J. Math. Soc. Japan
Vol. 21, No. 3, 1969

On the convergence of nonlinear semi-groups II

By Isao MIYADERA[1)]

(Received Oct. 2, 1968)

§ 1. Introduction.

Let X be a Banach space and let $\{T(\xi); \xi \geqq 0\}$ be a family of (nonlinear) operators from X into itself satisfying the following conditions:

(i) $T(0)=I$ (the identity) and $T(\xi+\eta)=T(\xi)T(\eta)$ for $\xi, \eta \geqq 0$.

(ii) For each $x \in X$, $T(\xi)x$ is strongly continuous in $\xi \geqq 0$.

(iii) There is a constant $\omega \geqq 0$ such that

$$\|T(\xi)x-T(\xi)y\| \leqq e^{\omega\xi}\|x-y\|$$

for $x, y \in X$ and $\xi \geqq 0$.

We call such a family $\{T(\xi); \xi \geqq 0\}$ simply *nonlinear semi-group of local type*. In particular, if $\omega=0$, it is called a *nonlinear contraction semi-group*. We define the *infinitesimal generator* A_0 of $\{T(\xi); \xi \geqq 0\}$ by

$$A_0 x = \lim_{\delta \to 0+} \delta^{-1}(T(\delta)-I)x \tag{1.1}$$

and the *weak infinitesimal generator* A' by

$$A'x = w\text{-}\lim_{\delta \to 0+} \delta^{-1}(T(\delta)-I)x, \tag{1.2}$$

where the notation "w-lim" means the weak limit in X.

Throughout this paper it is assumed that the dual X^* of X is uniformly convex. Our purpose is to prove the following theorem.

THEOREM 1. *Let $\{T^{(k)}(\xi); \xi \geqq 0\}_{k=1,2,3,\cdots}$ be a sequence of nonlinear semi-groups of local type satisfying the stability condition*

$$\|T^{(k)}(\xi)x-T^{(k)}(\xi)y\| \leqq e^{\omega\xi}\|x-y\| \tag{1.3}$$

for $\xi \geqq 0$, k and $x, y \in X$, where ω is a non-negative constant independent of x, y, ξ and k. Let $A^{(k)}$ be the weak infinitesimal generator of $\{T^{(k)}(\xi); \xi \geqq 0\}$ and assume $R(I-h_k A^{(k)})=X$ for some $h_k \in (0, 1/\omega)$, and define $Ax=\lim_k A^{(k)}x$.

Suppose that

(a) *$D(A)$ (the domain of A) is dense in X,*

(b) *$\overline{R(I-h_0 A)}=X$ for some $h_0 \in (0, 1/\omega)$,*

---

1) This work was partially supported by National Science Foundation Grant GP-8555.

where $\overline{R(I-h_0A)}$ *denotes the strong closure of the range* $R(I-h_0A)$.

Then the strong closure $\bar{A}$ *of* A, *which is not necessarily single-valued, generates a nonlinear semi-group* $\{T(\xi)\,;\,\xi\geqq 0\}$ *of local type* (in sense of Theorem 2); *and for each* $x\in X$

$$T(\xi)x=\lim_k T^{(k)}(\xi)x \quad \text{for} \quad \xi\geqq 0\,, \tag{1.4}$$

and the convergence is uniform with respect to ξ *in every finite interval.*

REMARKS. 1°. A multi-valued operator T is called to be the strong closure of A if $G(T)=\overline{G(A)}$, where the notation $G(\cdot)$ denotes the graph of operator; and we write $T=\bar{A}$.

2°. If $\{T^{(k)}(\xi)\,;\,\xi\geqq 0\}$ $(k=1,2,3,\cdots)$ are linear semi-groups (in this case, each $A^{(k)}$ becomes the infinitesimal generator and $R(I-h_kA^{(k)})=X$ holds automatically), then $\bar{A}$ is single-valued; and the theorem is a special case of Trotter's theorem (see [**9**]).

3°. If we omit the condition (a), then $\bar{A}$ generates a nonlinear semi-group $\{T(\xi)\,;\,\xi\geqq 0\}$ of local type defined on $\overline{D(A)}$ and (1.4) holds on $\overline{D(A)}$.

