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A characterization of the alternating groups of degrees 12, 13, 14, 15

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§ 1. Introduction.

The purpose of this paper is to characterize the alternating groups of degrees twelve, thirteen, fourteen and fifteen by the structure of the centralizer of an element of order 2 contained in the center of their Sylow 2-subgroups. Let A_n be the alternating group of degree n. Let $\hat{\alpha}$ denote the element of order 2 in A_n $(n \geqq 12)$ which has a cycle decomposition $(1, 2)(3, 4)(5, 6)(7, 8)(9, 10)(11, 12)$. We regard $A_{12} \subset A_{13} \subset A_{14} \subset A_{15}$ via the natural imbedding. Put $\hat{H}_1 = C_{A_{12}}(\hat{\alpha}) = C_{A_{13}}(\hat{\alpha})$, $\hat{H}_2 = C_{A_{14}}(\hat{\alpha})$ and $\hat{H}_3 = C_{A_{15}}(\hat{\alpha})$. The characterization of A_{12}, A_{13}, A_{14} and A_{15} is given by the following theorem.

THEOREM. *Let G_i be a finite group with the following two properties:*

(1) *G_i has no subgroup of index 2, and*

(2) *G_i contains an involution α which is contained in the center of a Sylow 2-subgroup of G_i such that the centralizer $C_{G_i}(\alpha)$ is isomorphic to $\hat{H}_i$.*

Then (i) $G_1 \cong A_{12}$ *or* A_{13} *or*

(ii) *G_1 has precisely four conjugacy classes of involutions*

and

(iii) $G_2 \cong A_{14}$,

(iv) $G_3 \cong A_{15}$.

REMARK. The third case of G_1 is non-empty. For example the group $PS_{p_6}(2)$, the projective symplectic group of six variables over the field of 2 elements, satisfies our conditions (1), (2) and has precisely four conjugacy classes of involutions. We will study this case in a subsequent paper.

In the course of our proof we show that a group G_i with properties (1) and (2) posesses precisely three or four conjugacy classes of involutions and determines the structure of the centralizers of involutions which are not conjugate to α. The identification of G_i with the alternating group is then accomplished by using a theorem of Kondo [**11**] which is a generalization of Wong's theorem [**14**] on A_8.

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We shall use the following notations which are fairly standard:

G'	the commutator subgroup of a group G.
$O^2(G)$, (resp. $G'(2)$)	the smallest normal subgroup N of G such that G/N is a (resp. abelian) 2-group.
$O_{2'}(G)$	the maximal normal subgroup of odd order of G.
$\Omega_1(P)$	the subgroup of a p-group P generated by the elements of order p.
$Z(G)$	the center of a group G.
$\langle x, y, \cdots\rangle$	the group generated by the elements $x, y, \cdots$.
Z_n	a cyclic group of order n.
$H<G$	H is a proper subgroup of G.
$H\lhd G$	H is a normal subgroup of G.
$H\lesssim G$	H is isomorphic to a subgroup of G.
$[x, y]$	$x^{-1}y^{-1}xy=x^{-1}(x)^{y^{-1}}$.
$x\sim y$ in G	an element x is conjugate to y in G.
$ccl_G(x)$	a conjugate class in a group G containing x.
A_n, S_n	the alternating (symmetric) group of degree n.
$GL(n, q)$	the general linear group of degree n over the field of q elements.
$PGL(n, q)$	$GL(n, q)/Z(GL(n, q))$.
$SL(n, q)$	the group of $n\times n$ matrices of determinant 1 over the field of q elements.
$PSL(n, q)$	$SL(n, q)/Z(GL(n, q))\cap SL(n, q)$.
$GF(q)$	the finite field of q elements.

§ 2. Some properties of $\hat{H}_i$.

The group $\hat{H}_1$ is generated by the following elements:

$$\hat{\pi}_1=(1, 2)(3, 4) \qquad \hat{\pi}_3=(9, 10)(11, 12) \qquad \hat{\pi}_2'=(5, 7)(6, 8)$$
$$\hat{\mu}=(1, 2)(5, 6) \qquad \hat{\pi}_1'=(1, 3)(2, 4) \qquad \hat{\sigma}'=(7, 9)(8, 10)$$
$$\hat{\pi}_2=(5, 6)(7, 8) \qquad \hat{\sigma}=(3, 5)(4, 6) \qquad \hat{\pi}_3'=(9, 11)(10, 12)$$
$$\hat{\mu}'=(1, 2)(9, 10) \qquad \hat{\alpha}=\hat{\pi}_1\hat{\pi}_2\hat{\pi}_3.$$

