J. Math. Soc. Japan
Vol. 21, No. 2, 1969

On the integrability of Killing-Yano's equation

By Shun-ichi TACHIBANA and Toyoko KASHIWADA

(Received March 21, 1968)

Introduction.

In a Riemannian space M^n, a Killing vector v^h is a vector field satisfying the Killing's equation:

$$\nabla_i v_j + \nabla_j v_i = 0\,,$$

where ∇_i denotes the operator of the Riemannian covariant derivation. A Killing vector generates (locally) a one parameter group of isometries. On the other hand a one parameter group of affine transformations induces an affine Killing vector v^h characterized by the equation:

$$\nabla_j \nabla_i v^h + R_{lji}{}^h v^l = 0\,.$$

K. Yano[1] have introduced a Killing tensor of order r as a skew symmetric tensor field $u_{i_1 \cdots i_r}$ satisfying

$$\nabla_{i_0} u_{i_1 \cdots i_r} + \nabla_{i_1} u_{i_0 i_2 \cdots i_r} = 0\,.$$

In a previous paper[2], one of the authors discussed on Killing tensor of order 2. We shall generalize the results to the case of order $r \geqq 2$. In § 1 a system of linear differential equations to be satisfied by a Killing tensor is obtained. This equation enable us to define an affine Killing tensor as a generalization of an affine Killing vector. It will be shown that an affine Killing tensor is a Killing tensor in a compact M^n. We shall devote § 2 to prove that M^n is a space of constant curvature if it admits sufficiently many Killing tensors. § 3 deals with the converse problem. Thus we have a new characterization of a space of constant curvature. In § 4 we shall give examples of Killing tensor in the Euclidean space and the Euclidean sphere.

§ 1. Killing tensor. Affine Killing tensor.

Let M^n be an n dimensional Riemannian space whose metric tensor is given by g_{ab}[3] in terms of local coordinates $\{x^h\}$. We can regard the com-

1) K. Yano, [3].
2) S. Tachibana, [2].
3) $a, b, \cdots, i, j, \cdots = 1, \cdots, n$.

ponents $u_{i_1\cdots i_r}$ of a skew symmetric tensor as the coefficients of an exterior differential form:

$$u=\frac{1}{r!}u_{i_1\cdots i_r}dx^{i_1}\wedge\cdots\wedge dx^{i_r}.$$

The coefficients $(du)_{i_0\cdots i_r}$ of the exterior derivative:

$$du=\frac{1}{(r+1)!}(du)_{i_0\cdots i_r}dx^{i_0}\wedge\cdots\wedge dx^{i_r}$$

are given by

$$(du)_{i_0\cdots i_r}=\sum_{h=0}^{r}(-1)^h\nabla_{i_h}u_{i_0\cdots\hat{i}_h\cdots i_r},$$

where $\hat{i}_h$ means that i_h is omitted.

A skew symmetric tensor $u_{i_1\cdots i_r}$ is called *a Killing tensor of order r*, if it satisfies the Killing-Yano's equation:

$$\nabla_{i_0}u_{i_1\cdots i_r}+\nabla_{i_1}u_{i_0 i_2\cdots i_r}=0.$$

Let us consider a Killing tensor $u_{i_1\cdots i_r}$, then we have

$$(du)_{i_0\cdots i_r}=(r+1)\nabla_{i_0}u_{i_1\cdots i_r}. \tag{1.1}$$

As $ddu=0$ it follows that

$$\begin{aligned}(ddu)_{abi_1\cdots i_r}&=0\\&=\nabla_a(du)_{bi_1\cdots i_r}-\nabla_b(du)_{ai_1\cdots i_r}+\sum(-1)^{k+1}\nabla_{i_k}(du)_{abi_1\cdots\hat{i}_k\cdots i_r},\end{aligned}$$

which turns to

$$\nabla_a\nabla_b u_{i_1\cdots i_r}-\nabla_b\nabla_a u_{i_1\cdots i_r}+\sum_{k=1}^{r}(-1)^{k+1}\nabla_{i_k}\nabla_a u_{bi_1\cdots\hat{i}_k\cdots i_r}=0 \tag{1.2}$$

by virtue of (1.1).

