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On doubly transitive permutation groups of degree n and order $4(n-1)n$*

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§1. Introduction.

Doubly transitive permutation groups of degree n and order $2(n-1)n$ were determined by N. Ito ([4]).

The object of this paper is to prove the following result.

THEOREM. *Let Ω be the set of symbols* $1, 2, \cdots, n$. *Let $\mathfrak{G}$ be a doubly transitive group on Ω of order* $4(n-1)n$ *not containing a regular normal subgroup and let $\mathfrak{K}$ be the stabilizer of the set of symbols* 1 *and* 2. *Assume that* $\mathfrak{K}\cap G^{-1}\mathfrak{K}G=1$ *or $\mathfrak{K}$ for every element G of $\mathfrak{G}$. Then we have the following results;*

(I) *If $\mathfrak{K}$ is a cyclic group, then $\mathfrak{G}$ is isomorphic to either* $PGL(2, 5)$ *or* $PSL(2, 9)$.

(II) *If K is an elementary abelian group, then $\mathfrak{G}$ is isomorphic to* $PSL(2, 7)$.

We use the standard notation. $C_{\mathfrak{X}}\mathfrak{T}$ denotes the centralizer of a subset $\mathfrak{T}$ in a group $\mathfrak{X}$ and $N_{\mathfrak{X}}\mathfrak{T}$ stands for the normalizer of $\mathfrak{T}$ in $\mathfrak{X}$. We denote the number of elements in $\mathfrak{T}$ by $|\mathfrak{T}|$.

§2. Proof of Theorem, (I).

1. Let $\mathfrak{H}$ be the stabilizer of the symbol 1. $\mathfrak{K}$ is of order 4 and it is generated by a permutation K whose cyclic structure has the form $(1)(2)\cdots$. Since $\mathfrak{G}$ is doubly transitive on Ω, it contains an involution I with the cyclic structure $(1\ 2)\cdots$. We may assume that I is conjugate to K^2. Then we have the following decomposition of $\mathfrak{G}$;

$$\mathfrak{G}=\mathfrak{H}+\mathfrak{H}I\mathfrak{H}.$$

Since I is contained in $N_{\mathfrak{G}}\mathfrak{K}$, it induces an automorphism of $\mathfrak{K}$ and (i) $\langle I\rangle\mathfrak{K}$ is an abelian 2-group of type $(2, 2^2)$ or (ii) $\langle I\rangle\mathfrak{K}$ is dihedral of order 8. If an element $H'IH$ of a coset $\mathfrak{H}IH$ of $\mathfrak{H}$ is an involution, then $IHH'I=(HH')^{-1}$ is contained in $\mathfrak{K}$. Hence, in case (i) the coset $\mathfrak{H}IH$ contains just two involutions, namely $H^{-1}IH$ and $H^{-1}K^2IH$, and, in case (ii), it contains just four involutions, namely $H^{-1}IH$, $H^{-1}KIH$, $H^{-1}K^2IH$ and $H^{-1}K^3IH$. Let $g(2)$ and $h(2)$ denote the numbers of involutions in $\mathfrak{G}$ and $\mathfrak{H}$, respectively. Then the following equality is obtained;

$$g(2)=h(2)+\alpha(n-1)\,, \tag{2.1}$$

where $\alpha=2$ and 4 for cases (i) and (ii), respectively.

2. Let $\mathfrak{K}$ keep i $(i\geqq 2)$ symbols of Ω, say $1, 2, \cdots, i$, unchanged. It is trivial by the assumption of $\mathfrak{K}$ that K has no transposition in its cyclic decomposition and that $N_{\mathfrak{G}}\mathfrak{K}=C_{\mathfrak{G}}K^2$. Put $\mathfrak{J}=\{1, 2, \cdots, i\}$. Then, by a theorem of Witt ([**6**], Th. 9.4), $N_{\mathfrak{G}}\mathfrak{K}/\mathfrak{K}$ can be considered as a doubly transitive permutation group on $\mathfrak{J}$. Since every permutation of $N_{\mathfrak{G}}\mathfrak{K}/\mathfrak{K}$ distinct from $\mathfrak{K}$ leaves by the definition of $\mathfrak{K}$ at most one symbol of $\mathfrak{J}$ fixed, $N_{\mathfrak{G}}\mathfrak{K}/\mathfrak{K}$ is a complete Frobenius group on $\mathfrak{J}$. Therefore i equals to a power of a prime number, say p^m, and the orders of $N_{\mathfrak{G}}\mathfrak{K}$ and $\mathfrak{H}\cap N_{\mathfrak{G}}\mathfrak{K}$ are equal to $4i(i-1)$ and $4(i-1)$, respectively. Hence there exist $(n-1)n/(i-1)i$ involutions in $\mathfrak{G}$ each of which is conjugate to K^2.

At first, let us assume that n is odd. Let $h^*(2)$ be the number of involutions in $\mathfrak{H}$ leaving only the symbol 1 fixed. Then from (2.1) and the above argument the following equality is obtained;

$$h^*(2)n+(n-1)n/(i-1)i=(n-1)/(i-1)+h^*(2)+\alpha(n-1)\,. \tag{2.2}$$

Since i is less than n, it follows from (2.2) that $h^*(2)<\alpha$. If $h^*(2)=1$, then there exists no group satisfying the conditions of the theorem. In fact, let J be the involution in $\mathfrak{H}$ leaving only the symbol 1 fixed. By [**2**, Cor. 1, p. 414], J is contained in $Z^*(\mathfrak{G})$, where $Z^*(\mathfrak{G})$ is the subgroup of $\mathfrak{G}$ containing the core of $\mathfrak{G}$, $K(\mathfrak{G})$, for which $Z^*(\mathfrak{G})/K(\mathfrak{G})=Z(\mathfrak{G}/K(\mathfrak{G}))$. If $K(\mathfrak{G})\neq 1$, then by the theorem of Feit-Thompson $K(\mathfrak{G})$ is solvable ([**1**]). Hence $\mathfrak{G}$ contains a regular normal subgroup ([**6**, Th. 11.5]). We have $K(\mathfrak{G})=1$ and J is an element of $Z(\mathfrak{G})$. Hence $Z(\mathfrak{G})\neq 1$. But $\mathfrak{G}$ must also contain a regular normal subgroup. Hence we may assume $h^*(2)\neq 1$. Thus there are three cases; (A) $\alpha-h^*(2)=1$, (B) $\alpha-h^*(2)=2$ and (C) $\alpha-h^*(2)=4$.