4°. It is easy to see that

$$(A^{(k)}x-A^{(k)}y,\, F(x-y))\leqq\omega\,\|x-y\|^2 \qquad \text{for} \quad x, y\in D(A^{(k)})\,,$$

where F is the duality map from X into X^*, i.e., $A^{(k)}-\omega I$ is dissipative; and hence the condition $R(I-h_kA^{(k)})=X$ shows that $A^{(k)}-\omega I$ is m-dissipative.[2] Conversely if $A^{(k)}-\omega I$ is single-valued m-dissipative with dense domain, then $A^{(k)}$ is the weak infinitesimal generator of a nonlinear semi-group $\{T^{(k)}(\xi)\,;\,\xi\geqq 0\}$ of local type with (1.3) (see T. Kato [**2**] and S. Oharu [**8**]).

5°. In the previous paper [**6**] we discussed the case of $R(I-h_0A)=X$ under slightly different conditions.

We use the recent results on nonlinear semi-groups generated by multi-valued m-dissipative operators, obtained by Y. Kōmura [**4, 5**], T. Kato [**3**], and M. G. Crandall and A. Pazy [**1**]. In § 2 we shall explain a part of their results related to ours. The proof of Theorem 1 is given in § 3.

§ 2. Generation of nonlinear semi-groups.

A multi-valued operator A with domain $D(A)$ and range $R(A)$ in X is said to be *dissipative* if

$$\mathrm{Re}\,(x'-y',\, F(x-y))\leqq 0 \qquad \text{for any} \quad x'\in Ax,\ y'\in Ay\,,$$

where F is the duality map from X into X^*. If A is dissipative and $R(I-\alpha_0A)$

2) See (3.1).

$=X$[3] for some $\alpha_0>0$, it is called to be *m-dissipative.*

In this section we shall sketch a construction and some properties of nonlinear semi-groups generated by multi-valued m-dissipative operators ([1], [3], [4] and [5]).

Throughout this section let $\omega\geqq 0$ and let $A-\omega I$ be m-dissipative. It is obtained that the set Ax is convex and weakly closed for each $x\in D(A)$. According to Kato [3] we define the canonical restriction A^0 of A by

$$A^0x=\{y'\,;\ y'\in Ax \text{ and } \|y'\|=\inf[\|x'\|\,;\ x'\in Ax]\} \tag{2.1}$$

for $x\in D(A)$. Since X is reflexive and Ax is weakly closed, $A^0x\neq\emptyset$ for $x\in D(A)$; so that A^0 is a multi-valued dissipative operator with $D(A^0)=D(A)$. In particular if X is strictly convex, then A^0 is single-valued.

From the dissipativity of $A-\omega I$ we get

$$\|x-y-\alpha(x'-y')\|\geqq(1-\alpha\omega)\|x-y\|$$

for $x'\in Ax$, $y'\in Ay$ and $\alpha\in(0,1/\omega)$; and hence for each $\alpha\in(0,1/\omega)$ $(I-\alpha A)^{-1}$ exists as a single-valued operator defined on X and

$$\|(I-\alpha A)^{-1}x-(I-\alpha A)^{-1}y\|\leqq(1-\alpha\omega)^{-1}\|x-y\| \tag{2.2}$$

for $x, y\in X$. If we put

$$J_n=(I-n^{-1}A)^{-1} \quad\text{and}\quad A_n=n(J_n-I)$$

for $n>\omega$, then

$$A_nx\in AJ_nx \qquad \text{for}\quad x\in X, \tag{2.3}$$

$$\begin{cases} \|A_nx-A_ny\|\leqq\dfrac{2n-\omega}{1-n^{-1}\omega}\|x-y\| \\ \operatorname{Re}(A_nx-A_ny, F(x-y))\leqq\omega(1-n^{-1}\omega)^{-1}\|x-y\|^2 \end{cases} \tag{2.4}$$

for $x, y\in X$,

$$\|A_nx\|\leqq(1-n^{-1}\omega)^{-1}|||Ax||| \qquad \text{for}\quad x\in D(A), \tag{2.5}$$

where $|||Ax|||=\inf\{\|x'\|\,;\ x'\in Ax\}$ (we note that $\|x'\|=|||Ax|||$ for all $x'\in A^0x$), and

$$\lim_{n\to\infty}J_nx=x \qquad \text{for}\quad x\in\overline{D(A)}. \tag{2.6}$$

