a self-injective artinian local ring for $i=1, 2, \cdots, n$. In this case, Q_i is an indecomposable injective Q-module for all i.

Set $\mathfrak{D}'=Q_2\oplus\cdots\oplus Q_n$ and set $\mathfrak{M}'=\mathfrak{M}''\oplus Q_2\oplus\cdots\oplus Q_n$, where $\mathfrak{M}''$ is the maximal ideal of Q_1. Then $Q_1=(Q/\mathfrak{B})/\mathfrak{D}'$ and $\mathfrak{D}'$ is an irreducible $\mathfrak{M}'$-primary ideal of $Q/\mathfrak{B}$ because $(Q/\mathfrak{B})/\mathfrak{D}'$ is an indecomposable injective $Q/\mathfrak{B}$-module. Let $\mathfrak{D}=h^{-1}(\mathfrak{D}')$ and let $\mathfrak{M}=h^{-1}(\mathfrak{M}')$. Then $\mathfrak{M}$ is a maximal ideal of Q and $\mathfrak{D}$ is an irreducible $\mathfrak{M}$-primary ideal of Q. Furthermore, $Q_1=(Q/\mathfrak{B})/\mathfrak{D}'\cong Q/\mathfrak{D}$, $Q/\mathfrak{D}\cong E_Q(Q/\mathfrak{M})$ by Theorem 2.4 of [6] because $Q/\mathfrak{D}$ is an indecomposable injective Q-module, and by Lemma 2, $Q/\mathfrak{D}=E_{Q_{\mathfrak{M}}}(Q_{\mathfrak{M}}/\mathfrak{M}Q_{\mathfrak{M}})$, and hence $Q_{\mathfrak{M}}$ is artinian local by Lemma 1 since $Q/\mathfrak{B}$ is a cyclic injective Q-module.

Now, $Q/\mathfrak{D}\to Q_{\mathfrak{M}}/\mathfrak{D}Q_{\mathfrak{M}}$ is monomorphic and $Q/\mathfrak{D}$ $(\cong E_{Q_{\mathfrak{M}}}(Q_{\mathfrak{M}}/\mathfrak{M}Q_{\mathfrak{M}}))$ is an indecomposable injective $Q_{\mathfrak{M}}$-module. Thus $Q/\mathfrak{D}=Q_{\mathfrak{M}}/\mathfrak{D}Q_{\mathfrak{M}}$. Since $Q_{\mathfrak{M}}$ is artinian local and $Q/\mathfrak{D}=Q_{\mathfrak{M}}/\mathfrak{D}Q_{\mathfrak{M}}$ is injective as a $Q_{\mathfrak{M}}$-module by Lemma 2, we have $\mathfrak{D}Q_{\mathfrak{M}}=0$. Thus $Q_{\mathfrak{M}}$ is a self-injective artinian local ring.

Now, set $\mathfrak{M}\cap R=\mathfrak{P}$. Then $\mathfrak{P}$ is not regular because $R/\mathfrak{P}\subseteqq Q/\mathfrak{M}$ is torsion-free. From this, the quotient field of $R/\mathfrak{P}$ is equal to $Q/\mathfrak{M}$. On the other hand, since $E_Q(Q/\mathfrak{M})$ is a finitely generated R-module, by the same method as in the proof of Theorem 2, $\mathfrak{P}$ is a maximal ideal of R. Moreover, $S(R) \subseteqq R-\mathfrak{P}$, and $R_{\mathfrak{P}} \to Q_{\mathfrak{M}}$ is monomorphic, and so, $R_{\mathfrak{P}} = Q_{\mathfrak{M}}$. Thus $R_{\mathfrak{P}}$ is injective as an R-module and is artinian local. By Lemma 2, $R_{\mathfrak{P}}$ is a self-injective artinian local ring. The proof is completed.

COROLLARY 3. *Let R be a ring. Then the following conditions are equivalent.*

(1) *R is a self-injective artinian ring.*

(2) *$Q(R)$ is a noetherian ring and for any simple R-module M, $E_R(M)$ is cyclic as an R-module.*

PROOF. (1)$\to$(2): It is immediate. (2)$\to$(1): Assume that $Q(R)$ is noetherian and, for any simple R-module, its injective hull is cyclic as an R-module. Let M ($\cong R/\mathfrak{M}$) be any simple R-module. Then, as is shown in proof of Theorem 3, $R_{\mathfrak{M}}$ is a self-injective artinian ring, and so R is a self-injective noetherian ring. Therefore, R is a self-injective artinian ring.