Put $\hat{\lambda}=(9, 10)(13, 14)$ and $\hat{\nu}=(13, 14, 15)$. Thus we have $\hat{H}_2=\langle\hat{H}_1, \hat{\lambda}\rangle$ and $\hat{H}_3=\langle\hat{H}_1, \hat{\lambda}, \hat{\nu}\rangle$. In the isomorphism from $\hat{H}_3$ onto $C_{G_3}(\alpha)$ let the images of $\hat{\pi}_1, \hat{\mu}, \hat{\pi}_2, \hat{\mu}', \hat{\pi}_3, \hat{\pi}_1', \hat{\sigma}, \hat{\pi}_2', \hat{\sigma}', \hat{\pi}_3', \hat{\lambda}, \hat{\nu}$ be $\pi_1, \mu, \pi_2, \mu', \pi_3, \pi_1', \sigma, \pi_2', \sigma', \pi_3', \lambda, \nu$ respectively. Then one has $\alpha=\pi_1\pi_2\pi_3$. Put $H_i=C_{G_i}(\alpha)$ for $i=1, 2, 3$. Hence $H_1=\langle\pi_1, \mu, \pi_2, \mu', \pi_3, \pi_1', \sigma, \pi_2', \sigma', \pi_3'\rangle$, $H_2=\langle H_1, \lambda\rangle$ and $H_3=\langle H_1, \lambda, \nu\rangle$; also we have $\lambda\nu\lambda^{-1}=\nu^{-1}$ and $[H_1, \nu]=1$. The group $M=\langle\mu, \mu', \pi_1, \pi_2, \pi_3, \lambda\rangle$ is an elementary abelian group of order 2^6 and is normal in H_2. The group $\langle\pi_1', \sigma, \pi_2', \sigma', \pi_3'\rangle$ is isomorphic to a symmetric group of degree six and satisfies the following relations:

$$\pi_1'^2=\sigma^2=\pi_2'^2=\sigma'^2=\pi_3'^2=1\,,$$
$$(\pi_1'\sigma)^3=(\sigma\pi_2')^3=(\pi_2'\sigma')^3=(\sigma'\pi_3')^3=1\,,$$
$$(\pi_1'\pi_2')^2=(\pi_1'\sigma')^2=(\pi_1'\pi_3')^2=(\sigma\sigma')^2=(\sigma\pi_3')^2=(\pi_2'\pi_3')^2=1\,.$$

The actions of the elements π_1', σ, π_2', σ', π_3' on M by conjugation are given by the following table.

M	π_1'	σ	π_2'	σ'	π_3'
π_1	π_1	μ	π_1	π_1	π_1
μ	$\mu\pi_1$	π_1	$\mu\pi_2$	μ	μ
π_2	π_2	$\mu\pi_1\pi_2$	π_2	$\mu\mu'$	π_2
μ'	$\mu'\pi_1$	μ'	μ'	$\mu\pi_2$	$\mu'\pi_3$
π_3	π_3	π_3	π_3	$\mu\mu'\pi_2\pi_3$	π_3
λ	λ	λ	λ	$\mu\mu'\lambda\pi_2$	$\lambda\pi_3$

Put $\alpha'=\pi_1'\pi_2'\pi_3'$, $\rho=\pi_1'\sigma$, $\xi=(\pi_1'\pi_2')^{\sigma}(\pi_2'\pi_3')^{\sigma'}$ and $\tau=(\pi_1'\pi_2')^{\sigma}$. Thus $\xi^3=\rho^3=\tau^2=1$ and the following relations of actions of ξ, ρ, τ by conjugation are satisfied.

	ξ	ρ	τ
π_1	π_2	$\mu\pi_1$	π_2
μ	$\mu\mu'$	π_1	μ
π_2	π_3	$\mu\pi_2$	π_1
μ'	μ	$\mu'\pi_1$	$\mu\mu'$
π_3	π_1	π_3	π_3
λ	$\mu'\lambda$	λ	λ
π_1'	π_2'	$\rho\pi_1'\rho^{-1}$	π_2'
π_2'	π_3'	$\rho\pi_2'\rho^{-1}$	π_1'
π_3'	π_1'	π_3'	π_3'
τ	$\xi\tau\xi^{-1}$	$\pi_1\pi_1'\pi_2\pi_2'$	τ