It follows from (2.4) that each A_n generates a nonlinear semi-group $\{T_n(\xi)\,;\ \xi\geqq 0\}$ of local type satisfying

$$\|T_n(\xi)x-T_n(\xi)y\|\leqq\exp\left(\frac{\omega\xi}{1-n^{-1}\omega}\right)\|x-y\| \tag{2.7}$$

for $x, y\in X$ and $\xi\geqq 0$, and

3) It is known that $R(I-\alpha_0A)=X$ implies $R(I-\alpha A)=X$ for all $\alpha>0$, if A is dissipative.

$$(2.8)\quad \begin{cases} \text{for each } x\in X,\ T_n(\xi)x\in C^1([0,\infty);X)^{4)} \text{ and} \\ (d/d\xi)T_n(\xi)x=A_nT_n(\xi)x \text{ for } \xi\geqq 0 \end{cases}$$

(for example, see the proof of Theorem 4.1 in [6]).

Notice that

$$(2.9)\quad \|A_nT_n(\xi)x\|\leqq e^{c_n\xi}\|A_nx\|\leqq e^{c_n\xi}d_n|||Ax|||$$

for $x\in D(A)$ and $\xi\geqq 0$, where $c_n=\omega(1-n^{-1}\omega)^{-1}$ and $d_n=(1-n^{-1}\omega)^{-1}$.

Let $x\in D(A)$ and let $z_{mn}(\xi)=T_n(\xi)x-T_m(\xi)x$. We shall now estimate $z_{mn}(\xi)$. Note that $c_n\leqq 2\omega$ and $d_n\leqq 2$ for $n>2\omega$. In the following let m and n be integers such that $m,n>2\omega$. From (2.9)

$$(2.10)\quad \|z_{mn}(\eta)\|\leqq\int_0^\eta\|A_nT_n(\tau)x-A_mT_m(\tau)x\|d\tau \leqq 4e^{2\omega\eta}|||Ax|||\eta\,,$$

$$(2.11)\quad \|z_{mn}(\eta)-u_{mn}(\eta)\|\leqq n^{-1}\|A_nT_n(\eta)x\|+m^{-1}\|A_mT_m(\eta)x\| \leqq\Big(\frac{1}{n-\omega}+\frac{1}{m-\omega}\Big)e^{2\omega\eta}|||Ax|||$$

for $\eta\geqq 0$, where $u_{mn}(\eta)=J_nT_n(\eta)x-J_mT_m(\eta)x$; and hence

$$(2.12)\quad \|u_{mn}(\eta)\|\leqq\Big(\frac{1}{n-\omega}+\frac{1}{m-\omega}\Big)e^{2\omega\eta}|||Ax|||+\|z_{mn}(\eta)\|\,.$$

Since $\mathrm{Re}\,(A_nT_n(\eta)x-A_mT_m(\eta)x,\ F(u_{mn}(\eta)))\leqq\omega\|u_{mn}(\eta)\|^2$ by (2.3),

$$\begin{aligned}&\mathrm{Re}\,(A_nT_n(\eta)x-A_mT_m(\eta)x,\ F(z_{mn}(\eta)))\\ &\leqq\mathrm{Re}\,(A_nT_n(\eta)x-A_mT_m(\eta)x,\ F(z_{mn}(\eta))-F(u_{mn}(\eta)))+\omega\|u_{mn}(\eta)\|^2\\ &\leqq 4e^{2\omega\eta}|||Ax|||\,\|F(z_{mn}(\eta))-F(u_{mn}(\eta))\|+\omega\|u_{mn}(\eta)\|^2\,;\end{aligned}$$

hence

$$(2.13)\quad \begin{cases}\|z_{mn}(\xi)\|^2=\int_0^\xi(d/d\eta)\|z_{mn}(\eta)\|^2d\eta\\ \qquad =2\int_0^\xi\mathrm{Re}\,(A_nT_n(\eta)x-A_mT_m(\eta)x,\ F(z_{mn}(\eta)))d\eta^{\ 5)}\\ \qquad \leqq 8e^{2\omega\xi}|||Ax|||\int_0^\xi\|F(z_{mn}(\eta))-F(u_{mn}(\eta))\|d\eta\\ \qquad\quad +2\omega\int_0^\xi\|u_{mn}(\eta)\|^2d\eta\,.\end{cases}$$

It follows from (2.12) and (2.13) that

4) $C^1([0,\infty);X)$ denotes the set of all strongly continuously differentiable X-valued functions on $[0,\infty)$.