REMARK. There exists a ring, which is not artinian, such that $E_R(M)$ is a cyclic R-module for any simple R-module M. In fact, set $R = \prod_{i=1}^{\infty} K_i$, where K_i is a field for $i = 1, 2, \cdots$. Then, for any maximal ideal $\mathfrak{M}$, $R_{\mathfrak{M}}$ is a field which is isomorphic to $E_R(R/\mathfrak{M})$ as an R-module. Let us show that $R_{\mathfrak{M}}$ is a field for any maximal ideal $\mathfrak{M}$ in R. Now, it is sufficient to show that for any x ($=(x_i)$) in $\mathfrak{M}$, there is an element r in $R-\mathfrak{M}$ such that $rx=0$. Set $r_i = 0$ if $i \in \{i \in I \mid x_i \neq 0\} = D(x)$, $r_i \neq 0$ in K_i if $i \notin D(x)$. Then $x+r$ ($r=(r_i)$) is a unit element in R and $x \in \mathfrak{M}$, and hence $r \notin \mathfrak{M}$. Furthermore, we have $rx=0$. Hence $R_{\mathfrak{M}}$ is a field.

§ 3. The ring properties of injective hulls.

Next we observe the ring property of an injective hull of a commutative ring. Let R be a ring and let M be an R-module. Then *the socle* of M is defined by the sum of all simple submodules of M. Thus the socle of a ring R is a direct sum of all distinct minimal ideals of R.

LEMMA 6. *Let R be an artinian local ring. Then R is self-injective if and only if the socle of R is simple.*

PROOF. If R is self-injective, then by Proposition 3.1 of [**6**] $R = E_R(R/\mathfrak{M})$ ($\mathfrak{M}$ is the maximal ideal of R) since R is indecomposable as an R-module. Thus by Theorem 3.4 of [**6**] the socle of R is equal to A_1 ($=\{x \in R \mid x\mathfrak{M}=0\}$) $\cong R/\mathfrak{M}$. Conversely, assume that the socle of R is simple. Then $A_i = \{x \in E_R(R/\mathfrak{M}) \mid x\mathfrak{M}^i = 0\} \supseteqq \mathfrak{M}^{n-i}$ ($\mathfrak{M}^n \neq 0$ and $\mathfrak{M}^{n+1}=0$) and by Lemma 1 of [**8**] $A_{i+1}/A_i \cong \mathfrak{M}^i/\mathfrak{M}^{i+1}$ as a vector space over $R/\mathfrak{M}$. Furthermore, R is an essential extension of the socle of R, and so $E_R(R/\mathfrak{M}) \cong E_R(R)$. Hence $E_R(R)$ has the same length as R. Thus $R = E_R(R)$.

PROPOSITION 2. *Let R be a ring with the noetherian total quotient ring. Then R can be embedded into a direct sum of finitely many self-injective artinian local rings, which is an essential extension of R.*

PROOF. Let $(0)=\mathfrak{Q}_1\cap\mathfrak{Q}_2\cap\cdots\cap\mathfrak{Q}_n$ ($\mathfrak{Q}_i$: $\mathfrak{P}_i$-primary) be an irredundant irreducible primary decomposition of (0) in the total quotient ring Q of R. Then by Theorem 2.3 of [6], we have $E_Q(Q)=\Sigma\oplus E_Q(Q/\mathfrak{Q}_i)$.

On the other hand, $E_Q(Q/\mathfrak{P}_i)=E_Q(Q/\mathfrak{Q}_i)$ by Proposition 3.1 of [6], and $E_Q(Q/\mathfrak{P}_i)\cong E_{Q_{\mathfrak{P}_i}}(Q_{\mathfrak{P}_i}/\mathfrak{P}_iQ_{\mathfrak{P}_i})$ by Lemma 2, for all i.

