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On Stiefel manifolds

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Introduction.

T. T. Frankel [5] applied Morse theory to the Stiefel manifolds using the trace function. The critical sets in this case are Grassmann manifolds. In this note we apply Morse theory to the Stiefel manifolds using "length function". We think of Stiefel manifolds as imbedded in Euclidean spaces and use methods similar to R. Bott [3]. Finally, using some results on P. A. Smith theory of periodic transformations we show that the Morse inequalities are equalities. This method is due to Frankel [5]. The CW-decomposition and the Poincaré polynomials obtained for Stiefel manifolds are, of course, known. For this reason detailed proofs are omitted.

The referee points out that "the length function" is essentially the same as the function used by Takeuchi and Kobayashi [7] generalizing the trace function of Frankel [5]. The author is grateful to the referee for this and other valuable suggestions and comments.

Preliminaries.

Let F be R, the field of real numbers, C the field of complex numbers or Q, the quaternions. Let $U(n;F)=\{A\mid A\bar{A}^t=I_n\}$ where A is an $n\times n$ matrix with coefficients in F. The 'bar' denotes complex conjugation or the quaternionic conjugation as the case may be. Let $U_0(n;F)$ be the identity component of $U(n;F)$. Hence $U_0(n;F)$ is $SO(n)$ if $F=R$, is $U(n)$ if $F=C$, and is $Sp(n)$ if $F=Q$. Let $\underline{u}(n;F)$ be the Lie algebra of $U(n;F)$. Let $V_{p+q,p}(F)=\dfrac{U_0(p+q;F)}{U_0(q;F)}$ be Stiefel manifold over F. If $q=0$, we get the classical groups; $V_{p+q,p}(F)$ is the set of all orthogonal p-frames in F^{p+q} space with respect to the standard metric $\sum x_i\bar{x}_i$.

The Stiefel manifolds are imbedded in Euclidean spaces as follows: Let G be a compact connected Lie group with an invariant Riemannian metric. (We will take G to be $U_0(n;F)$). Let $\sigma: G\to G$ be an involution with the full

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fixed group K. The Lie algebra $\underline{g}$ of G splits into a direct sum $\underline{g}=\underline{k}\oplus\underline{p}$ where $\underline{k}$ is the Lie algebra of K. Of course $\underline{p}$ will be the eigenspace of eigenvalue -1 for the involution on $\underline{g}$ induced by σ. The group K acts on $\underline{p}$ by adjoint action. The Stiefel manifolds arise as orbits for this action.

Let $G=U_0(n;F)$ with $n=2p+q$ and consider the involution $X\to I(p+q,p)XI(p+q,p)$ on G where $I(p+q,p)$ is a diagonal matrix with the first $p+q$ entries equal to 1 and the next p entries equal to -1. The full fixed set K is $\{U(p+q;F)\times U(p;F)\}\cap U_0(2p+q;F)$.

$$\text{Hence } \underline{p}=\begin{array}{cc} \begin{array}{cc} p+q & p \end{array} & \\ \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix} & \begin{array}{c} p+q \\ p \end{array} \end{array} \subset \underline{u}(n;F).$$

Let $\tau=\left(\begin{array}{ccc|ccc} & & & 1 & & \\ & & & & \ddots & \\ & 0 & & 0 & & 1 \\ \hline -1 & & & & & \\ & \ddots & 0 & & 0 & \\ & & -1 & & & \end{array}\right)\in\underline{p}$. Then the orbit of τ under the adjoint action of K is K/K^*, where K^* is

$$\begin{array}{cc} & \begin{array}{ccc} p & q & p \end{array} \\ \begin{array}{c} p \\ q \\ p \end{array} & \begin{pmatrix} X & 0 & 0 \\ 0 & Y & 0 \\ 0 & 0 & X \end{pmatrix} \end{array} \in U_0(n;F).$$

LEMMA. *Stiefel manifold $V_{p+q,p}(F)$ can be identified with K/K^*.*

PROOF. This lemma is essentially due to Ehresmann. The following proof is a slight modification of the proof in M. Takeuchi [8]. The group K acts on $U_0(p+q;F)$ by

$$\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}\cdot U=XU\begin{pmatrix} Y & 0 \\ 0 & I_q \end{pmatrix}^{-1} \qquad \begin{array}{l} X=(p+q)\times(p+q) \\ Y=p\times p \end{array}$$

for $U\in U_0(p+q;F)$. This action is transitive and induces a transitive action of K on $\dfrac{U_0(p+q;F)}{U_0(q;F)}$. If $k=\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}\in K$ leaves the identity coset fixed then

