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A generalization of F. Schur's theorem

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The following theorem, due to F. Schur, is well-known:

THEOREM A. *Let M be a Riemannian manifold with* $\dim M \geqq 3$. *If the sectional curvature K of M is constant at each point of M, then K is actually constant on M.*

There are several other theorems of this type; we mention a few of them.

THEOREM B. *Let M be an Einstein manifold, that is, assume the Ricci curvature of M is a scalar multiple λ of the metric tensor of M. If* $\dim M \geqq 3$, *then λ is constant.*

THEOREM C (Thorpe [2]). *Let M be a Riemannian manifold with* $\dim M \geqq 2p+1$. *If the $2p$th sectional curvature γ_{2p} is constant at each point of M, then γ_{2p} is constant on M.*

THEOREM D. *Let M be a Kähler manifold with* $\dim M \geqq 4$. *If the holomorphic sectional curvature K_h is pointwise constant, then it is actually constant.*

THEOREM E (M. Berger, unpublished). *Let M be a Riemannian manifold with metric tensor g_{ij} and Riemann curvature tensor R_{ijkl}. Suppose*

$$\sum_{i,j,k} R_{ijks}R^{ijkt} = \lambda g_{st}\,.$$

If $\dim M \geqq 5$, *then λ is constant.*

In this paper we prove a result (theorem 2) which includes theorems A, B, C, and D as special cases. Although theorem E is not a consequence of theorem 2, it almost is, in the sense that it would be if a slightly different contraction were used.

We shall use the notation of [1]. Recall that a *double form* of type (p, q) is a function $\omega\colon \mathfrak{X}(M)^{p+q} \to \mathcal{F}(M)$ which is skew-symmetric in the first p variables and also in the last q variables. Here, as usual, $\mathfrak{X}(M)$ denotes the Lie algebra of vector fields on the C^∞ manifold M and $\mathcal{F}(M)$ the ring of C^∞ real valued functions on M. We write $\omega(X_1, \cdots, X_p)(Y_1, \cdots, Y_q)$ for the value of ω on $X_1, \cdots, X_p, Y_1, \cdots, Y_q$. If $p=q$ and

$$\omega(X_1, \cdots, X_p)(Y_1, \cdots, Y_p) = \omega(Y_1, \cdots, Y_p)(X_1, \cdots, X_p)$$

for

$$X_1, \cdots, X_p, Y_1, \cdots, Y_p \in \mathfrak{X}(M)\,,$$

the double form ω is said to be *symmetric*.

Now assume that M has a Riemannian metric g, and let ∇ be the corresponding connection. Then if ω is a double form of type (p, p), so is $\nabla_X(\omega)$ for $X \in \mathfrak{X}(M)$ (see [1]). Furthermore a double form $D\omega$ of type $(p+1, q)$ is defined by

$$(D\omega)(X_1, \cdots, X_{p+1}) = \sum_{j=1}^{p+1} (-1)^{j+1} \nabla_{X_j}(\omega)(X_1, \cdots, \hat{X}_j, \cdots, X_{p+1}).$$

Here D is an analog of the exterior derivative d; however, unlike d, D is not independent of ∇.

It will also be necessary to define the notion of the *contraction operator* C on double forms. If ω is a double form of type (p, q), then $C\omega$ is the double form of type $(p-1, q-1)$ defined by

$$(C\omega)(X_1, \cdots, X_{p-1})(Y_1, \cdots, Y_{q-1}) = \sum_{i=1}^{n} \omega(X_1, \cdots, X_{p-1}, E_i)(Y_1, \cdots, Y_{q-1}, E_i)$$

where $n = \dim M$ and $\{E_1, \cdots, E_n\}$ is any orthonormal frame field defined on an open subset of M. Then C^r, $r = 0, 1, 2, \cdots$ are defined inductively. We shall agree that if $p = 0$ or $q = 0$, then $C\omega = 0$.

We shall need the following result.

THEOREM 1. *Let A be a double form of type (p, q) such that $DA = 0$. Then*

$$(1) \qquad (DC^r A)(U, X_1, \cdots, X_{p-r})(Y_1, \cdots, Y_{q-r})$$
$$= (-1)^{p-r} r \sum_{i=1}^{n} \nabla_{E_i}(C^{r-1}A)(U, X_1, \cdots, X_{p-r})(Y_1, \cdots, Y_{q-r}, E_i)$$

for $U, X_1, \cdots, X_{p-r}, Y_1, \cdots, Y_{q-r} \in \mathfrak{X}(M)$, where $\{E_1, \cdots, E_n\}$ is a local orthonormal frame field on M.

PROOF. We induct on r. The assumption $DA = 0$ implies that (1) is true for $r = 0$. Next suppose that (1) is true for general r. Then

$$0 = (CDC^r A)(U, X_1, \cdots, X_{p-r-1})(Y_1, \cdots, Y_{q-r-1})$$
$$+ (-1)^{p-r} r \sum_{i=1}^{n} \nabla_{E_i}(C^r A)(U, X_1, \cdots, X_{p-r-1})(Y_1, \cdots, Y_{q-r-1}, E_i)$$
$$= (DC^{r+1}A)(U, X_1, \cdots, X_{p-r-1})(Y_1, \cdots, Y_{q-r-1})$$
$$+ (-1)^{p-r}(r+1) \sum_{i=1}^{n} \nabla_{E_i}(C^r A)(U, X_1, \cdots, X_{p-r-1})(Y_1, \cdots, Y_{q-r-1}, E_i).$$

Hence (1) is true for $r+1$. This completes the proof.