Let D_i be a Sylow 2-subgroup of G_i contained in H_i. We may assume that $D_1=\langle\pi_1, \pi_1', \pi_2, \pi_2', \pi_3, \pi_3'\rangle\langle\tau, \mu, \mu'\rangle$ and $D_2=D_3=\langle\pi_1, \pi_1', \pi_2, \pi_2', \pi_3, \pi_3'\rangle\langle\tau, \mu, \mu'\rangle\langle\lambda\rangle$. Moreover, $Z(D_i)=\langle\pi_1\pi_2, \pi_3\rangle$, $D_i'=\langle\pi_1, \pi_2, \pi_3, \pi_1'\pi_2', \mu\rangle$ and $(D_i')'=\langle\pi_1\pi_2\rangle$.

The group $\langle\tau, \mu, \mu'\rangle$ is a dihedral group of order 8 with center $\langle\mu\rangle$. Put $S=\langle\pi_1, \pi_1', \pi_2, \pi_2', \pi_3, \pi_3'\rangle$. Then S is an elementary abelian normal subgroup of order 2^6 in D_i and $N_{H_1}(S)=D_1\langle\xi\rangle$, $N_{H_2}(S)=D_2\langle\xi\rangle$, $N_{H_3}(S)=D_3\langle\xi, \nu\rangle$. There are ten conjugacy classes of involutions of H_1 and they are as follows:

π_1	π_1'	$\pi_1\pi_2$	$\pi_1\pi_2'$	$\pi_1'\pi_2'$	α	$\alpha\pi_1'$	$\alpha\pi_1'\pi_2'$	$\alpha\alpha'$	α'
3	6	3	12	12	1	6	12	4	4

Table I.

The first and second entries in the column give respectively, representative of the class and the cardinality of the intersection of the class and S. This implies that every involution of D_1 is conjugate to an element of S in H_1. In H_2 and H_3, we have $\alpha'\sim\alpha\alpha'$ and some involution is not conjugate to an element of S.

§ 3. Conjugacy classes of involutions of G_i.

In this section we determine the fusion of the conjugate classes of involutions in G_i. By Kondo's theorem [12] it is sufficient to determine them in G_1.

LEMMA 1. *The involution α is conjugate in G_1 to an element of D_1 distinct from α.*

PROOF. Assume that the element α is not conjugate to an involution of D_1 distinct from α in G_1. Then by the theorem of Glauberman [4] we have $\alpha\in Z(G_1 \bmod O_{2'}(G_1))$ and so $G_1\triangleright\langle\alpha\rangle O_{2'}(G_1)$. It follows from Frattini argument that $G_1=C_{G_1}(\alpha)O_{2'}(G_1)$. $H_1>O^2(H_1)$ implies that $G_1>O^2(G_1)$. This contradicts our assumption.

Since every conjugate class of involution of H_1 intersects S non-trivially Lemma 1 implies that the involution α is conjugate in G_1 to an element of S distinct from α.

LEMMA 2. *The group S is the only elementary abelian subgroup of order 2^6 in D_1. If two elements of S are conjugate in G_1 they are conjugate in $N_{G_1}(S)$.*

PROOF. Since $C_S(\tau)=\langle\pi_1\pi_2, \pi_1'\pi_2', \pi_3, \pi_3'\rangle$, $C_S(\mu)=\langle\pi_1, \pi_2, \pi_3, \pi_3'\rangle$ and $C_S(\mu')=\langle\pi_1, \pi_2, \pi_3, \pi_2'\rangle$ the first part is obvious. Since S is weakly closed in D_1 with respect to G_1 and $Z(S)=S$ the result follows from Burnside's argument.

LEMMA 3. $N_{G_1}(D_1)=D_1$.

PROOF. $Z(D_1)=\langle\pi_1\pi_2, \pi_3\rangle$ implies that $C_{G_1}(Z(D_1))=\langle D_1, \rho\rangle$. It follows from $\rho\notin N_{G_1}(D_1)$ that $C_{G_1}(Z(D_1))\cap N_{G_1}(D_1)=D_1$. On the other hand $Z(D_1)\cong Z_2\times Z_2$ yields $N_{G_1}(Z(D_1))=C_{G_1}(Z(D_1))$ or $N_{G_1}(Z(D_1))=\langle\omega\rangle C_{G_1}(Z(D_1))$ where ω acts on $Z(D_1)$ as an element of order 3. If $N_{G_1}(D_1)>D_1$ then $N_{G_1}(D_1)=\langle\omega\rangle D_1$. Since $(D')'=\langle\pi_1\pi_2\rangle$ is a characteristic subgroup of D_1, ω centralizes $\pi_1\pi_2$. This contradicts the choice of the element ω. Therefore we get $N_{G_1}(D_1)=D_1$.