The following equalities are obtained from (2.2) for cases (A), (B) and (C), respectively.

(A) $\quad n=i^2=p^{2m} \qquad (p\colon \text{odd})\,,$

(B) $\quad n=i(2i-1)=p^m(2p^m-1) \qquad (p\colon \text{odd})$

and

(C) $\quad n=i(4i-3)=p^m(4p^m-3) \qquad (p\colon \text{odd})\,.$

Next let us assume that n is even. Let $g^*(2)$ be the number of involutions in $\mathfrak{G}$ leaving no symbol of Ω fixed. Then corresponding to (2.2) the following equality is obtained from (1);

$$g^*(2)+(n-1)n/(i-1)i=(n-1)/(i-1)+\alpha(n-1)\,. \tag{2.3}$$

Let J be an involution in $\mathfrak{G}$ leaving no symbol of Ω fixed. Let $C_{\mathfrak{G}}J$ be the centralizer of J in $\mathfrak{G}$. Assume that the order of $C_{\mathfrak{G}}J$ is divisible by a prime factor q of $n-1$. Then $C_{\mathfrak{G}}J$ contains a permutation Q of order q. Since q is odd, Q must leave just one symbol of Ω fixed. This shows that Q cannot be commutative with J. Hence $g^*(2)$ is a multiple of $n-1$. It follows from (2.3) that $g^*(2)<\alpha(n-1)$. Thus there are four cases; (D) $\alpha-g^*(2)/(n-1)=1$, (E) $\alpha-g^*(2)/(n-1)=2$, (F) $\alpha-g^*(2)/(n-1)=3$ and (G) $\alpha-g^*(2)/(n-1)=4$.

The following equalities are obtained from (2.3) for cases (D), (E), (F) and (G), respectively;

(D) $\quad n=i^2=2^{2m}\,,$

(E) $\quad n=i(2i-1)=2^m(2^{m+1}-1)\,,$

(F) $\quad n=i(3i-2)=2^{m+1}(3\cdot 2^{m-1}-1)$

and

(G) $\quad n=i(4i-3)=2^m(2^{m+2}-3)\,.$

3. Let us assume that n is odd. Let $\mathfrak{P}$ be a Sylow p-subgroup of $N_{\mathfrak{G}}\mathfrak{K}$. Then, since $N_{\mathfrak{G}}\mathfrak{K}/\mathfrak{K}$ is a complete Frobenius group of degree p^m and $\mathfrak{K}$ is cyclic, $\mathfrak{P}$ is elementary abelian and normal in $N_{\mathfrak{G}}\mathfrak{K}$.

4. Case (A). Let $\mathfrak{M}$ be a subgroup of $\mathfrak{G}$ such that its Sylow 2-subgroup $\mathfrak{K}'$ is conjugate to subgroup of $\mathfrak{K}$. Then, since $\mathfrak{K}$ is cyclic, $\mathfrak{K}'$ has a normal 2-complement in $\mathfrak{M}$. By this fact it can be proved in the same way of Case (A) in [4] that there exists no group satisfying the conditions of the theorem in Case (A) (see [4], p. 411).

5. Case (B) and (C) ($p\neq 3$ for Case (C)). $\mathfrak{P}$ is also a Sylow p-subgroup of $\mathfrak{G}$ in these cases. Let the orders of $N_{\mathfrak{G}}\mathfrak{P}$ and $C_{\mathfrak{G}}\mathfrak{P}$ be $4(p^m-1)p^m x$ and $4p^m y$, respectively. If $x=1$, then from Sylow's theorem it should hold that $(2p^m-1)(2p^m+1)\equiv 1 \pmod p$ and $(4p^m-3)(4p^m+1)\equiv 1 \pmod p$ for Cases (B) and (C), respectively, which, since p is odd, is a contradiction. Thus x is greater than one. If $y=1$, then $\mathfrak{K}$ would be normal in $N_{\mathfrak{G}}\mathfrak{P}$, and this would imply that $x=1$. Thus y is greater than one. If y is even, then let $\mathfrak{S}$ be a Sylow 2-subgroup of $C_{\mathfrak{G}}\mathfrak{P}$. Since the order of $\mathfrak{S}$ must be greater than four, $\mathfrak{S}$ leaves just one symbol of Ω fixed. Hence $\mathfrak{S}$ cannot be contained in $C_{\mathfrak{G}}\mathfrak{P}$. Thus y is odd and y is a factor of $2p^m-1$ and $4p^m-3$ for Cases (B) and (C), respectively. $\mathfrak{P}$ has a normal complement $\mathfrak{A}$ in $C_{\mathfrak{G}}\mathfrak{P}$ and, since $\mathfrak{K}$ is cyclic, $\mathfrak{K}$ has also a normal complement $\mathfrak{B}$ in $C_{\mathfrak{G}}\mathfrak{P}$. Let $\mathfrak{Y}$ be the intersection of $\mathfrak{A}$ and $\mathfrak{B}$. $\mathfrak{Y}$ is a normal Hall subgroup of $C_{\mathfrak{G}}\mathfrak{P}$ of order y. Then $\mathfrak{Y}$ is normal even in

$N_{\mathfrak{G}}\mathfrak{P}$.