On the other hand we have by Ricci's identity

$$\nabla_a\nabla_b u_{i_1\cdots i_r}-\nabla_b\nabla_a u_{i_1\cdots i_r}=-\sum_{h=1}^{r}R_{abi_h}{}^{c}u_{i_1\cdots c\cdots i_r},$$

$$\begin{aligned}\nabla_{i_k}\nabla_a u_{bi_1\cdots\hat{i}_k\cdots i_r}&=(-1)^k\nabla_a\nabla_b u_{i_1\cdots i_r}+(-1)^kR_{i_kab}{}^{d}u_{i_1\cdots d\cdots i_r}\\&\quad+\sum_{h(\neq k)}R_{i_kai_h}{}^{c}u_{bi_1\cdots c\cdots\hat{i}_k\cdots i_r},\end{aligned}$$

where the lower indices c and d appear at the h-th and k-th position respectively. Substituting these equations into (1.2) we can get

$$\begin{aligned}-r\nabla_a\nabla_b u_{i_1\cdots i_r}&-\sum R_{abi_h}{}^{c}u_{i_1\cdots c\cdots i_r}-\sum R_{i_hab}{}^{c}u_{i_1\cdots c\cdots i_r}\\&+\sum_k(-1)^{k+1}\sum_{h(\neq k)}R_{i_kai_h}{}^{c}u_{bi_1\cdots c\cdots\hat{i}_k\cdots i_r}=0.\end{aligned}$$

Thus we know that a Killing tensor $u_{i_1\cdots i_r}$ satisfies the following equation:

(1.3)
$$r\nabla_a\nabla_b u_{i_1\cdots i_r}+\sum_h R_{i_h ba}{}^c u_{i_1\cdots c\cdots i_r} - \sum_{h<k} R_{i_h i_k a}{}^c u_{i_1\cdots c\cdots i_{k-1} b i_{k+1}\cdots i_r}=0\,,$$

or equivalently

$$r\nabla_a\nabla_{i_1} u_{i_2\cdots i_{r+1}}+\sum_{h<k}(-1)^{k+1} R_{i_h i_k a}{}^c u_{i_1\cdots c\cdots \hat{i}_k\cdots i_{r+1}}=0\,,$$

where the lower index c appears at the h-th position.

Now we shall define *an affine Killing tensor of order* r as a skew symmetric tensor $u_{i_1\cdots i_r}$ satisfying (1.3). Any Killing tensor is an affine Killing tensor. The converse is also true for a compact Riemannian space. That is we have

THEOREM 1. *In a campact Riemannian space, an affine Killing tensor is a Killing tensor.*

PROOF. Transvecting (1.3) with g^{ab}, we get

(1.4)
$$r\nabla^a\nabla_a u_{i_1\cdots i_r}+\sum R_{i_h}{}^c u_{i_1\cdots c\cdots i_r} + \sum_{h<k} R_{i_h i_k}{}^{cb} u_{i_1\cdots c\cdots b\cdots i_r}=0\,.$$

Next by transvection (1.3) with g^{bi_1}, it follows that $\nabla_a\nabla^b u_{bi_2\cdots i_r}=0$. Then we have

(1.5)
$$(\nabla^a u_a{}^{i_2\cdots i_r})(\nabla^b u_{bi_2\cdots i_r})=\nabla^a(u_a{}^{i_2\cdots i_r}\nabla^b u_{bi_2\cdots i_r})\,.$$

Without loss of generality we can assume that M^n is orientable and then applying the Green's theorem to (1.5) we get

(1.6)
$$\nabla^b u_{bi_2\cdots i_r}=0\,.$$

Equations (1.4) and (1.6) are sufficient conditions for a skew symmetric tensor $u_{i_1\cdots i_r}$ to be a Killing tensor[4]. Q. E. D.

§ 2. A sufficient condition for M^n to be of constant curvature.

In this section we shall show that if a Riemannian space admits sufficiently many Killing tensor fields then it is a space of constant curvature.

For a Killing tensor $u_{i_1\cdots i_r}$ it holds that

(2.1)
$$r\nabla_a\nabla_b u_{i_1\cdots i_r}+\sum_h R_{i_h ba}{}^c u_{i_1\cdots c\cdots i_r} - \sum_{h<k} R_{i_h i_k a}{}^c u_{i_1\cdots c\cdots b\cdots i_r}=0\,.$$

Interchanging the indices a and b and subtracting the equation from (2.1),

4) K. Yano and S. Bochner, [4], p. 76.

we have

$$(r-1)\sum_{h} R_{abi_h}{}^{c}u_{i_1\cdots c\cdots i_r} + \sum_{h<k}(R_{i_h i_k a}{}^{c}u_{i_1\cdots c\cdots b\cdots i_r} - R_{i_h i_k b}{}^{c}u_{i_1\cdots c\cdots a\cdots i_r}) = 0 . \tag{2.2}$$