5) See T. Kato [2; Lemma 1.3].

$$\|z_{mn}(\xi)\|^2 \leqq 8e^{2\omega\xi} ||| Ax ||| \int_0^\xi \|F(z_{mn}(\eta))-F(u_{mn}(\eta))\| d\eta$$

$$+4\omega\Big(\frac{1}{n-\omega}+\frac{1}{m-\omega}\Big)^2 e^{4\omega\xi} ||| Ax |||^2 \xi + 4\omega \int_0^\xi \|z_{mn}(\eta)\|^2 d\eta .$$

Consequently for any fixed $\beta > 0$ we have

$$\|z_{mn}(\xi)\|^2 \leqq K_{mn}(\beta) + 4\omega \int_0^\xi \|z_{mn}(\eta)\|^2 d\eta$$

for $\xi \in [0, \beta]$, where

$$K_{mn}(\beta) = 8e^{2\omega\beta} ||| Ax ||| \int_0^\beta \|F(z_{mn}(\eta))-F(u_{mn}(\eta))\| d\eta$$

$$+4\omega\Big(\frac{1}{n-\omega}+\frac{1}{m-\omega}\Big)^2 e^{4\omega\beta} ||| Ax |||^2 \beta .$$

From this integral inequality we get

(2.14) $$\|z_{mn}(\xi)\| \leqq \sqrt{K_{mn}(\beta)}\, e^{2\omega\xi} \qquad \text{for} \quad \xi \in [0, \beta] .$$

Since F is uniformly continuous on any bounded set of X (see [**2**; Lemma 1.2]), (2.10) and (2.11) show that $K_{mn}(\beta) \to 0$ as $m, n \to \infty$. Therefore it follows from (2.14) that

(2.15) $$\lim_{m,n} \|T_n(\xi)x - T_m(\xi)x\| = 0 \qquad \text{uniformly in} \quad \xi \in [0, \beta] .$$

By (2.7), the above (2.15) holds good for each $x \in \overline{D(A)}$.

Now we define $\{T(\xi)\,;\, \xi \geqq 0\}$ by

(2.16) $$T(\xi)x = \lim_n T_n(\xi)x \qquad \text{for} \quad \xi \geqq 0 \text{ and } x \in \overline{D(A)} .$$

It is clear that $\{T(\xi)\,;\, \xi \geqq 0\}$ is a nonlinear semi-group of local type defined on $\overline{D(A)}$ such that

$$\|T(\xi)x - T(\xi)y\| \leqq e^{\omega\xi} \|x-y\|$$

for $x, y \in \overline{D(A)}$ and $\xi \geqq 0$. The following results are due to T. Kato [3].

THEOREM 2. (I) *The above* $\{T(\xi)\,;\, \xi \geqq 0\}$ *is a unique semi-group of local type satisfying the following conditions;*

(a) *for each* $x \in D(A)$, $T(\xi)x$ *is strongly absolutely continuous on every finite interval,*

(b) *for each* $x \in D(A)$ $(= D(A^0))$, $T(\xi)x \in D(A)$ *for all* $\xi \geqq 0$ *and*

$$(d/d\xi)T(\xi)x \in A^0 T(\xi)x (\subset AT(\xi)x) \qquad \textit{for a. e. } \xi ,$$

where $(d/d\xi)T(\xi)x$ *denotes the strong derivative of* $T(\xi)x$.

(II) *In particular if* X *is uniformly convex, then*

(c) *for each* $x \in D(A)$

$$D^+ T(\xi)x = A^0 T(\xi)x \qquad \textit{for all} \quad \xi \geqq 0$$

and $A^0T(\xi)x$ *is strongly right-hand continuous in* $\xi \geqq 0$, *where* $D^+T(\xi)x$ *denotes the strong right-hand derivative of* $T(\xi)x$,

(d) *for each* $x \in D(A)$ *the strong derivative* $(d/d\xi)T(\xi)x = A^0T(\xi)x$ *exists and is strongly continuous except at a countable number of values* ξ.

REMARKS. 1°. In case of $\omega=0$ (i.e., A is m-dissipative), the above results have been given by T. Kato [3] (in this case, of course, $\{T(\xi); \xi \geqq 0\}$ is a nonlinear contraction semi-group). And his results can be extended to our case (i.e., $A-\omega I$ $(\omega \geqq 0)$ is m-dissipative).

2°. In (I), if A is single-valued (so that $A^0=A$), then it is known that A is the weak infinitesimal generator of $\{T(\xi); \xi \geqq 0\}$ and for each $x \in D(A)$ $AT(\xi)x$ is weakly continuous in $\xi \geqq 0$ (see T. Kato [2] and S. Oharu [8]).