Let $\mathfrak{V}$ be a Sylow p-complement of $N_{\mathfrak{G}}\mathfrak{K}$ of order $4(p^m-1)$. Then $\mathfrak{V}$ is contained in $N_{\mathfrak{G}}\mathfrak{Y}$. Since y is a factor of n, any permutation $(\neq 1)$ of $\mathfrak{Y}$ does not leave any symbol of Ω fixed. On the other hand every element $(\neq 1)$ of $\mathfrak{V}$ leaves a symbol of Ω fixed. Therefore every permutation $(\neq 1)$ of $\mathfrak{V}$ is not commutative with any permutation $(\neq 1)$ of $\mathfrak{Y}$. This implies that y is not less than $4p^m-3$. Thus there exists no group satisfying the conditions of the theorem in Case (B). In Case (C) y is equal to $4p^m-3$. All permutations $(\neq 1)$ of $\mathfrak{Y}$ are conjugate under $\mathfrak{V}$. Therefore $4p^m-3$ must be equal to a power of a prime, say q^l, and $\mathfrak{Y}$ must be an elementary abelian q-group. It is easily seen that $C_{\mathfrak{G}}\mathfrak{Y}=\mathfrak{P}\mathfrak{Y}$. Hence $N_{\mathfrak{G}}\mathfrak{Y}$ is contained in $N_{\mathfrak{G}}\mathfrak{P}$ and therefore we obtain that $N_{\mathfrak{G}}\mathfrak{Y}=N_{\mathfrak{G}}\mathfrak{P}$. It can be easily seen that the set of involutions in $N_{\mathfrak{G}}\mathfrak{P}$ each of which is conjugate to K^2 in $N_{\mathfrak{G}}\mathfrak{P}$ is equal to the set of involutions in $C_{\mathfrak{G}}\mathfrak{P}$ each of which is conjugate to K^2 in $C_{\mathfrak{G}}\mathfrak{P}$. It is trivial that the intersection of $N_{\mathfrak{G}}\mathfrak{K}$ and $C_{\mathfrak{G}}\mathfrak{P}$ is equal to $\mathfrak{K}\mathfrak{P}$. Therefore we obtain that the index of $\mathfrak{K}\mathfrak{P}$ in $C_{\mathfrak{G}}\mathfrak{P}$ is equal to the index of $N_{\mathfrak{G}}\mathfrak{K}$ in $N_{\mathfrak{G}}\mathfrak{K}C_{\mathfrak{G}}\mathfrak{P}$. Thus $N_{\mathfrak{G}}\mathfrak{P}$ is equal to $N_{\mathfrak{G}}\mathfrak{K}C_{\mathfrak{G}}\mathfrak{P}$ and therefore the index of $N_{\mathfrak{G}}\mathfrak{P}$ in $\mathfrak{G}$ is equal to $4p^m+1$. Then we must have that $4p^m+1\equiv 4 \pmod q$, which contradicts the theorem of Sylow. Thus there exists no group satisfying the conditions of the theorem in Case (C).

6. Case (C) for $p=3$. At first we shall prove that the order of $C_{\mathfrak{G}}\mathfrak{P}$ is equal to $4\cdot 3^{m+1}y$, where y is a factor of $4\cdot 3^{m-1}-1$. $\mathfrak{K}$ is contained in $C_{\mathfrak{G}}\mathfrak{P}$. If the order of $C_{\mathfrak{G}}\mathfrak{P}$ is equal to $4\cdot 3^m$, then $N_{\mathfrak{G}}\mathfrak{P}$ is contained in $N_{\mathfrak{G}}\mathfrak{K}$. On the other hand the order of $N_{\mathfrak{G}}\mathfrak{P}$ is divisible by 3^{m+1}. Thus the order of $C_{\mathfrak{G}}\mathfrak{P}$ is greater than $4\cdot 3^m$. Assume that the order of $C_{\mathfrak{G}}\mathfrak{P}$ is equal to $4\cdot 3^m\cdot y'$, where y' is not divisible by 3 and it is a factor of $4\cdot 3^{m-1}-1$. Likewise in 5 there exists a normal subgroup $\mathfrak{Y}'$ of $C_{\mathfrak{G}}\mathfrak{P}$ of order y' and it is normal even in $N_{\mathfrak{G}}\mathfrak{P}$. Let $\mathfrak{V}$ be a Sylow 3-complement of $N_{\mathfrak{G}}\mathfrak{Y}$ of order $4(3^m-1)$. Since every permutation $(\neq 1)$ of $\mathfrak{Y}'$ leaves no symbol of Ω fixed and it is not commutative with any permutation $(\neq 1)$ leaving a symbol of Ω fixed, every permutation $(\neq 1)$ of $\mathfrak{Y}'$ is not commutative with any permutation $(\neq 1)$ of $\mathfrak{V}$. Hence y' is no less than $4\cdot 3^m-3$. This is a contradiction. Thus the order of $C_{\mathfrak{G}}\mathfrak{P}$ is equal to $4\cdot 3^{m+1}y$. Let $\mathfrak{P}'$ be a Sylow 3-subgroup of $C_{\mathfrak{G}}\mathfrak{P}$ of order 3^{m+1}. Since $\mathfrak{P}$ is contained in $C_{\mathfrak{G}}(\mathfrak{P}')$, $\mathfrak{P}'$ is abelian.