Now we put

$$B_{abi_1\cdots i_r}{}^{j_1\cdots j_r} = (r-1)\sum_{h} R_{abi_h}{}^{c}\delta_{i_1}{}^{j_1}\cdots\delta_{c}{}^{j_h}\cdots\delta_{i_r}{}^{j_r}$$
$$+\sum_{h<k}(R_{i_h i_k a}{}^{c}\delta_{i_1}{}^{j_1}\cdots\delta_{c}{}^{j_h}\cdots\delta_{b}{}^{j_k}\cdots\delta_{i_r}{}^{j_r} - R_{i_h i_k b}{}^{c}\delta_{i_1}{}^{j_1}\cdots\delta_{c}{}^{j_h}\cdots\delta_{b}{}^{j_k}\cdots\delta_{i_r}{}^{j_r}) ,$$

so as to write (2.2) as

$$B_{abi_1\cdots i_r}{}^{j_1\cdots j_r}u_{j_1\cdots j_r} = 0 . \tag{2.3}$$

THEOREM 2. *For any point p of a Riemannian space M^n and any skew symmetric constants $c_{i_1\cdots i_r}$, if there exists (locally) a Killing tensor $u_{i_1\cdots i_r}$ of order r $(2\leqq r<n)$ satisfying $u_{i_1\cdots i_r}(p)=c_{i_1\cdots i_r}$, then M^n is a space of constant curvature.*

PROOF. From (2.3) and the assumption of theorem, we have

$$\sum_{\sigma\in\mathfrak{S}} \text{sign}\,\sigma B_{abi_1\cdots i_r}{}^{\sigma(1)\cdots\sigma(r)} = 0 \tag{2.4}$$

on M^n, where

$$\mathfrak{S} = \left\{\text{permutation } \sigma \,|\, \sigma = \begin{pmatrix} j_1 & \cdots & j_r \\ \sigma(1) & \cdots & \sigma(r)\end{pmatrix}\right\}^{5)}.$$

Putting

$$\delta_{i_1\cdots i_r}^{j_1\cdots j_r} = \sum_{\sigma\in\mathfrak{S}} \text{sign}\,\sigma\delta_{i_1}{}^{\sigma(1)}\cdots\delta_{i_r}{}^{\sigma(r)} = \begin{vmatrix} \delta_{i_r}{}^{j_1} & \cdots & \delta_{i_1}{}^{j_r} \\ \vdots & & \vdots \\ \delta_{i_r}{}^{j_1} & \cdots & \delta_{i_r}{}^{j_r}\end{vmatrix},$$

we have from (2.4)

$$(r-1)\sum_{h} R_{abi_h}{}^{c}\delta_{i_1\cdots c\cdots i_r}^{j_1\cdots j_h\cdots j_r} + \sum_{h<k}(R_{i_h i_k a}{}^{c}\delta_{i_1\cdots c\cdots b\cdots i_r}^{j_1\cdots\cdots\cdots j_r} - R_{i_h i_k b}{}^{c}\delta_{i_1\cdots c\cdots a\cdots i_r}^{j_1\cdots\cdots\cdots j_r}) = 0 .$$

Contracting with respect to i_2 and $j_2, \cdots, i_r$ and j_r, we can get

$$(n-1)R_{abi_1}{}^{j_1} - R_{i_1 b}\delta_{a}{}^{j_1} + R_{i_1 a}\delta_{b}{}^{j_1} = 0 , \tag{2.5}$$

after some complicated computations where we have used $n>r\geqq 2$ and the identity

$$\delta_{i_1\cdots i_k j_{k+1}\cdots j_r}^{j_1\cdots j_k j_{k+1}\cdots j_r} = \frac{(n-k)!}{(n-r)!}\delta_{i_1\cdots i_k}^{j_1\cdots j_k} .$$

Thus we have

5) $\sigma(s)$ means $\sigma(j_s)$.

$$R_{abi_1}{}^{j_1} = \frac{R}{n(n-1)} (g_{bi_1}\delta_a{}^{j_1} - g_{ai_1}\delta_b{}^{j_1})$$

from (2.5) and hence M^n is of constant curvature.

§ 3. The converse problem.

We shall show the converse of Theorem 2 is true. Namely we have

THEOREM 3. *If M^n is a space of constant curvature, then there exists (locally) Killing tensor $u_{i_1\cdots i_r}$ of order r satisfying*

$$u_{i_1\cdots i_r}(p) = c_{i_1\cdots i_r}, \qquad (\nabla_{i_1} u_{i_2\cdots i_{r+1}})(p) = d_{i_1\cdots i_{r+1}},$$

where p is any given point and $c_{i_1\cdots i_r}$ and $d_{i_1\cdots i_{r+1}}$ are any given skew symmetric constants.