If ω is a double form of type (p, q), then ω' is a double form of type $(p+1, q-1)$ defined by

$$\omega'(X_1, \cdots, X_{p+1})(Y_2, \cdots, Y_q) = \sum_{j=1}^{p+1} (-1)^{j+1} \omega(X_1, \cdots, \hat{X}_j, \cdots, X_{p+1})(X_j, Y_2, \cdots, Y_q)$$

for $X_1, \cdots, X_{p+1}, Y_2, \cdots, Y_q \in \mathfrak{X}(M)$. We define a *Riemannian double form* (as in [1]) to be a symmetric double form such that $D\omega = \omega' = 0$.

The best known examples of Riemannian double forms are the metric tensor g (type (1, 1)) and the Riemannian curvature tensor R (type (2, 2)). (The Bianchi identities state that $R' = DR = 0$.) In [1] the notion of exterior products of double forms is defined. In particular $g^p = g \wedge \cdots \wedge g$ and $R^p = R \wedge \cdots \wedge R$ (each p times) are double forms of types (p, p) and $(2p, 2p)$ respectively.

We are now ready to prove our main result.

THEOREM 2. *Let A and B be Riemannian double forms of types (p, p) and (r, r) respectively and assume that* (a) *B is parallel (that is $\nabla_X B = 0$ for all $X \in \mathfrak{X}(M)$)*, (b) *$C^{r-1}B = \alpha g$ for some $\alpha \in \mathcal{F}(M)$, not identically* 0, (c) *$p < n = \dim M$*, (d) *there exist $\lambda \in \mathcal{F}(M)$ and an integer q such that for all $X_1, \cdots, X_{p-q} \in \mathfrak{X}(M)$ we have*

$$(C^q A)(X_1, \cdots, X_{p-q})(X_1, \cdots, X_{p-q}) = \lambda(C^{r-p+q}B)(X_1, \cdots, X_{p-q})(X_1, \cdots, X_{p-q}).$$

Then λ is constant on M.

PROOF. Since B is parallel, so is $C^{r-1}B$, and thus α is a nonzero constant. Furthermore $n\alpha = C^r B$. According to [1] condition (d) is equivalent to $C^q A = \lambda C^{r-p+q} B$. Hence for $U \in \mathfrak{X}(M)$ we have

$$\begin{aligned} 0 &= (DC^p A - D(\lambda C^r B))(U) \\ &= p \sum_{i=1}^{n} \nabla_{E_i}(\lambda C^{r-1}B)(U)(E_i) - (U\lambda) C^r B \\ &= p \sum_{i=1}^{n} (E_i \lambda)(C^{r-1}B)(U)(E_i) - (U\lambda) C^r B \\ &= (p-n)\alpha(U\lambda). \end{aligned}$$

Since U is arbitrary, it follows that λ is constant.

The following is an important special case of theorem 2.

THEOREM 3. *Let A be a Riemannian double form of type (p, p) with $p < n = \dim M$, and assume that for some $q \leq p-1$ and $\lambda \in \mathcal{F}(M)$ we have*

$$(C^q A)(X_1, \cdots, X_{p-q})(X_1, \cdots, X_{p-q}) = \lambda g^{p-q}(X_1, \cdots, X_{p-q})(X_1, \cdots, X_{p-q}),$$

for all $X_1, \cdots, X_{p-q} \in \mathfrak{X}(M)$. Then λ is constant on M.

PROOF. In theorem 2 we take $B = g^{p-q}$. We have the general formula

$$C^s g^t = \frac{t!\,(n-t+s)!}{(t-s)!\,(n-t)!} g^{t-s}$$

for all integers s and t with $0 \leq s \leq t$. Thus condition (b) of theorem 2 is satisfied. Furthermore g^t is parallel for all t (see [1]) and so condition (a) of theorem (2) holds. We conclude that λ must be constant.

We obtain theorems A and B from theorem 3 by taking $A = R$ and $q = 0$

and 1, respectively. Theorem C is also obtained from theorem 3, using $A=R^p$, $q=0$.

Furthermore we have the following generalization of theorems A, B and C.

THEOREM 4. *Suppose* $p<n$, $q<2p-1$, *and*

$$C^q R^p(X_1, \cdots, X_{2p-q})(X_1, \cdots, X_{2p-q}) = \lambda g^{2p-q}(X_1, \cdots, X_{2p-q})(X_1, \cdots, X_{2p-q})$$

for all $X_1, \cdots, X_{2p-q} \in \mathfrak{X}(M)$. *Then* λ *is constant.*

However, to prove theorem D we must use theorem 2 with $A=R$, $q=0$, and B defined by

$$\begin{aligned} B(W, X)(Y, Z) = {} & g(W, Y)g(X, Z) - g(W, Z)g(X, Y) \\ & + g(JW, Y)g(JX, Z) - g(JW, Z)g(JX, Y) \\ & + 2g(JW, X)g(JY, Z) \end{aligned}$$

for $W, X, Y, Z \in \mathfrak{X}(M)$, where J denotes the almost complex structure of the Kähler manifold M. It seems plausible that an analog of theorem D holds for $2p$th holomorphic sectional curvature; however, the author has been unable to prove this.

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References

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[2] J. Thorpe, Sectional curvatures and characteristic classes, Ann. of Math., 80 (1964), 429–443.