Let us assume $y>1$. Let $\mathfrak{A}$ be a normal 2-complement in $C_{\mathfrak{G}}\mathfrak{P}$. It is trivial that $C_{\mathfrak{G}}\mathfrak{P}'$ is contained in $\mathfrak{A}$. An element of $(\mathfrak{A}\frown N_{\mathfrak{G}}\mathfrak{P}')/C_{\mathfrak{G}}\mathfrak{P}'$ induces trivial automorphism of $\mathfrak{P}$ and $\mathfrak{P}'/\mathfrak{P}$. Therefore $(\mathfrak{A}\frown N_{\mathfrak{G}}\mathfrak{P}')/C_{\mathfrak{G}}\mathfrak{P}'$ must be 3-group. Thus we have $\mathfrak{A}\frown N_{\mathfrak{G}}\mathfrak{P}'=C_{\mathfrak{G}}\mathfrak{P}'$. By the splitting theorem of Burnside $\mathfrak{P}'$ has a normal complement $\mathfrak{Y}$ in $\mathfrak{A}$. Since $\mathfrak{Y}$ is a Hall subgroup of $C_{\mathfrak{G}}\mathfrak{P}$, it is normal in $N_{\mathfrak{G}}\mathfrak{P}$. Since every permutation $(\neq 1)$ of $\mathfrak{Y}$ is not commutative with any permutation ($\neq 1$) of $\mathfrak{V}$, y is no less than $4\cdot 3^m-3$. This is a contradiction. Therefore y must be equal to 1 and then $C_{\mathfrak{G}}\mathfrak{P}$ is equal to $\mathfrak{P}'\mathfrak{K}$.

The order of the group of automorphisms of $\mathfrak{P}'/\mathfrak{P}$ is equal to 2. Therefore K^2 must induce the trivial automorphism of $\mathfrak{P}'/\mathfrak{P}$. Since K is contained in $C_{\mathfrak{G}}\mathfrak{P}$, K^2 is commutative with every element of $\mathfrak{P}'$. By the assumption of theorem $\mathfrak{P}'$ must be contained in $N_{\mathfrak{G}}\mathfrak{K}$. Since $\mathfrak{P}$ is a Sylow 3-subgroup of $N_{\mathfrak{G}}\mathfrak{K}$, this is a contradiction.

Thus there exists no group satisfying the conditions of the theorem in Case (C) for $p=3$.

7. Case (D). It can be proved in the same way as in Case (C) in [4] that there exists no group satisfying the conditions of the theorem in Case (D).

8. Case (E), (F) and (G). Let $\mathfrak{S}$ be a Sylow 2-subgroup of $N_{\mathfrak{G}}\mathfrak{K}$. Since $N_{\mathfrak{G}}\mathfrak{K}/\mathfrak{K}$ is a complete Frobenius group on $\mathfrak{J}$, $\mathfrak{S}$ is normal in $N_{\mathfrak{G}}\mathfrak{K}$. Therefore $C_{\mathfrak{G}}\mathfrak{K}$ contains $\mathfrak{S}$ or is contained in $\mathfrak{S}$.

In the case $\alpha=4$, I is not contained in $C_{\mathfrak{G}}\mathfrak{K}$. Thus $\mathfrak{S}$ contains $C_{\mathfrak{G}}\mathfrak{K}$. Since the index of $\mathfrak{S}$ in $N_{\mathfrak{G}}\mathfrak{K}$ is equal to 2^m-1, we have $m=1$. Therefore it can be easily seen that $\mathfrak{G}$ is isomorphic to $PGL(2, 5)$ in Case (E) for $\alpha=4$ and that $\mathfrak{G}$ is isomorphic to $PSL(2, 9)$ in Case (G). In Case (F), since $n-i=6$ and $n-i$ must be divisible by 4, there exists no group satisfying the conditions of the theorem.

Next we shall consider Case (E) for $\alpha=2$. Let $\mathfrak{V}$ be a Sylow 2-complement of $N_{\mathfrak{G}}\mathfrak{K}$ of order 2^m-1. Since all the elements ($\neq 1$) of $\mathfrak{S}/\mathfrak{K}$ are conjugate under $\mathfrak{V}\mathfrak{K}/\mathfrak{K}$, every permutation ($\notin \mathfrak{K}$) of $\mathfrak{S}$ can be represented uniquely in the form $V^{-1}IV$, $V^{-1}IVK$, $V^{-1}IVK^2$ or $V^{-1}IVK^3$, where V is any permutation of $\mathfrak{V}$. Thus $S^2=K^2$ for any permutation S of order 4 in $\mathfrak{S}$. Since I is contained in $C_{\mathfrak{G}}\mathfrak{K}$, K is contained $C_{\mathfrak{G}}I$. Let $\mathfrak{S}'$ be a Sylow 2-subgroup of $C_{\mathfrak{G}}I$. Then, since $C_{\mathfrak{G}}I$ is conjugate to $C_{\mathfrak{G}}K^2=N_{\mathfrak{G}}\mathfrak{K}$, $\mathfrak{S}'$ contains K. Thus we must have $K^2=I$. This is a contradiction.

§ 3. Proof of Theorem, (II).