§ 3. Proof of Theorem 1.

For $x, y \in D(A^{(k)})$ $\mathrm{Re}\,(A^{(k)}x - A^{(k)}y, F(x-y))$

$$= \lim_{\xi \to 0+} (\xi^{-1}[T^{(k)}(\xi)x - x] - \xi^{-1}[T^{(k)}(\xi)y - y], F(x-y)) \leqq \omega \|x-y\|^2 ;$$

this shows that $A^{(k)}-\omega I$ are dissipative. Moreover it follows from the assumption $R(I-h_kA^{(k)})=X$ that $R(I-\alpha_k(A^{(k)}-\omega I))=X$ for each k, where $\alpha_k = h_k(1-h_k\omega)^{-1}$. Thus we have

$$A^{(k)}-\omega I \text{ are } m\text{-dissipative}. \tag{3.1}$$

Fix k. From the arguments in § 2, for each $n > \omega$

$$\begin{cases} J_n^{(k)} = (I-n^{-1}A^{(k)})^{-1} \text{ exists and} \\ \|J_n^{(k)}x - J_n^{(k)}y\| \leqq (1-n^{-1}\omega)^{-1}\|x-y\| \quad \text{for} \quad x, y \in X, \end{cases} \tag{3.2}$$

and if we put

$$A_n^{(k)} = n(J_n^{(k)} - I) \ (= A^{(k)}J_n^{(k)}, \text{ because } A^{(k)} \text{ is single-valued}), \tag{3.3}$$

then

$$\begin{cases} A_n^{(k)} \text{ is the infinitesimal generator of a nonlinear semi-group} \\ \{T_n^{(k)}(\xi); \xi \geqq 0\} \text{ of local type such that } \|T_n^{(k)}(\xi)x - T_n^{(k)}(\xi)y\| \\ \leqq \exp\left(\dfrac{\omega\xi}{1-n^{-1}\omega}\right)\|x-y\| \text{ for } x, y \in X \text{ and } \xi \geqq 0; \end{cases} \tag{3.4}$$

and for each $x \in \overline{D(A^{(k)})}$

$$T^{(k)}(\xi)x = \lim_n T_n^{(k)}(\xi)x \quad \text{for} \quad \xi \geqq 0. \tag{3.5}$$

Let $x \in D(A^{(k)})$ and put

$$z_{mn}^{(k)}(\eta) = T_n^{(k)}(\eta)x - T_m^{(k)}(\eta)x \quad \text{for} \quad \eta \geqq 0,$$

where m and n are integers such that $m, n > 2\omega$.
From (2.10), (2.12) and (2.14)

$$\|z_{mn}^{(k)}(\eta)\| \leqq 4e^{2\omega\eta}\|A^{(k)}x\|\eta \quad \text{for} \quad \eta \geqq 0\,, \tag{3.6}$$

$$\|z_{mn}^{(k)}(\eta)-u_{mn}^{(k)}(\eta)\| \leqq \Big(\frac{1}{n-\omega}+\frac{1}{m-\omega}\Big)e^{2\omega\eta}\|A^{(k)}x\| \tag{3.7}$$

for $\eta \geqq 0$, where $u_{mn}^{(k)}(\eta)=J_n^{(k)}T_n^{(k)}(\eta)x-J_m^{(k)}T_m^{(k)}(\eta)x$, and

$$\|z_{mn}^{(k)}(\xi)\| \leqq \sqrt{K_{mn}^{(k)}(\beta)}\, e^{2\omega\xi} \quad \text{for} \quad \xi \in [0, \beta]\,, \tag{3.8}$$

where

$$K_{mn}^{(k)}(\beta)=8e^{2\omega\beta}\|A^{(k)}x\|\int_0^\beta \|F(z_{mn}^{(k)}(\eta))-F(u_{mn}^{(k)}(\eta))\|d\eta \tag{3.9}$$

$$+4\omega\Big(\frac{1}{n-\omega}+\frac{1}{m-\omega}\Big)^2 e^{4\omega\beta}\|A^{(k)}x\|^2\beta\,.$$

(We note that $|||A^{(k)}x|||=\|A^{(k)}x\|$ because $A^{(k)}$ is single-valued.)