LEMMA 4. *$\pi_1\pi_2 \not\sim \pi_3$, $\pi_1\pi_2 \not\sim \alpha$ and $\pi_3 \not\sim \alpha$, consequently G_1 has at least three conjugacy classes of involutions.*

PROOF. Since $Z(D_1)=\langle \pi_1\pi_2, \pi_3\rangle$, the result follows from Lemma 3 and Burnside's argument.

DEFINITION. $n(\alpha)=(N_{G_1}(S): C_{G_1}(\alpha)\cap N_{G_1}(S))$. Note by Lemma 2 that $n(\alpha)$ is the number of elements of S which are conjugate in G_1 to α.

LEMMA 5. (i) $n(\alpha)=7$, 15 *or* 27.

(ii) $\alpha \sim \alpha\pi_1'$ *or* $\alpha \sim \pi_1'$.

PROOF. Put $\mathfrak{N}=N_{G_1}(S)/S$ and $\tilde{\mathfrak{N}}=\mathfrak{N}/O_{2'}(\mathfrak{N})$. In the following sequence of natural epimorphisms $N_{G_1}(S)\to\mathfrak{N}\to\tilde{\mathfrak{N}}$, put $\mu\to\bar{\mu}\to\tilde{\mu}$, $\mu'\to\bar{\mu}'\to\tilde{\mu}'$, $\tau\to\bar{\tau}\to\tilde{\tau}$ and $\xi\to\bar{\xi}\to\tilde{\xi}$. $\mathfrak{N}$ is isomorphic to a subgroup of the full linear group $GL(6, 2)$. Representing $\bar{\mu}$, $\bar{\mu}'$ on the vector space S over the finite field $GF(2)$ we get in terms of the besis π_1', $\pi_1\pi_1'$, π_2', $\pi_2\pi_2'$, π_3', $\pi_3\pi_3'$:

$$\bar{\mu} \longrightarrow \left[\begin{array}{cc|cc|cc} 0 & 1 & & & & \\ 1 & 0 & \multicolumn{2}{c|}{0} & \multicolumn{2}{c}{0} \\ \hline & & 0 & 1 & & \\ \multicolumn{2}{c|}{0} & 1 & 0 & \multicolumn{2}{c}{0} \\ \hline & & & & 1 & 0 \\ \multicolumn{2}{c|}{0} & \multicolumn{2}{c|}{0} & 0 & 1 \end{array}\right] \qquad \bar{\mu}' \longrightarrow \left[\begin{array}{cc|cc|cc} 0 & 1 & & & & \\ 1 & 0 & \multicolumn{2}{c|}{0} & \multicolumn{2}{c}{0} \\ \hline & & 1 & 0 & & \\ \multicolumn{2}{c|}{0} & 0 & 1 & \multicolumn{2}{c}{0} \\ \hline & & & & 0 & 1 \\ \multicolumn{2}{c|}{0} & \multicolumn{2}{c|}{0} & 1 & 0 \end{array}\right]$$