$$XU_0(q;F)\begin{pmatrix} Y & 0 \\ 0 & I_q \end{pmatrix}^{-1}=U_0(q;F)$$

$$X\begin{pmatrix} I_p & 0 \\ 0 & U_0(q;F) \end{pmatrix}=\begin{pmatrix} I_p & 0 \\ 0 & U_0(q;F) \end{pmatrix}\begin{pmatrix} Y & 0 \\ 0 & I_q \end{pmatrix}=\begin{pmatrix} Y & 0 \\ 0 & U_0(q;F) \end{pmatrix}$$

$$X\in\begin{pmatrix} Y & 0 \\ 0 & U_0(q;F)\end{pmatrix}\begin{pmatrix} I_p & 0 \\ 0 & U_0(q;F)\end{pmatrix}^{-1},\ X=\begin{pmatrix} Y & 0 \\ 0 & V\end{pmatrix},\ V\in U_0(q;F).$$

Hence $k=\begin{pmatrix} Y & & \\ & V & \\ & & Y\end{pmatrix}\in K^*$. Thus the Stiefel manifold $V_{p+q,p}(F)$ can be identified with K/K^*.

Length function and its critical set.

Let M^m be an m-dimensional manifold differentiably imbedded in a Euclidean space R^n. Let $P\in R^n-M^m$. Let $L_P(x)=$square of the distance between $x\in M^m$ and the fixed point P. Then $Q\in M^m$ is a critical point for $L_P(x)$ if and only if the straight line PQ is perpendicular to the tangent space M_Q of M at Q. For details see [**3**] or [**6**].

We are considering the Stiefel manifolds $V_{p+q,p}(F)$ imbedded in $\underline{p}$. (For notations, see the previous section.) It is known that for the decomposition $\underline{g}=\underline{k}\oplus\underline{p}$, $[\underline{k},\underline{k}]\subset\underline{k}$, $[\underline{p},\underline{k}]\subset\underline{p}$ and $[\underline{p},\underline{p}]\subset\underline{k}$. A maximal sub-algebra $\underline{h}$ of $\underline{p}$ is abelian and is called a *Cartan subalgebra*. In our cases $\underline{h}$ will be of the form

$$\begin{array}{cc} & \begin{array}{cc} p+q & p \end{array} \\ \begin{array}{c} p+q \\ \\ p \end{array} & \left(\begin{array}{c|c} 0 & \begin{matrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{matrix} \\ \hline \begin{matrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{matrix} & 0 \end{array}\right) \end{array} \quad \text{(all entries real)}.$$

We choose $P=\left(\begin{array}{c|c} 0 & \begin{matrix} 1 & & 0 \\ & 2 & \\ & & \ddots \\ 0 & & p \end{matrix} \\ \hline \begin{matrix} -1 & & 0 \\ & -2 & \\ & & \ddots \\ 0 & & -p \end{matrix} & \begin{matrix} 0 & 0 \end{matrix} \end{array}\right)\in\underline{h}$. Such a point having distinct entries on the diagonal is called a *general point.*

The orbit of a general point is $\dfrac{U_0(p+q;F)}{U(1;F)\times\cdots\times U(1;F)\cap U_0(q;F)}$.

It is well known that the tangent space to the orbit of P at P is $XP-PX$ for all $X\in\underline{k}$, the Lie algebra of $K=\{U(p+q;F)\times U(p;F)\}\cap U_0(2p+q;F)$. If we take P to be general point, then the normal space to the orbit of P at P is precisely $\underline{h}$. Also, if a straight line is perpendicular to an orbit at a point, then it is perpendicular to all the orbits it meets [**3, 4**]. Hence all the critical points on the Stiefel manifold for the function $L_P(x)$ are matrices of the form

$$\left(\begin{array}{cc|cc} 0 & & \pm 1 & 0 \\ & & & \pm 1 \\ & & 0 & \\ \hline \mp 1 & & 0 & 0 \\ 0 & \mp 1 & & \end{array}\right).$$

There are 2^p such isolated critical points.

It is clear that choosing a point in $\underline{p}$ suitably orbits such as $\frac{U_0(p+q\,;F)}{U(q_1\,;F)\times\cdots\times U(q_k\,;F)\cap U_0(q\,;F)}$ $q_1+\cdots+q_k=q$ can be obtained, and these spaces can be studied in a similar way as Stiefel manifolds. These orbits would have $2^p\frac{q\,!}{q_1\,!\,q_2\,!\cdots q_k\,!}$ critical points.

REMARK. These imbeddings of the orbits obtained by the adjoint action of K on $\underline{p}$ have been studied by Kobayashi and Takeuchi [7] and shown to be minimal in the sense of total curvature.

Non-degeneracy and Index of the critical points.