From the above estimations we have the following

LEMMA 1. *Let $\beta>0$. For each $x\in D(A)$ the convergence* (3.5) *is uniform with respect to k and $\xi \in [0, \beta]$.*

PROOF. Let $x \in D(A)$. Since $\lim_k A^{(k)}x=Ax$, there exist k_0 and $M>0$ such that $x\in D(A^{(k)})$ and $\|A^{(k)}x\|\leqq M$ for $k \geqq k_0$. It follows from (3.6) and (3.7) that the set

$$B=\{z_{mn}^{(k)}(\eta),\ u_{mn}^{(k)}(\eta)\,;\ \eta\in[0,\beta],\ k\geqq k_0 \text{ and } m, n>2\omega\}$$

is bounded. Since F is uniformly continuous on B, for every $\varepsilon>0$ there is $\delta=\delta_\varepsilon>0$ such that $z, u\in B$ and $\|z-u\|<\delta$ imply $\|F(z)-F(u)\|<2^{-1}K\varepsilon^2$, where $K=(8e^{2\omega\beta}M\beta)^{-1}$. Choose an integer $N(=N_\varepsilon)$ such that $N>2\omega$ and $2(N-\omega)^{-1}e^{2\omega\beta}M \leqq \min(\delta, \varepsilon/\sqrt{8\omega\beta})$.

Let $m, n>N$. By (3.7)

$$\|z_{mn}^{(k)}(\eta)-u_{mn}^{(k)}(\eta)\|<2(N-\omega)^{-1}e^{2\omega\beta}M\leqq\delta$$

for $\eta\in[0,\beta]$ and $k\geqq k_0$, so that

$$\|F(z_{mn}^{(k)}(\eta))-F(u_{mn}^{(k)}(\eta))\|<2^{-1}K\varepsilon^2$$

for $\eta\in[0,\beta]$ and $k\geqq k_0$. Hence

$$8e^{2\omega\beta}\|A^{(k)}x\|\int_0^\beta\|F(z_{mn}^{(k)}(\eta))-F(u_{mn}^{(k)}(\eta))\|d\eta$$

$$\leqq 8e^{2\omega\beta}M\beta 2^{-1}K\varepsilon^2=\varepsilon^2/2\,,$$

and

$$4\omega\Big(\frac{1}{n-\omega}+\frac{1}{m-\omega}\Big)^2 e^{4\omega\beta}\|A^{(k)}x\|^2\beta$$

$$\leqq 4\omega\Big(\frac{2}{N-\omega}e^{2\omega\beta}M\Big)^2\beta\leqq\varepsilon^2/2\,.$$

Consequently $K_{mn}^{(k)}(\beta)\leqq\varepsilon^2$ for $k\geqq k_0$. Therefore it follows from (3.8) that

$$\sup_{\xi \in [0,\beta], k \geqq k_0} \|T_n^{(k)}(\xi)x - T_m^{(k)}(\xi)x\| \leqq e^{2\omega\beta}\varepsilon \quad \text{for} \quad n, m > N. \tag{3.10}$$

Q. E. D.

Since $A^{(k)} - \omega I$ are dissipative (see (3.1)), the limit operator $A - \omega I$ is also dissipative. Combining this and $\overline{R(I-h_0 A)} = X$ (the assumption (b)) we have the following

LEMMA 2. *For each* $n > \omega$

$$\begin{cases} (I-n^{-1}A)^{-1} \text{ has a unique extension } J_n \text{ defined on } X \\ \text{such that } \|J_n x - J_n y\| \leqq (1-n^{-1}\omega)^{-1}\|x-y\| \text{ for } x, y \in X, \end{cases} \tag{3.11}$$

and

$$\bar{A} - \omega I \text{ is m-dissipative and } J_n = (I - n^{-1}\bar{A})^{-1}. \tag{3.12}$$

PROOF. At first we remark that

$$\overline{R(I-n^{-1}A)} = X \quad \text{for all} \quad n > \omega \quad (\text{see S. Oharu [7; Lemma 4]}). \tag{3.13}$$

From the dissipativity of $A - \omega I$, for each $n > \omega$ $(I-n^{-1}A)^{-1}$ exists and

$$\|(I-n^{-1}A)^{-1}x - (I-n^{-1}A)^{-1}y\| \leqq (1-n^{-1}\omega)\|x-y\|$$

for $x, y \in R(I-n^{-1}A)$. Thus (3.11) follows from (3.13).