Direct computations show that $|C_{\mathfrak{N}}(\bar{\mu})|=2^3$, $2^3\cdot 3$ or $2^3\cdot 3^2$ and $O_{2'}(\mathfrak{N})\cap C_{\mathfrak{N}}(\bar{\mu}, \bar{\mu}')=1$. Since $\bar{\xi}\in\mathfrak{N}$ and then $\bar{\mu}\sim\bar{\mu}'\sim\bar{\mu}\bar{\mu}'$ in $\mathfrak{N}$, Brauer and Wielandt's theorem [**13**] implies that $|O_{2'}(\mathfrak{N})|=|C(\bar{\mu})\cap O_{2'}(\mathfrak{N})|^3$ and so $|O_{2'}(\mathfrak{N})|=1$ or 3^3 because of $\mathfrak{N}\subsetneqq GL(6, 2)$. Since $\langle\tilde{\tau}, \tilde{\mu}, \tilde{\mu}'\rangle$ is a dihedral group with center $\langle\tilde{\mu}\rangle$, $C_{\tilde{\mathfrak{N}}}(\tilde{\mu})$ is $2'$-closed by a transfer theorem and $[\bar{\mu}, O_{2'}(C_{\tilde{\mathfrak{N}}}(\tilde{\mu}))]\subset O_{2'}(\mathfrak{N})$. This implies that $|O_{2'}(C_{\tilde{\mathfrak{N}}}(\tilde{\mu}))|=1$, 3 or 3^2 and then $|C_{\tilde{\mathfrak{N}}}(\tilde{\mu})|=2^3$, $2^3\cdot 3$ or $2^3\cdot 3^2$. Since $C_{\tilde{\mathfrak{N}}}(\tilde{\mu})$ has an abelian 2-complement we may now apply a theorem of Gorenstein and Walter [**5**]. If $\tilde{\mathfrak{N}}\cong\langle\tau, \mu, \mu'\rangle$ then $\bar{\xi}\in O_{2'}(\mathfrak{N})$ and so $\langle\bar{\xi}\rangle \lhd \langle\bar{\mu}, \bar{\mu}'\rangle\langle\bar{\xi}\rangle\cong A_4$ which is impossible. Thus $\tilde{\mathfrak{N}}$ is not a 2-group and we get $\tilde{\mathfrak{N}}\cong PGL(2, q)$, $PSL(2, q)$ or A_7. Noting that $q\pm 1$ divides $|C_{\tilde{\mathfrak{N}}}(\tilde{\mu})|$ and that $|\mathfrak{N}|$ divides $|GL(6, 2)|$ we have the following table which is selfexplanatory.

$\tilde{\mathfrak{N}}$	$\lvert\tilde{\mathfrak{N}}\rvert$	$\lvert O_{2'}(\mathfrak{N})\rvert$	$\lvert\mathfrak{N}\rvert$	$n(\alpha)$
$PGL(2, 3)$	$2^3\cdot 3$	1	$2^3\cdot 3$	1
		3^3	$2^3\cdot 3^4$	3^3
$PGL(2, 5)$	$2^3\cdot 3\cdot 5$	1	$2^3\cdot 3\cdot 5$	5
		3^3	$2^3\cdot 3^4\cdot 5$	$3^5\cdot 5$

$PSL(2, 7)$	$2^3\cdot 3\cdot 7$	1	$2^3\cdot 3\cdot 7$	7
		3^3	$2^3\cdot 3^4\cdot 7$	$3^7\cdot 7$
$PSL(2, 9)$	$2^3\cdot 3^2\cdot 5$	1	$2^3\cdot 3^2\cdot 5$	$3\cdot 5$
		3^3	$2^3\cdot 3^5\cdot 5$	$3^4\cdot 5$
A_7	$2^3\cdot 3^2\cdot 5\cdot 7$	1	$2^3\cdot 3^2\cdot 5\cdot 7$	$3\cdot 5\cdot 7$
		3^3	$2^3\cdot 3^5\cdot 5\cdot 7$	$3^4\cdot 5\cdot 7$

Since $1< n(\alpha)\leqq |S|-1=63$ by Lemma 1, we have $n(\alpha)=5$, 7, 15 or 27. Assume that $n(\alpha)=5$. We have $\alpha\sim\alpha'$ or $\alpha\sim\alpha\alpha'$ by Table I. If $\alpha\sim\alpha'$, then there exists an element x of order 5 in $N_{G_1}(S)$ and x permutes cyclically α, $\alpha'\pi_1\pi_2$, $\alpha'\pi_2\pi_3$, α', $\alpha'\pi_1\pi_3$. Put $X=\langle\alpha', \pi_1, \pi_2, \pi_3\rangle$. It follows from $|X|=2^4$ that x is fixed-point-free on X. Since $\pi_1\sim\pi_2\sim\pi_3$, $\pi_1\pi_2\sim\pi_2\pi_3\sim\pi_1\pi_3$, $\alpha\alpha'\sim\alpha'\pi_1\sim\alpha'\pi_2\sim\alpha'\pi_3$ and $\alpha\sim\alpha'\sim\alpha'\pi_1\pi_2\sim\alpha'\pi_2\pi_3\sim\alpha'\pi_1\pi_3$, we must have $\pi_1\sim\pi_1\pi_2$ which is impossible by Lemma 4. Similarly we can treat the case $\alpha\sim\alpha\alpha'$. Thus we have proved that $n(\alpha)=7$, 15 or 27. Lemma 4 and Table I imply that

$$n(\alpha)=7=1+6\,.$$
$$15=1+6+4+4 \text{ or}$$
$$27=1+6+4+4+12\,.$$

In any cases 6 appears in the direct summand of $n(\alpha)$. Therefore we get $\alpha\sim\alpha\pi_1'$ or $\alpha\sim\pi_1'$ by Table I. The proof is complete.