1. Let $\mathfrak{H}$, $\mathfrak{K}$ and I be as in § 2. Then in this case $\mathfrak{K}$ is elementary abelian and it is generated by two involutions, say K_1, K_2, leaving the symbols 1, 2 fixed. We may assume that I is conjugate to a permutation of $\mathfrak{K}$. Then we have the following decomposition of $\mathfrak{G}$;

$$\mathfrak{G}=\mathfrak{H}+\mathfrak{H}I\mathfrak{H}.$$

Since I is contained in $N_{\mathfrak{G}}\mathfrak{K}$, (i) $\langle I\rangle\ \mathfrak{K}$ is an abelian 2-group of type (2, 2, 2) or (ii) $\langle I\rangle\ \mathfrak{K}$ is dihedral of order 8. If an element $H'IH$ of a coset $\mathfrak{H}IH$ of $\mathfrak{H}$ is an involution, then $IHH'I=(HH')^{-1}$ is contained in $\mathfrak{K}$. Hence, in case (i), the coset $\mathfrak{H}IH$ contains just four involutions namely $H^{-1}IH$, $H^{-1}K_1IH$, $H^{-1}K_2IH$ and $H^{-1}K_1K_2IH$ and, in case (ii), it contains just two involutions, namely $H^{-1}IH$ and $H^{-1}K_1K_2IH$, $H^{-1}K_2IH$ or $H^{-1}K_1IH$. Let $g(2)$, $g^*(2)$, $h(2)$ and $h^*(2)$ be as in § 2. Then the following equality is obtained;

$$g(2)=h(2)+\alpha(n-1)\,, \tag{3.1}$$

where $\alpha=4$ and 2 for cases (i) and (ii), respectively.

2. Let $\mathfrak{I}$ be as in § 2. Then $N_{\mathfrak{G}}\mathfrak{K}/\mathfrak{K}$ can be considered as a complete Frobenius group on $\mathfrak{I}$ and i equals a power of a prime number, say p^m, and the orders of $N_{\mathfrak{G}}\mathfrak{K}$ and $N_{\mathfrak{G}}\mathfrak{K}\cap\mathfrak{H}$ are equal to $4i(i-1)$ and $4(i-1)$, respectively. Hence, since $\mathfrak{K}$ has just three involutions, there exist $3(n-1)n/(i-1)i$ involutions in $\mathfrak{G}$ each of which is conjugate to an involution in $\mathfrak{K}$.

At first, let us assume that n is odd. Then from (3.1) the following equality is obtained;

$$h^*(2)n+3(n-1)n/(i-1)i=3(n-1)/(i-1)+h^*(2)+\alpha(n-1)\,. \tag{3.2}$$

It follows from (3.2) that $h^*(2)<\alpha$. Likewise in § 2.2 we may assume $h^*(2)\neq 1$. Thus there are three cases; (A) $\alpha-h^*(2)=1$, (B) $\alpha-h^*(2)=2$ and (C) $\alpha-h^*(2)=4$. The following equalities are obtained from (3.2) for cases (A), (B) and (C), respectively;

(A) $\quad n=\frac{1}{3}i(i+2)=\frac{1}{3}p^m(p^m+2) \qquad (p\colon \text{odd})\,,$

(B) $\quad n=\frac{1}{3}i(2i+1)=\frac{1}{3}p^m(2p^m+1) \qquad (p\colon \text{odd})$

and

(C) $\quad n=\frac{1}{3}i(4i-1)=\frac{1}{3}p^m(4p^m-1) \qquad (p\colon \text{odd})\,.$

Next let us assume that n is even. Corresponding to (2.2) the following equality is obtained from (3.1);

$$g^*(2)+3(n-1)n/(i-1)i=3(n-1)/(i-1)+\alpha(n-1)\,. \tag{3.3}$$

Likewise in § 2 $g^*(2)$ is multiple of $n-1$. It follows from (3.3) that $g^*(2)<\alpha(n-1)$. Thus there are four cases; (D) $\alpha-g^*(2)/(n-1)=3$, (E) $\alpha-g^*(2)/(n-1)=1$, (F) $\alpha-g^*(2)/(n-1)=2$ and (G) $\alpha-g^*(2)/(n-1)=4$.

The following equalities are obtained from (3.3) for cases (D), (E), (F) and (G), respectively;

(D) $\quad n=i^2=2^{2m}\,,$

(E) $\quad n=\frac{1}{3}i(i+2)=\frac{1}{3}2^{m+1}(2^{m-1}+1)\,,$

(F) $\quad n=\frac{1}{3}i(2i+1)=\frac{1}{3}2^m(2^{m+1}+1)$

and

(G) $\quad n=\frac{1}{3}i(4i-1)=\frac{1}{3}2^m(2^{m+2}-1)\,.$

3. Let us assume that n is odd. Let $\mathfrak{P}$ be a Sylow p-subgroup of $N_{\mathfrak{G}}\mathfrak{K}$ and let $\mathfrak{V}$ be the subgroup of $N_{\mathfrak{G}}\mathfrak{K}$ consisting of permutations leaving the symbol 1 fixed. Then the order of $\mathfrak{V}$ is equal to $4(p^m-1)$. Since $N_{\mathfrak{G}}\mathfrak{K}/\mathfrak{K}$ is a complete Frobenius group of degree p^m, $\mathfrak{P}$ is elementary abelian of order p^m and $\mathfrak{P}\mathfrak{K}$ is normal in $N_{\mathfrak{G}}\mathfrak{K}$. Since $C_{\mathfrak{G}}\mathfrak{K}$ is normal in $N_{\mathfrak{G}}\mathfrak{K}$, $C_{\mathfrak{G}}\mathfrak{K}$ contains $\mathfrak{P}\mathfrak{K}$ or $\mathfrak{P}\mathfrak{K}$ is greater than $C_{\mathfrak{G}}\mathfrak{K}$. It is trivial that the index of $C_{\mathfrak{G}}\mathfrak{K}$ in $N_{\mathfrak{G}}\mathfrak{K}$ is a factor of 6. If $\mathfrak{P}\mathfrak{K}$ is greater than $C_{\mathfrak{G}}\mathfrak{K}$, we must have $p=3$ and $m=1$.