We shall now prove (3.12). Let $m, n > \omega$. For $x \in R(I-m^{-1}A)$

$$(I-n^{-1}A)(I-m^{-1}A)^{-1}x = (1-m/n)(I-m^{-1}A)^{-1}x + (m/n)x,$$

so that

$$(I-m^{-1}A)^{-1}x = (I-n^{-1}A)^{-1}\{(1-m/n)(I-m^{-1}A)^{-1}x + (m/n)x\}$$

i. e.,

$$J_m x = J_n\{(1-m/n)J_m x + (m/n)x\}$$

for $x \in R(I-m^{-1}A)$. From $\overline{R(I-m^{-1}A)} = X$ we have

$$J_m x = J_n\{(1-m/n)J_m x + (m/n)x\} \quad \text{for all} \quad x \in X. \tag{3.14}$$

Consequently

$$R(J_n) = R(J_m), \tag{3.15}$$

$$n(x - J_n^{-1}x) = m(x - J_m^{-1}x) \quad \text{for} \quad x \in D, \tag{3.16}$$

where D is the set $R(J_n)$ independent of $n > \omega$ and J_n^{-1} are multi-valued mappings defined by $J_n^{-1}x = \{y; J_n y = x\}$ (see S. Oharu [7; Lemma 6]).

Define $\tilde{A}$ by

$$\tilde{A}x = n(x - J_n^{-1}x) \quad \text{for} \quad x \in D. \tag{3.17}$$

It is easy to see that $\tilde{A} \supset A$ (i. e., $D \supset D(A)$ and $\tilde{A}x \ni Ax$ for $x \in D(A)$) and the graph $G(\tilde{A})$ of $\tilde{A}$ is closed. Hence $G(\tilde{A}) \supset \overline{G(A)}$. Moreover $G(\tilde{A}) \subset \overline{G(A)}$. In fact, let $y \in \tilde{A}x$. There is $x' \in X$ such that $x = J_n x'$ and $y = n(x - x')$. Since

$\overline{R(I-n^{-1}A)}=X$, there exists a sequence $\{x_k\}$ in $D(A)$ such that $(I-n^{-1}A)x_k\to x'$ as $k\to\infty$. Hence

$$x_k=J_n(I-n^{-1}A)x_k\to J_nx'=x\,, \qquad \text{and}$$
$$Ax_k\to n(x-x')=y\,.$$

Thus $G(\tilde{A})=\overline{G(A)}$ i.e., $\tilde{A}=\bar{A}$ (the strong closure of A). And then we get $J_n=(I-n^{-1}\bar{A})^{-1}$.

Finally we shall prove that $\bar{A}-\omega I$ is m-dissipative. For $x'\in\bar{A}x$ and $y'\in\bar{A}y$ there exist $\{x_k\}$ and $\{y_k\}$ in $D(A)$ such that $x_k\to x$, $Ax_k\to x'$ and $y_k\to y$, $Ay_k\to y'$. Since $\operatorname{Re}((A-\omega I)x_k-(A-\omega I)y_k, F(x_k-y_k))\leqq 0$, it follows from the continuity of F that

$$\operatorname{Re}((x'-\omega x)-(y'-\omega y), F(x-y))\leqq 0\,.$$

This shows that $\bar{A}-\omega I$ is dissipative. From $R(I-n^{-1}\bar{A})=X$ for $n>\omega$ we have $R(I-\alpha(\bar{A}-\omega I))=X$ for $\alpha>0$. Thus $\bar{A}-\omega I$ is m-dissipative. Q. E. D.

REMARK. The above lemma is also true for multi-valued operators; i.e., if $A-\omega I$ is multi-valued dissipative and if $\overline{R(I-h_0A)}=X$ for some $h_0\in(0, 1/\omega)$, then $\overline{R(I-hA)}=X$ for all $h\in(0, 1/\omega)$, and (3.11) and (3.12) hold good.

By Lemma 2 and Theorem 2, $\bar{A}$ generates a nonlinear semi-group $\{T(\xi)\,;\ \xi\geqq 0\}$[6] of local type; and for each $x\in X$

$$T(\xi)x=\lim_n T_n(\xi)x \tag{3.18}$$

uniformly with respect to ξ in every finite interval, where $\{T_n(\xi)\,;\ \xi\geqq 0\}$ is a nonlinear semi-group of local type generated by $A_n=n(J_n-I)$ and

$$\|T_n(\xi)x-T_n(\xi)y\|\leqq\exp\left(\frac{\omega\xi}{1-n^{-1}\omega}\right)\|x-y\| \tag{3.19}$$

for $x, y\in X$ and $\xi\geqq 0$.