PROOF. It suffices to verify that the following system (3.1)$\sim$(3.4) of partial differential equations with unknown functions $u_{i_1\cdots i_r}$, $u_{i_1\cdots i_{r+1}}$ is completely integrable.

$$u_{i_1\cdots i_h\cdots i_k\cdots i_r} + u_{i_1\cdots i_k\cdots i_h\cdots i_r} = 0, \tag{3.1}$$

$$u_{i_1\cdots i_h\cdots i_k\cdots i_{r+1}} + u_{i_1\cdots i_k\cdots i_h\cdots i_{r+1}} = 0, \tag{3.2}$$

$$\nabla_{i_1} u_{i_2\cdots i_{r+1}} = u_{i_1\cdots i_{r+1}}, \tag{3.3}$$

$$\nabla_a u_{i_1\cdots i_{r+1}} = (1/r)\sum_{h<k}(-1)^k R_{i_h i_k a}{}^c u_{i_1\cdots c\cdots \hat{i}_k\cdots i_{r+1}}. \tag{3.4}$$

As M^n is a space of constant curvature, we can replace (3.4) by the following equation:

$$\nabla_a u_{i_1\cdots i_{r+1}} = \frac{R}{n(n-1)} \sum_{k=1}^{r+1} (-1)^k g_{i_k a} u_{i_1\cdots \hat{i}_k\cdots i_{r+1}}. \tag{3.4$'$}$$

The equations obtained from (3.1) by differentiation:

$$\partial_a u_{i_1\cdots i_h\cdots i_k\cdots i_r} + \partial_a u_{i_1\cdots i_k\cdots i_h\cdots i_r} = 0$$

are satisfied identically by (3.1), (3.3) and (3.2). The equations obtained from (3.2) by differentiation:

$$\partial_a u_{i_1\cdots i_h\cdots i_k\cdots i_{r+1}} + \partial_a u_{i_1\cdots i_k\cdots i_h\cdots i_{r+1}} = 0$$

are satisfied identically by (3.2), (3.4)$'$ and (3.1).

Next the integrability condition of (3.3):

$$\nabla_a \nabla_b u_{i_1\cdots i_r} - \nabla_b \nabla_a u_{i_1\cdots i_r} = -\sum_h R_{abi_h}{}^c u_{i_1\cdots c\cdots i_r}$$

follows from (3.3), (3.4)$'$ and (3.1) identically. Similarly the integrability condition of (3.4)$'$:

$$\nabla_a \nabla_b u_{i_1 \cdots i_{r+1}} - \nabla_b \nabla_a u_{i_1 \cdots i_{r+1}} = -\sum_h R_{abi_h}{}^c u_{i_1 \cdots c \cdots i_{r+1}}$$

follows from (3.4)′, (3.3) and (3.2) identically.

Thus the system (3.1)$\sim$(3.3) and (3.4)′ is completely integrable and then there exists (locally) Killing tensor of order r with the stated initial conditions. Q. E. D.

§4. Examples of Killing tensors.

(i) Let E^{n+1} be a Euclidean space and $\{y^\lambda\}$ $(\lambda=1, \cdots, n+1)$ an orthogonal coordinate system. A Killing tensor in E^{n+1} is a skew symmetric tensor $u_{\lambda_1 \cdots \lambda_r}$ such that

$$\partial_\mu u_{\lambda_1 \cdots \lambda_r} + \partial_{\lambda_1} u_{\mu \lambda_2 \cdots \lambda_r} = 0, \qquad (\partial_\lambda = \partial/\partial y^\lambda). \tag{4.1}$$

For such a tensor we have by virtue of (1.3)

$$\partial_\nu \partial_\mu u_{\lambda_1 \cdots \lambda_r} = 0.$$

Integrating the last equation we get as the general solution of (4.1)

$$u_{\lambda_1 \cdots \lambda_r} = y^\alpha a_{\alpha \lambda_1 \cdots \lambda_r} + b_{\lambda_1 \cdots \lambda_r}, \tag{4.2}$$

where $a_{\alpha \lambda_1 \cdots \lambda_r}$ and $b_{\lambda_1 \cdots \lambda_r}$ are skew symmetric constant tensors.