Furthermore, by Proposition 3.1 of [6], $E_{Q_{\mathfrak{P}_i}}(Q_{\mathfrak{P}_i}/\mathfrak{P}_iQ_{\mathfrak{P}_i})=E_{Q_{\mathfrak{P}_i}}(Q_{\mathfrak{P}_i}/\mathfrak{Q}_iQ_{\mathfrak{P}_i})$.

Since by Lemma 2.1 of [5], $E_R(R)=E_Q(Q)$, we have $E_R(R)=\Sigma\oplus E_Q(Q_{\mathfrak{P}_i}/\mathfrak{P}_iQ_{\mathfrak{P}_i})$. Thus $\Sigma\oplus Q_{\mathfrak{P}_i}/\mathfrak{Q}_iQ_{\mathfrak{P}_i}$ is an essential extension of R because we can embed R in $\Sigma\oplus Q_{\mathfrak{P}_i}/\mathfrak{Q}_iQ_{\mathfrak{P}_i}$. Moreover by Lemma 6 $Q_{\mathfrak{P}_i}/\mathfrak{Q}_iQ_{\mathfrak{P}_i}$ is a self-injective artinian local ring for all i.

EXAMPLES. (1) Let R be an integral domain. Then an injective hull of R is isomorphic to the quotient field of R. Thus $E_R(R)$ can be made into an R-algebra and R is contained in the center of $E_R(R)$. (2) Let R be a ring and $M\supset N$ be R-modules. Then M is called a rational extension of N in case for any endomorphism f of M, if f is trivial on N, then f is trivial. Now, if $E_R(R)$ is a rational extension of R, then $E_R(R)$ can be taken into an R-algebra and moreover, R is contained in the center of $E_R(R)$. In fact, since $E_R(R)$ is a rational extension of R, any R-homomorphism of R into $E_R(R)$ can be uniquely extended to an R-endomorphism of $E_R(R)$. Thus $\mathrm{Hom}_R(E_R(R), E_R(R))\cong\mathrm{Hom}_R(R, E_R(R))\cong E_R(R)$. The canonical embedding φ of R into $\mathrm{Hom}_R(E_R(R), E_R(R))$ is given by $\varphi(r)=T_r$, where T_r is defined by $T_R(x)=rx$ for all x in $E_R(R)$. Hence R is contained in the center of $E_R(R)$.

LEMMA 7. *Let R be a noetherian local ring and suppose that the socle of R is not simple. Then $E_R(R)$ can not be made into an R-algebra.*

PROOF. Let S_1 and S_2 be two distinct simple submodules of R. Then $E_R(S_1)$ is a direct summand of $E_R(R)$. Set $S_1=Rx_1$ and set $S_2=Rx_2$ ($x_1, x_2\in R$). Let $f_i\in\mathrm{Hom}_R(Rx_1\oplus Rx_2, E_R(Rx_1))$ such that $f_i(x_i)=x_1$ and $f_i(x_j)=0$ if $i\neq j$, for $i,j=1,2$. Then there is $g_i\in\mathrm{Hom}_R(R, E_R(Rx_1))$ such that the restriction of g_i to $Rx_1\oplus Rx_2$ is f_i, for $i=1,2$. Set $g_i(1)=a_i$ for $i=1,2$. Then $x_1a_1=x_1=x_2a_2$ and $x_1a_2=x_2a_1=0$. Hence $(x_2a_2)\cdot a_1=x_1\cdot a_1=x_1\neq 0$, but $a_2\cdot(x_2a_1)=a_2\cdot 0$, and so $E_R(R)$ can not be made into an R-algebra because $E_R(Rx_1)$ is a direct summand of $E_R(R)$.

THEOREM 4. *Let R be an artinian local ring. Then an injective hull of R can be made into an R-algebra if and only if R is a self-injective ring.*

PROOF. If R is not self-injective, then the socle of R is not simple by Lemma 6. Thus by Lemma 7, $E_R(R)$ can not be made into an R-algebra. The converse is trivial.

Let R be a ring, $\mathfrak{P}$ a prime ideal, and let M be an R-module. Then let us call the socle of $M_{\mathfrak{P}}$ *the socle of* M *at* $\mathfrak{P}$.