R. Bott [2] has outlined a procedure to find the indexes of the critical points. Let P be a general point and let σ be a critical point. Then find all points Q between P and σ where σP meets an orbit of lower dimension. (Recall P is on an orbit of maximum dimension.) The index of $\sigma=\sum\dim 0(P)-\dim 0(Q)$ where $0(P)=$ orbit of P and the index can be readily computed. Also if P is a general point then σ will be a non-degenerate critical point.

As a concrete example consider $\frac{U(4)}{U(2)}$. Here $p=q=2$. The four critical points for $L_P(x)$ are

$$\begin{array}{c} 4\times 4 \\ \left(\begin{array}{cccc|cc} & & & & \pm 1 & 0 \\ & 0 & & & 0 & \pm 1 \\ & & & & 0 & 0 \\ & & & & 0 & 0 \\ \hline \mp 1 & 0 & 0 & 0 & & \\ 0 & \mp 1 & 0 & 0 & 0 & \end{array}\right). \\ 2\times 2 \end{array}$$

For simplicity we will write them as $(\pm 1, \pm 1)$. Also (a, b) would mean the matrix

$$\left(\begin{array}{cccc|cc} & & & & a & 0 \\ & & 0 & & 0 & b \\ & & & & 0 & 0 \\ & & & & 0 & 0 \\ \hline -a & 0 & 0 & 0 & & \\ 0 & -b & 0 & 0 & & 0 \end{array}\right).$$

Thus $P=(1, 2)$.

Critical points	*Points Q where P meet orbits of lower dimension*	*Index*
$(1, 1)$	—	0
$(-1, 1)$	$(0, 3/2)$	5
$(1, -1)$	$(1, 0)$, $(1, 1)$	$5+2=7$
$(-1, -1)$	$(0, 1/2)$, $(-1/5, 1/5)$ $(-1/3, 0)$	$5+2+5=12$

The relationship between critical points and homology is stated in the form of (weak) Morse inequalities i. e., $b_i\ (V_{p+q,q}(F))\leqq c_i$, where b_i is the ith Betti number with a field as coefficients. (If $F=R$ we use Z_2 as coefficients and in the other two cases any field may be used). Also c_i is the number of critical points of index i.

Applications of fixed point theory.

In this last section we show that the Morse inequalities are equalities. For this we use results on P. A. Smith theory of periodic maps. (This method of showing the Morse inequalities to be equalities is due to Frankel [**5**].) In particular we use the following two theorems. More general results and proofs can be found in [**1**].

Let $\Gamma=\left(\begin{array}{c|c} \begin{matrix}\Gamma' & \\ & I_q\end{matrix} & 0 \\ \hline 0 & \Gamma' \end{array}\right)$, where $\Gamma'=\begin{pmatrix}\pm 1 & & 0\\ & \ddots & \\ 0 & & \pm 1\end{pmatrix}$, $p\times p$ matrix .

THEOREM I. *If Γ acts on a compact differentiable manifold M, if F is the fixed set then* $\sum b_i(F\,;Z_2)\leqq \sum_i b_i(M\,;Z_2)$.

This theorem is applied to $M=V_{p+q,p}(R)$. For the adjoint action of Γ on M, the fixed set is precisely the set of critical points for the function $L_P(x)$. These are 2^p in number and we get $2^p\leqq \sum_i b_i(M\,;Z_2)$. Hence the Morse inequalities for $V_{p+q,p}(R)$ become equalities.

For the case $V_{p+q,p}(F)$, $F=C$ or Q, we use

THEOREM II. *If a toral group operates on a compact differentiable manifold M and if F in the fixed set, then* $\sum_i b_i(F\,;K)\leqq \sum_i b_i(M\,;K)$ *where $K=R$ or Z_p, p prime.*

$$\text{Let } T=\left(\begin{array}{ccc|ccc} \cos\theta_1 & & 0 & \sin\theta & & \\ & \ddots & & & \ddots & \\ 0 & & \cos\theta_p & & & \sin\theta_p \\ & & & I_q & 0 & \\ \hline & & & 0 & & \\ -\sin\theta & & & \cos\theta_1 & & \\ & \ddots & & & \ddots & 0 \\ & & -\sin\theta_p & 0 & & \cos\theta_p \end{array}\right)$$

T operates on $V_{p+q,p}(F)$ by adjoint action and fixed points are the critical points. Thus $2^p \leqq \sum_i b_i(M; K)$. Hence the Morse inequalities in these two cases become equalities. Further $V_{p+q,p}(F)$, $F=C$ or Q, has no torsion.

By induction the Poincaré polynomials for the three Stiefel manifolds can be obtained.

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