LEMMA 6. *$\pi_1\not\sim\alpha\pi_1'$ or $\pi_1\not\sim\pi_1'$.*

PROOF. It is $\pi_1\sim\pi_3$ and $\alpha\pi_1'\sim\alpha\pi_3'$ in H_1. Assume by way of contradiction that $\pi_3\sim\alpha\pi_3'\sim\pi_3'$. Put $W=S\langle\tau, \mu\rangle=C_{D_1}(\alpha\pi_3')$. It is $Z(W)=\langle\pi_1\pi_2, \pi_3, \pi_3'\rangle\lhd D_1$. Let D be a Sylow 2-subgroup of G_1 with $W\subset D\subset C_{G_1}(\alpha\pi_3')$. Put $N=\langle D_1, D\rangle$. Then $N\rhd S$, $Z(W)$ and it follows from $\xi\pi_3\xi^{-1}=\pi_1\notin Z(W)$ that $C(Z(W))\cap N=W$. Since N/W is isomorphic to a subgroup of $GL(3, 2)$ and is not a 2-group, we have $|N|=2^9\cdot 3$ (cf. Dickson [2]). $\xi\notin N$ and $N_{H_1}(S)=D_1\langle\xi\rangle$ imply that $\alpha\notin Z(N)$. Thus we have $(N:C_{G_1}(\alpha)\cap N)=3$. On the other hand, the conjugacy classes of involutions in D_1 which contain at most two elements are $ccl_{D_1}(\alpha)$, $ccl_{D_1}(\pi_1\pi_2)$, $ccl_{D_1}(\pi_3)$, $ccl_{D_1}(\pi_3')$ and $ccl_{D_1}(\alpha\pi_3')$. Since $\alpha\not\sim\pi_1\pi_2$ and $\alpha\not\sim\pi_3$, $(N:C_{G_1}(\alpha)\cap N)=3$ forces to be $\alpha\sim\pi_3'$ or $\alpha\sim\alpha\pi_3'$. Hence $\alpha\sim\pi_1'$ or $\alpha\sim\alpha\pi_1'$. This is impossible because of Lemma 4.

LEMMA 7. *We may assume that $\pi_1\not\sim\alpha\pi_1'$.*

PROOF. If we replace π_1', σ, π_2', σ', π_3' with $\alpha\pi_1'$, $\alpha\sigma$, $\alpha\pi_2'$, $\alpha\sigma'$, $\alpha\pi_3'$ in this order the same relations hold in H_1. The result follows from Lemma 6.

LEMMA 8. *$\alpha\sim\alpha\pi_1'$ and $\pi_1\sim\pi_1'$.*

PROOF. By Lemma 5 we have $\alpha\sim\alpha\pi_1'$ or $\alpha\sim\pi_1'$. Since $\pi_1'\sim\pi_3'$, assume by way of contradiction that $\alpha\sim\pi_3'$. Put $W=S\langle\tau,\mu\rangle=C_{D_1}(\pi_3')$. It is $Z(W)=\langle\pi_1\pi_2,\pi_3,\pi_3'\rangle$ and $W'=\langle\pi_1,\pi_2,\pi_1'\pi_2'\rangle$. Similarly as in Lemma 6, define the group N. Thus we have $|N|=2^9\cdot 3$ and $(N: C_{G_1}(\alpha)\cap N)=3$. Hence $\alpha\sim\pi_3'$ or $\alpha\sim\alpha\pi_3'$ in N by the same reason as in Lemma 6. $\langle\pi_1\pi_2\rangle=W'\cap Z(W)$ implies that $\pi_1\pi_2\in Z(N)$. If $\alpha^x=\pi_3'$ for some $x\in N$, then $\pi_3^x=\pi_1\pi_2\pi_3'\sim\alpha\pi_1'$ which is impossible by Lemma 7. If $\alpha^x=\alpha\pi_3'$ for some $x\in N$, then $\alpha\sim\pi_3'\sim\alpha\pi_3'$ by our assumption. In other words $n(\alpha)=1+6+6+\cdots$ by Table I. This contradicts Lemma 5. Therefore $\alpha\nsim\pi_3'$ and then $\alpha\sim\alpha\pi_1'$ by Table I. Since $\alpha\sim\alpha\pi_3'$ and $W=S\langle\tau,\mu\rangle=C_{D_1}(\alpha\pi_3')$, define N as above. It is $N\triangleright Z(W)$ and $|N|=2^9\cdot 3$. Let x be an element of order 3 in N. $\alpha\notin Z(N)$ implies that $[\alpha, x]\neq 1$. Since $[x, Z(W)]\subset Z(W)$ and $Z(W)$ contains exactly three elements α, $\alpha\pi_3'$, $\pi_1\pi_2\pi_3'$ which are conjugate to α, we may assume that $\alpha^x=\alpha\pi_3'$. It follows from $\alpha^{x^2}=\pi_1\pi_2\pi_3'$ that $\alpha^x\cdot\pi_3'^x=\alpha\pi_3'\cdot\pi_3'^x=\pi_1\pi_2\pi_3'$ and so $\pi_3'^x=\pi_3$. It is $\pi_1\sim\pi_3$ and $\pi_1'\sim\pi_3'$ in H_1. This proves our lemma.