THEOREM 5. *Let R be a ring with the noetherian total quotient ring. Then if an injective hull of R can be regarded as an R-algebra, then the socle of $Q=Q(R)$ at any prime ideal of Q is simple or empty.*

PROOF. Suppose that the socle of Q at some prime ideal $\mathfrak{P}$ of Q is not simple and not empty, then there are two elements x_1, x_2 in Q such that $Qx_1 \cap Qx_2 = 0$ and the annihilator ideal of Qx_i is equal to $\mathfrak{P}$, for $i=1, 2$. In fact, since the socle of $Q_{\mathfrak{P}}$ is not simple and not empty, there are at least two simple submodules S_1, S_2 in $Q_{\mathfrak{P}}$. Thus there are y_1, y_2 in Q such that $Q_{\mathfrak{P}}y_i = S_i$ for $i=1, 2$. Since $\mathfrak{P}$ is finitely generated, there is an r in $Q-\mathfrak{P}$ such that $r\mathfrak{P}y_i = \mathfrak{P}ry_i = 0$ for $i=1, 2$, and so $x_i = ry_i$ $(i=1, 2)$ answer the question.

Let f_1 and f_2 be two Q-homomorphism of Qx_1+Qx_2 into $E_Q(Qx_1)$ defined by $f_1(x_1)=f_2(x_2)=x_1$ and $f_1(x_2)=f_2(x_1)=0$. Since $E_Q(Qx_1)$ is injective, there is a Q-homomorphism g_i of into $E_Q(Qx_1)$ such that the restriction of g_i to $Qx_1 +Qx_2$ is f_i for $i=1, 2$. Set $g_i(1)=a_i$ for $i=1, 2$. Then $x_1a_1=x_2a_2=x_1$ and $x_1a_2=x_2a_1=0$. Now, we have $(x_2a_2)\cdot a_1 = x_1\cdot a_1 = x_1$ and $a_2\cdot(x_2a_1)=a_2\cdot 0=0$. Thus $E_Q(Q)$ can not be made into a Q-algebra since $E_Q(Qx_1)$ is a direct summand of $E_Q(Q)$. Therefore $E_R(R)$ can not be made into an R-algebra. This is a contradiction to the hypothesis that $E_R(R)$ is an R-algebra.

Let R be a ring with the total quotient ring which is artinian. Then we call such a ring a *(qa)-ring*. It is well known that if R is noetherian then R is a (qa)-ring if and only if the prime divisors of (0) in R are all minimal.

THEOREM 6. *Let R be a (qa)-ring. Then an injective hull $E_R(R)$ of R can be made into an R-algebra if and only if $E_R(R)=Q(R)$.*

PROOF. Assume that $E_R(R)$ is an R-algebra. As $E_R(R)$ is a torsion-free and divisible R-module, $E_R(R)$ can be regarded as a Q-module, and hence $E_R(R)$ is injective as a Q-module by Lemma 2.1 of [**5**]. On the other hand, Q is an essential extension of R, and so we have $E_R(R)=E_R(Q)$. Furthermore by Lemma 2.1 of [**5**] $E_R(R)=E_Q(Q)$. Since R is a (qa)-ring, we may write $Q=\Sigma\oplus Q_i$, where Q_i is artinian local for all i. Thus $E_Q(Q)\cong\Sigma\oplus E_Q(Q_i) =\Sigma\oplus E_{Q_i}(Q_i)$.

If Q_i is not a self-injective artinian local ring, then by Lemma 6 there are at least two distinct minimal ideals S_1, S_2 in Q_i, a fortiori, of Q. Thus there are two elements x_1, x_2 in Q_i such that $Qx_j=S_j$ for $j=1, 2$. By the same method as in proof of Theorem 5, we can find two elements a_1, a_2 in $E_Q(Qx_1)$ such that $x_1a_1=x_2a_2=x_1$ and $x_1a_2=x_2a_1=0$. Now, we have $(x_2a_2)\cdot a_1=x_1\cdot a_1 =x_1$ and $a_2\cdot(x_2a_1)=a_2\cdot 0=0$. Thus $E_Q(Q)$ can not be made into a Q-algebra because $E_Q(Qx_1)$ is a direct summand of $E_Q(Q)$. Thus $E_R(R)=E_Q(Q)$ can not be taken into an R-algebra. This is a contradiction. The converse is trivial.

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