4. Cases (A), (B) and (C). At first let us assume $p=3$. Since the order of $N_{\mathfrak{G}}\mathfrak{K}$ is equal to $4\cdot 3^m(3^m-1)$, the order of $\mathfrak{G}$ is divisible by 3^m. But in Cases (A) and (C) it is not divisible by 3^m. In Case (B) m must be equal to 1 and it can be easily checked that $\mathfrak{G}$ is isomorphic to $PSL(2,7)$ as a permutation group of degree 7. Hence it will be assumed hereafter that p is greater than 3 and therefore $\mathfrak{P}\mathfrak{K}$ is contained in $C_{\mathfrak{G}}\mathfrak{K}$.

It is trivial that $\mathfrak{P}$ is normal in $\mathfrak{P}\mathfrak{K}$. Therefore $\mathfrak{P}$ is normal even in $N_{\mathfrak{G}}\mathfrak{K}$. Let the orders of $N_{\mathfrak{G}}\mathfrak{P}$ and $C_{\mathfrak{G}}\mathfrak{P}$ be $4(p^m-1)p^mx$ and $4p^my$, respectively. If $x=1$, from Sylow's theorem it should hold that $\frac{1}{9}(p^m+2)(p^m+3)\equiv 1 \pmod{p}$, $\frac{1}{9}(2p^m+1)(2p^m+3)\equiv 1 \pmod{p}$ and $\frac{1}{9}(4p^m-1)(4p^m+3)\equiv 1 \pmod{p}$ for Cases (A), (B) and (C), respectively, which, since p is greater than 3, is a contradiction. Thus x is greater than 1. If $y=1$, then $\mathfrak{K}$ would be normal in $N_{\mathfrak{G}}\mathfrak{P}$, and this would imply that $x=1$. Thus y is greater than 1. Since y is a factor of n, we have $N_{\mathfrak{G}}\mathfrak{K}\cap C_{\mathfrak{G}}\mathfrak{P}=C_{\mathfrak{G}}\mathfrak{K}\cap C_{\mathfrak{G}}\mathfrak{P}=\mathfrak{K}\mathfrak{P}$. Therefore $C_{\mathfrak{G}}\mathfrak{P}$ contains a normal subgroup $\mathfrak{Y}$ of order y. $\mathfrak{Y}$ is normal even in $N_{\mathfrak{G}}\mathfrak{P}$.

Let us consider the subgroup $\mathfrak{Y}\mathfrak{V}$. Since $\mathfrak{Y}$ is subgroup of $C_{\mathfrak{G}}\mathfrak{P}$, any permutation $(\neq 1)$ of $\mathfrak{Y}$ does not leave any symbol of Ω fixed. Therefore every permutation $(\neq 1)$ of $\mathfrak{V}$ is not commutative with any permutation $(\neq 1)$ of $\mathfrak{Y}$. This imply that y is not less than $4\cdot p^m-3$. But y is a factor of $\frac{1}{3}(p^m+2)$, $\frac{1}{3}(2p^m+1)$ and $\frac{1}{3}(4p^m-1)$ for Cases (A), (B) and (C), respectively, which is a contradiction.

5. Let us assume that n is even. Since n is integer, we may assume that m is even for Cases (E), (F) and (G). Let $\mathfrak{S}$ be a Sylow 2-group of $N_{\mathfrak{G}}\mathfrak{K}$ of order 2^{m+2} and let $\mathfrak{V}$ be a Sylow 2-complement of $N_{\mathfrak{G}}\mathfrak{K}\cap\mathfrak{H}$ of order 2^m-1. Then $\mathfrak{S}/\mathfrak{K}$ is elementary abelian. Likewise in § 2, 8 every permutation $(\notin\mathfrak{K})$ of $\mathfrak{S}$ can be represented uniquely in the form $V^{-1}IV$, $V^{-1}IVK_1$, $V^{-1}IVK_2$ or

$V^{-1}IVK_1K_2$, where V is any permutation of $\mathfrak{V}$. Then if I is contained in $C_{\mathfrak{G}}\mathfrak{K}$, every permutation $(\neq 1)$ of $\mathfrak{S}$ is an involution and therefore $\mathfrak{S}$ is elementary abelian and it is contained in $C_{\mathfrak{G}}\mathfrak{K}$. Let β be the number of involutions of $\mathfrak{S}$ leaving just i symbols of Ω fixed. It is clear that every permutation $(\notin \mathfrak{K})$ is conjugate under $\mathfrak{V}$ to I, IK_1, IK_2 or IK_1K_2. Thus β is equal to $(2^m-1)+3$, $2(2^m-1)+3$, $3(2^m-1)+3$ or $4(2^m-1)+3$.

Now let us assume that $\mathfrak{S}$ is greater than $C_{\mathfrak{G}}\mathfrak{K}$. Then we have $m=2$ and the orders of $N_{\mathfrak{G}}\mathfrak{K}$, $C_{\mathfrak{G}}\mathfrak{K}$ and $\mathfrak{S}$ are $16\cdot 3$, 8 and 16, respectively. It is easily seen that the number of involutions of $\mathfrak{S}$ is equal to 9. But there exists no non-abelian group of order 16 satisfying the above condition. Hence it will be assumed that $\mathfrak{S}$ is contained in $C_{\mathfrak{G}}\mathfrak{K}$. Let us consider the order of $N_{\mathfrak{G}}\mathfrak{S}$. If $G^{-1}\mathfrak{S}G$ contains $\mathfrak{K}$ for some $G\in\mathfrak{G}$, then $G\in N_{\mathfrak{G}}(\mathfrak{S})$. In fact, since $\mathfrak{S}$ is elementary abelian and normal in $N_{\mathfrak{G}}\mathfrak{K}$, $G^{-1}\mathfrak{S}G$ is contained in $N_{\mathfrak{G}}\mathfrak{K}$ and $G\in N_{\mathfrak{G}}(\mathfrak{S})$. Let γ be the number of subgroups of $\mathfrak{S}$ each of which is conjugate to $\mathfrak{K}$ in $\mathfrak{G}$. Then we have

$$[\mathfrak{G}:N_{\mathfrak{G}}(\mathfrak{K})]=\gamma[\mathfrak{G}:N_{\mathfrak{G}}(\mathfrak{S})]\,.$$