We shall show

$$\lim_k J_n^{(k)}x=J_nx \qquad \text{for} \quad x\in X \text{ and } n\,. \tag{3.20}$$

In fact, for $y=(I-n^{-1}A)x$

$$\begin{aligned}\|J_n^{(k)}y-J_ny\|&=\|J_n^{(k)}y-J_n^{(k)}(I-n^{-1}A^{(k)})x\|\\&\leqq(1-n^{-1}\omega)^{-1}\|y-(I-n^{-1}A^{(k)})x\|\\&=n^{-1}(1-n^{-1}\omega)^{-1}\|A^{(k)}x-Ax\|\to 0 \text{ as } k\to\infty\,.\end{aligned}$$

Then (3.20) follows from $\overline{R(I-n^{-1}A)}=X$ (see (3.13)). Hence

$$\lim_k A_n^{(k)}x=A_nx \qquad \text{for} \quad x\in X \text{ and } n\,. \tag{3.21}$$

6) From the assumption (a) ($D(A)$ is dense in X), $\{T(\xi)\,;\ \xi\geqq 0\}$ is defined on X.

Since each A_n is Lipschitz continuous uniformly in $x \in X$ (see (2.4)), we have

$$R(I-hA_n)=X \quad \text{for sufficiently small } h>0. \tag{3.22}$$

(This is really true for $h \in (0, (1/\omega)-n^{-1})$.)

Consequently, by Theorem 2.3 in [6], for each n we have

$$\sup_{0 \leqq \xi \leqq \beta} \|T_n^{(k)}(\xi)x - T_n(\xi)x\| \to 0 \text{ (as } k \to \infty) \tag{3.23}$$

for any $\beta > 0$ and $x \in X$.

We can now prove the convergence (1.4). Let $\beta > 0$ be arbitrarily fixed, and let $x \in D(A)$. From Lemma 1 and (3.18), for each $\varepsilon > 0$ there is an integer N $(= N_\varepsilon)$ such that

$$\sup_{0 \leqq \xi \leqq \beta} \|T^{(k)}(\xi)x - T_n^{(k)}(\xi)x\| < \varepsilon/2 \qquad \text{for} \quad n > N \text{ and } k,$$

$$\sup_{0 \leqq \xi \leqq \beta} \|T_n(\xi)x - T(\xi)x\| < \varepsilon/2 \qquad \text{for} \quad n > N.$$

Thus for $n > N$ and k

$$\sup_{0 \leqq \xi \leqq \beta} \|T^{(k)}(\xi)x - T(\xi)x\| < \varepsilon + \sup_{0 \leqq \xi \leqq \beta} \|T_n^{(k)}(\xi)x - T_n(\xi)x\|.$$

Going $k \to \infty$, it follows from (3.23) that

$$\sup_{0 \leqq \xi \leqq \beta} \|T^{(k)}(\xi)x - T(\xi)x\| \to 0 \text{ as } k \to \infty. \tag{3.24}$$

Finally, by the stability condition (1.3) and $\overline{D(A)} = X$, (3.24) holds good for every $x \in X$. This completes the proof of Theorem 1.

Georgetown University, Washington, D.C.
and
Waseda University, Tokyo

References

[1] M.G. Crandall and A. Pazy, Nonlinear semi-groups of contractions and dissipative sets, to appear.
[2] T. Kato, Nonlinear semi-groups and evolution equations, J. Math. Soc. Japan, **19** (1967), 508-520.
[3] T. Kato, Remarks on nonlinear accretive operators in Banach spaces (Address), Symposium on Nonlinear Functional Analysis, Chicago, April 1968.
[4] Y. Kōmura, Nonlinear semi-groups in Hilbert space, J. Math. Soc. Japan, **19** (1967), 493-507.
[5] Y. Kōmura, Differentiability of nonlinear semi-groups, J. Math. Soc. Japan, **21** (1969), 375-402.
[6] I. Miyadera, On the convergence of nonlinear semi-groups, to appear in Tôhoku Math. J., **21** (1969).
[7] S. Oharu, Note on the representation of semi-groups of nonlinear operators, Proc. Japan Acad., **42** (1966), 1149-1154.
[8] S. Oharu, Nonlinear semi-groups in Banach space, to appear.
[9] H.F. Trotter, Approximation of semi-groups of operators, Pacific J. Math., **8** (1958), 887-919.