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On the group of units of an absolutely cyclic number field of prime degree

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Let K be a cyclic extension of odd prime degree p over $\boldsymbol{Q}$ with Galois group G generated by s and let $\boldsymbol{E}$ be the group of units of K of norm 1 (so that the group of units of K is the direct product of $\boldsymbol{E}$ and $\{\pm 1\}$). It was shown by Hasse ([1]) in case p is 3 and in a recent paper by Morikawa ([3]) for $p=5$ that we can find a unit ε in $\boldsymbol{E}$ which together with its conjugates generates $\boldsymbol{E}$. We shall call such a unit a *Minkowski unit* for K. We have the following generalization of the above results.

THEOREM. *Let h be the class number of K. Consider the set $\boldsymbol{A}$ of integral ideals $\boldsymbol{a}$ in the cyclotomic field $\boldsymbol{Q}_p$ of p^{th} roots of unity such that $h=N(\boldsymbol{a})$, where N denotes the absolute norm.*

i) *If all ideals $\boldsymbol{a}$ in $\boldsymbol{A}$ are principal, then K has a Minkowski unit;*

ii) *If no ideal $\boldsymbol{a}$ in $\boldsymbol{A}$ is principal, then K has no Minkowski unit.*

COROLLARY. *If p is at most 19, then K has a Minkowski unit since $\boldsymbol{Q}_p$ has class number 1 in those cases.*

REMARK. The second assertion suggests that a fearless computer would have no problem finding fields K with no Minkowski units.

PROOF OF THE THEOREM. Clearly $\boldsymbol{E}$ is a module over $\boldsymbol{Z}[G]/(1+s \cdots +s^{p-1})$ which is isomorphic with the ring of integers $\boldsymbol{O}$ in $\boldsymbol{Q}_p$ by the map sending s on a fixed primitive p^{th} root of unity. The cyclotomic units of K form a G-submodule $\boldsymbol{H}$ of $\boldsymbol{E}$ of index h by the analytic class number formulae (cf. [2]). In fact, $\boldsymbol{H}$ is the free $\boldsymbol{O}$-module generated by a cyclotomic unit η; since $\boldsymbol{O}$ is Dedekind and $\boldsymbol{E}$ has rank $p-1$, the isomorphism of $\boldsymbol{O}$ with $\boldsymbol{H}$ induced by sending 1 to η extends uniquely to an isomorphism of $\boldsymbol{a}^{-1}$ with $\boldsymbol{E}$ for a suitable integral ideal $\boldsymbol{a}$ of $\boldsymbol{Q}_p$. Hence we have $h=[\boldsymbol{E}:\boldsymbol{H}]=[\boldsymbol{a}^{-1}:\boldsymbol{O}]=N(\boldsymbol{a})$ which proves the theorem since K has a Minkowski unit if and only if $\boldsymbol{E}$ is a free $\boldsymbol{O}$-module, i.e. if and only if $\boldsymbol{a}$ is principal.

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[3] R. Morikawa, On the unit group for absolutely cyclic number fields of degree five, J. Math. Soc. Japan, **20** (1968), 263–265.