On the other hand, since $\mathfrak{K}\cap G^{-1}\mathfrak{K}G=1$ for every $G\notin N_{\mathfrak{G}}\mathfrak{K}$, 3γ is equal to β. Hence we have the following equality;

$$|\mathfrak{G}|/|N_{\mathfrak{G}}\mathfrak{S}|=3|\mathfrak{G}|/\beta|N_{\mathfrak{G}}\mathfrak{K}|\,. \tag{3.4}$$

6. Case (D). Since $3|\mathfrak{G}|/\beta|N_{\mathfrak{G}}\mathfrak{K}|=3\cdot 2^m(2^m+1)/\beta$ is integer, we have $\beta=6$ for $m=2$, $3\cdot 2^m$ or 15 for $m=2$. If $m=2$ and $\beta=6$, then $\mathfrak{H}\cap N_{\mathfrak{G}}\mathfrak{S}=\mathfrak{K}\mathfrak{V}$. If $m=2$ and $\beta=15$, then $|\mathfrak{H}|=4\cdot 3\cdot 5$ and $|N_{\mathfrak{G}}\mathfrak{S}|=16\cdot 3\cdot 5$. Since $\mathfrak{H}\cap N_{\mathfrak{G}}\mathfrak{S}$ contains $\mathfrak{K}$, $|\mathfrak{H}\cap N_{\mathfrak{G}}\mathfrak{S}|=4\cdot 3\cdot 5$. Hence $\mathfrak{H}$ is contained in $N_{\mathfrak{G}}\mathfrak{S}$ and the index of $\mathfrak{H}$ in $N_{\mathfrak{G}}\mathfrak{S}$ is equal to 4. Let $\mathfrak{W}$ be a Sylow 5-group of $\mathfrak{H}$. Then, since $N_{\mathfrak{G}}\mathfrak{W}$ is contained in $\mathfrak{H}$, by Sylow's theorem the index of $N_{\mathfrak{G}}\mathfrak{W}$ in $\mathfrak{H}$ is equal to 1 or 6. Therefore the index of $N_{\mathfrak{G}}\mathfrak{W}$ in $N_{\mathfrak{G}}\mathfrak{S}$ must be equal to 4 or 24. This is a contradiction. Next if $\beta=3\cdot 2^m$, then $|N_{\mathfrak{G}}\mathfrak{S}|$ is equal to $2^{2m+2}(2^m-1)$ from (3.4). Hence $\mathfrak{H}\cap N_{\mathfrak{G}}\mathfrak{S}=\mathfrak{K}\mathfrak{V}$. In any case we may assume that $\mathfrak{H}\cap N_{\mathfrak{G}}\mathfrak{S}=\mathfrak{K}\mathfrak{V}$.

Since $N_{\mathfrak{G}}\mathfrak{K}/\mathfrak{K}$ is a complete Frobenius group of degree 2^m, all the Sylow subgroups of $\mathfrak{V}$ are cyclic. Let l be the least prime factor of the order of $\mathfrak{V}$. Let $\mathfrak{L}$ be a Sylow l-subgroup of $\mathfrak{V}$. Then $\mathfrak{L}$ is cyclic and clearly leaves only the symbol 1 fixed. Hence $N_{\mathfrak{G}}\mathfrak{L}$ is contained in $\mathfrak{H}$. We shall show that $N_{\mathfrak{G}}\mathfrak{L}$ $=C_{\mathfrak{G}}\mathfrak{L}$. We shall assume that $l=3$. Let x be the index of $N_{\mathfrak{G}}\mathfrak{L}\cap N_{\mathfrak{G}}\mathfrak{S}$ in $\mathfrak{K}\mathfrak{V}$. If x is divisible by 4, then the order of $N_{\mathfrak{G}}\mathfrak{L}$ is odd. Since the index of $C_{\mathfrak{G}}\mathfrak{L}$ in $N_{\mathfrak{G}}\mathfrak{L}$ is equal to 1 or 2, we have $N_{\mathfrak{G}}\mathfrak{L}=C_{\mathfrak{G}}\mathfrak{L}$. If x is even and not divisible by 4 or if x is odd, then the order of $N_{\mathfrak{G}}\mathfrak{L}\cap\mathfrak{K}\mathfrak{V}$ is even. Let τ be an involution in $N_{\mathfrak{G}}\mathfrak{L}\cap\mathfrak{K}\mathfrak{V}$. Then τ is a permutation in $\mathfrak{K}$. Since $\tau\mathfrak{L}\tau=\mathfrak{L}$ and $\mathfrak{K}\mathfrak{V}$ is a semi-direct product, $\mathfrak{L}$ is contained in $C_{\mathfrak{G}}\tau$. Since $N_{\mathfrak{G}}\mathfrak{K}$ contains $C_{\mathfrak{G}}\tau$, the index of $C_{\mathfrak{G}}\mathfrak{K}$ in $C_{\mathfrak{G}}\tau$ is equal to 1 or 2. On the other hand, since $\mathfrak{S}$ is a