(ii) Let M^{n+1} be an $n+1$ dimensional Riemannian space and M^n be its hypersurface represented locally by $y^\lambda = y^\lambda(x^h)$ in terms of local coordinates $\{y^\lambda\}$ in M^{n+1} and $\{x^h\}$ in M^n. Putting $B_a{}^\lambda = \partial y^\lambda / \partial x^a$, the induced metric g_{ab} is given by $g_{ab} = B_a{}^\lambda B_b{}^\mu G_{\lambda\mu}$, where $G_{\lambda\mu}$ means the Riemannian metric of M^{n+1}. The second fundamental tensor $H_{ab}{}^\lambda$ is defined by

$$H_{ab}{}^\lambda \equiv \nabla_a B_b{}^\lambda \equiv \partial B_b{}^\lambda / \partial x^a - \left\{ {c \atop ab} \right\} B_c{}^\lambda + \left\{ {\lambda \atop \mu\nu} \right\} B_a{}^\mu B_c{}^\nu .$$

Let $u^{\lambda_1 \cdots \lambda_r}$ be a skew symmetric tensor field in M^{n+1} and assume that it is tangent to M^n at any point of M^n, that is, there exists a skew symmetric tensor field $v^{a_1 \cdots a_r}$ on M^n such that

$$u^{\lambda_1 \cdots \lambda_r} = B_{a_1}{}^{\lambda_1} \cdots B_{a_r}{}^{\lambda_r} v^{a_1 \cdots a_r} \tag{4.3}$$

holds good over M^n. Defining $B^a{}_\lambda$ by $B^a{}_\lambda = g^{ab} G_{\lambda\mu} B_b{}^\mu$, (4.3) reduces to

$$u_{\lambda_1 \cdots \lambda_r} = B^{a_1}{}_{\lambda_1} \cdots B^{a_r}{}_{\lambda_r} v_{a_1 \cdots a_r}$$

in terms of covariant components of the tensors.

Differentiating the last equation covariantly along M^n and transvecting this with $B_e{}^{\lambda_1}$ we can get

(3.4) $$B_e^{\lambda_1}B_c^{\nu}\nabla_\nu u_{\lambda_1\cdots\lambda_r}=\sum_h B^{a_2}{}_{\lambda_2}\cdots H_c{}^{a_h}{}_{\lambda_h}B^{a_{h+1}}{}_{\lambda_{h+1}}\cdots B^{a_r}{}_{\lambda_r}v_{ea_2\cdots a_r}$$
$$+B^{a_2}{}_{\lambda_2}\cdots B^{a_r}{}_{\lambda_r}\nabla_c v_{ea_2\cdots a_r}.$$

Now we assume that our M^n is totally umbilic. Then there exists a vector field C^λ on M^n locally such as $H_{ab}{}^\lambda=C^\lambda g_{ab}$ and hence (4.4) reduces to

$$B_e^{\lambda_1}B_c^{\nu}\nabla_\nu u_{\lambda_1\cdots\lambda_r}=\sum_h B^{a_2}{}_{\lambda_2}\cdots C_{\lambda_h}B^{a_{h+1}}{}_{\lambda_{h+1}}\cdots B^{a_r}{}_{\lambda_r}v_{ea_2\cdots c\cdots a_r}$$
$$+B^{a_2}{}_{\lambda_2}\cdots B^{a_r}{}_{\lambda_r}\nabla_c v_{ea_2\cdots a_r}.$$

This equation shows that $v_{a_1\cdots a_r}$ is a Killing tensor on M^n provided that $u_{\lambda_1\cdots\lambda_r}$ is Killing.

Next let $M^{n+1}=E^{n+1}$ and apply the above argument to the sphere $S^n:\sum(y^\lambda)^2=1$.

The condition in order that a skew symmetric tensor $u_{\lambda_1\cdots\lambda_r}$ to be tangent to S^n everywhere is $y^\alpha u_{\alpha\lambda_1\cdots\lambda_r}=0$. As a Killing tensor in E^{n+1} is the form (4.2), we know that

$$u_{\lambda_1\cdots\lambda_r}=y^\alpha a_{\alpha\lambda_1\cdots\lambda_r}$$

is a Killing tensor defined globally on S^n, where $a_{\alpha\lambda_1\cdots\lambda_r}$ is a skew symmetric constant tensor in E^{n+1}.

Ochanomizu University

Bibliography

[1] L. P. Eisenhart, Continuous group of transformations, Princeton, 1933.
[2] S. Tachibana, On Killing tensors in a Riemannian space, to appear.
[3] K. Yano, Some remarks on tensor fields and curvature, Ann. of Math., **55** (1952), 328–347.
[4] K. Yano and S. Bochner, Curvature and Betti numbers, Ann. of Math. Studies. No. 32, 1953.