There exist precisely three subgroups $S\langle\mu,\mu'\rangle$, $S\langle\tau,\mu\rangle$ and $S\langle\tau\mu'\rangle$ of order 2^8 containing S in D_1. The center and the commutator subgroups of these groups are as follows:

$$Z(S\langle\mu,\mu'\rangle)=(S\langle\mu,\mu'\rangle)'=\langle\pi_1,\pi_2,\pi_3\rangle$$
$$Z(S\langle\tau,\mu\rangle)=\langle\pi_1\pi_2,\pi_3,\pi_3'\rangle,\quad (S\langle\tau,\mu\rangle)'=\langle\pi_1,\pi_2,\pi_1'\pi_2'\rangle$$
$$Z(S\langle\tau\mu'\rangle)=\langle\pi_1\pi_2,\pi_3\rangle,\quad (S\langle\tau\mu'\rangle)'=\langle\pi_1,\pi_2,\pi_3,\pi_1'\pi_2'\rangle\,.$$

Hence $S\langle\mu,\mu'\rangle$ is the only nilpotent subgroup of class 2 in these groups and so D_1 is not generated by nilpotent subgroup of class 2 of order 2^8 containing S. We use this fact for the proof of the next lemma.

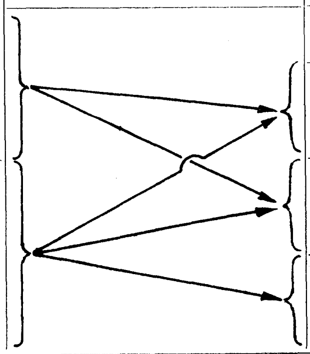
LEMMA 9. (i) *If $S\langle\tau\rangle\nsim S\langle\mu\rangle$, then $\pi_1\pi_2\sim\pi_1\pi_2'$ and $\alpha\sim\alpha'\sim\alpha\alpha'$.*

(ii) *If $S\langle\tau\rangle\sim S\langle\mu\rangle$, then $\pi_1\sim\pi_1'\pi_2'$ and we may assume that $\pi_1\pi_2\sim\alpha\alpha'$, $\pi_1'\pi_2\sim\alpha\pi_1'\pi_2'\sim\alpha'$.*

PROOF. (i) Put $W=S\langle\mu\mu'\rangle=C_{D_1}(\pi_1')$. The group W is of order 2^7 and $Z(W)=\langle\pi_1,\pi_2,\pi_3,\pi_1'\rangle$, $W'=\langle\pi_2,\pi_3\rangle$. Let D be a group of order 2^8 with $W\subset D\subset C_{G_1}(\pi_1')$ and $Z(D)\cong Z_2\times Z_2\times Z_2$. Since $S\langle\tau\rangle\nsim S\langle\mu\rangle$ and $\mu\mu'\sim\mu$ we may assume that D is a nilpotent group of class 2. Put $N=\langle W\langle\mu\rangle, D\rangle$. Then $N\triangleright S, Z(W), W'$ and $N(S)\cap H_1=D_1\langle\xi\rangle$ implies that $C_{G_1}(Z(W))\cap N=W$. The group N/W is isomorphic to a subgroup of $GL(4,2)$ and so $(N: W)$ divides $2^2\cdot 3^2\cdot 5\cdot 7$. If N is a 2-group, N must be a Sylow 2-subgroup of G_1. Since $W\langle\mu\rangle$ and D are nilpotent subgroups of class 2 it follows from the remark before this Lemma that N is not a 2-group. Hence $[\xi, W']\not\subset W'$ implies that $\alpha\notin Z(N)$. If there exists an element x of order 5 or 7 in $N-W$, x centralizes W'. Since the action of x on $Z(W)$ is completely reducible there exists x-invariant subgroup K_x such that $Z(W)=W'\times K_x$. $[x, K_x]=1$ yields $[x, Z(W)]=1$. This is impossible because of $C_{G_1}(Z(W))\cap N=W$. Therefore 3 divides