Sylow 2-subgroup of $N_{\mathfrak{G}}\mathfrak{K}$ and $C_{\mathfrak{G}}\mathfrak{K}$ contains $\mathfrak{S}$, the index of $C_{\mathfrak{G}}\mathfrak{K}$ in $N_{\mathfrak{G}}\mathfrak{K}$ is equal to 1 or 3. Hence $C_{\mathfrak{G}}\tau = C_{\mathfrak{G}}\mathfrak{K}$. Thus $\mathfrak{L}$ is contained in $C_{\mathfrak{G}}\mathfrak{K}$ and, therefore, $N_{\mathfrak{G}}\mathfrak{K} = C_{\mathfrak{G}}\mathfrak{K}$. If $l \neq 3$, then $N_{\mathfrak{G}}\mathfrak{K} = C_{\mathfrak{G}}\mathfrak{K}$. If $N_{\mathfrak{G}}\mathfrak{K} = C_{\mathfrak{G}}\mathfrak{K}$, then $C_{\mathfrak{G}}\mathfrak{L}$ contains $\mathfrak{K}$. Using Sylow's theorem, we obtain that $N_{\mathfrak{G}}\mathfrak{L} = C_{\mathfrak{G}}\mathfrak{L}(N_{\mathfrak{G}}\mathfrak{K} \cap N_{\mathfrak{G}}\mathfrak{L}) = C_{\mathfrak{G}}\mathfrak{L}(\mathfrak{K}\mathfrak{V} \cap N_{\mathfrak{G}}\mathfrak{L})$. Then it is easily seen that $N_{\mathfrak{G}}\mathfrak{L} = C_{\mathfrak{G}}\mathfrak{L}$.

In any case we have that $N_{\mathfrak{G}}\mathfrak{L} = C_{\mathfrak{G}}\mathfrak{L}$. By the splitting theorem of Burnside $\mathfrak{G}$ has the normal l-complement. Continuing in the similar way, it can be shown that $\mathfrak{G}$ has the normal subgroup $\mathfrak{A}$, which is a complement of $\mathfrak{B}$. Since the order of $\mathfrak{H} \cap \mathfrak{A}$ is equal to $4(2^m+1)$, $\mathfrak{K}$ has a normal complement $\mathfrak{V}$ of order 2^m+1 in $\mathfrak{H} \cap \mathfrak{A}$. $\mathfrak{H} \cap \mathfrak{A} = \mathfrak{K}\mathfrak{V}$. Let τ be an involution of $\mathfrak{K}$. Since $C_{\mathfrak{G}}\tau = C_{\mathfrak{G}}\mathfrak{K}$ and the order of $\mathfrak{V}$ is relatively prime to the order of $N_{\mathfrak{G}}\mathfrak{K}$, it is clear that every permutation $(\neq 1)$ of $\mathfrak{V}$ and, hence, τ induces a fixed-point-free automorphism of $\mathfrak{V}$. Thus $\mathfrak{V}$ has three fixed-point-free-automorphisms of order two. But, since the order of $\mathfrak{V}$ is odd, this is a contradiction.

Thus there exists no group satisfying the conditions of the theorem in Case (D).

7. Case (E). From (3.4) we have the following equality;

$$|\mathfrak{G}|/|N_{\mathfrak{G}}\mathfrak{S}| = 2(2^{m-1}+1)(2^m+3)/3\beta\,.$$

Since the order of a Sylow 2-subgroup of $\mathfrak{G}$ is equal to 2^{m+3}, β must be even, but not divisible by 4. Hence we have that $\beta = 2(2^{m-1}+1)$. Therefore the index of $N_{\mathfrak{G}}\mathfrak{S}$ in $\mathfrak{G}$ is equal to $(2^m+3)/3$. But this is not integer. This is a contradiction.

Thus there exists no group satisfying the conditions of the theorem in Case (E).

8. Case (F). In this case $\mathfrak{S}$ is also a Sylow 2-group of $\mathfrak{G}$. Every involution of $\mathfrak{S}$ leaving i symbols of Ω fixed is conjugate to an involution of $\mathfrak{K}$. Since $\mathfrak{S}$ is elementary abelian, it is conjugate already in $N_{\mathfrak{G}}\mathfrak{S}$. If the index of $C_{\mathfrak{G}}\mathfrak{K}$ in $N_{\mathfrak{G}}\mathfrak{K}$ is equal to 3, then the index of $C_{\mathfrak{G}}\mathfrak{K}$ in $N_{\mathfrak{G}}\mathfrak{S}$ is equal to β. If $N_{\mathfrak{G}}\mathfrak{K} = C_{\mathfrak{G}}\mathfrak{K}$, then the index of $C_{\mathfrak{G}}\mathfrak{K}$ in $N_{\mathfrak{G}}\mathfrak{S}$ is equal to $\beta/3$. On the other hand, since $\mathfrak{S}$ is a Sylow 2-group of $\mathfrak{G}$ and $g^*(2) \neq 0$, β must be equal to $2^{m+1}+1$. Therefore the order of $N_{\mathfrak{G}}\mathfrak{S}$ is equal to $2^{m+2}(2^m-1)(2^{m+1}+1)/3$. Hence the index of $N_{\mathfrak{G}}\mathfrak{S}$ in $\mathfrak{G}$ is equal to $(2^{m+1}+3)/3$, which is a contradiction.

Thus there exists no group satisfying the conditions of the theorem in Case (F).

9. Case (G). Since $g^*(2) = 0$, we have $\beta = 2^{m+2}-1$. Therefore likewise in Case (G) it is easily seen that the order of $N_{\mathfrak{G}}\mathfrak{S}$ is equal to $2^{m+2}(2^m-1)(2^{m+2}-1)/3$. Hence the index of $N_{\mathfrak{G}}\mathfrak{S}$ in $\mathfrak{G}$ is equal to $(2^{m+2}+3)/3$, which is a contradiction.

Thus there exists no group satisfying the conditions or the theorem in Case (G).

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