$|N|$. Let y be an element of order 3 in $N-W$. $[y, W']\subset W'$ and $\pi_2 \not\sim \pi_2\pi_3$ imply that $[y, W']=1$. Since $[y, Z(W)]\neq 1$, twelve involutions in $Z(W)-W'$ are divided into four associated classes by y. On the other hand since $\pi_1\sim\pi_1'$ $\sim\pi_1\pi_1'$, $\alpha\sim\alpha\pi_2'\sim\pi_1\pi_2'\pi_3$, $\pi_1\pi_2'\sim\pi_2'\pi_3\sim\pi_2\pi_2'\pi_3\sim\pi_1\pi_2\pi_2'$ and $\pi_1\pi_2\sim\pi_1\pi_3$ by Lemma 8 we must have $\pi_1\pi_2\sim\pi_1\pi_2'$ by Lemma 4. Assume that $n(\alpha)=7$. It follows from $|N_{G_1}(S)|=2^9\cdot 3\cdot 7$ that $\xi\sim y$ in $N_{G_1}(S)$. Since $C_S(\xi)=\langle\alpha, \alpha'\rangle$ we have $\langle\alpha, \alpha'\rangle\sim\langle\pi_2, \pi_3\rangle=W'=C_S(y)$. This contradicts Lemma 4. Hence $n(\alpha)\neq 7$ and so Lemma 5 implies that $n(\alpha)=15$ or 27. In both cases 4 appears twice in the direct summand of $n(\alpha)$ by Table I. Thus we get $\alpha\sim\alpha'\sim\alpha\alpha'$ by Table I.

(ii) It is $(S\langle\tau\rangle)'=\langle\pi_1\pi_2, \pi_1'\pi_2'\rangle$, $(S\langle\mu\rangle)'=\langle\pi_1, \pi_2\rangle$ and $Z(S\langle\mu\rangle)=\langle\pi_1, \pi_2, \pi_3, \pi_3'\rangle$, $Z(S\langle\tau\rangle)=\langle\pi_1\pi_2, \pi_1'\pi_2', \pi_3, \pi_3'\rangle$. Since $\pi_1\not\sim\pi_1\pi_2$ by Lemma 4, $(S\langle\tau\rangle)'\sim(S\langle\mu\rangle)'$ implies that $\pi_1\sim\pi_1'\pi_2'$ and so $(N_{G_1}(S):C_{G_1}(\pi_1)\cap N_{G_1}(S))\geqq 21$ by Table I. Assume that $n(\alpha)=15$. Thus $|N_{G_1}(S)|=2^9\cdot 3^2\cdot 5$ and $(N_{G_1}(S):C_{G_1}(\pi_1)\cap N_{G_1}(S))=45$, $(N_{G_1}(S):C_{G_1}(\pi_1\pi_2)\cap N_{G_1}(S))=3$ by using Table I and examining the possibilities of the orbits of π_1 and $\pi_1\pi_2$. Moreover we have $\alpha\sim\alpha'\sim\alpha\alpha'$ by Table I. Let x be an element in $N_{G_1}(S)$ with $\alpha^x=\alpha'$. (Such element x exists by Lemma 2.) It follows from $\pi_3^x=\alpha'(\pi_1\pi_2)^x$ that $\pi_3^x=\alpha'\pi_1\pi_2$, $\alpha'\pi_2\pi_3$ or $\alpha'\pi_1\pi_3$ because of $(N_{G_1}(S):C_{G_1}(\pi_1\pi_2)\cap N_{G_1}(S))=3$. This is impossible because $\alpha\sim\alpha'\pi_1\pi_2\sim\alpha'\pi_2\pi_3$ $\sim\alpha'\pi_1\pi_3\sim\alpha'$. Therefore $n(\alpha)\neq 15$ and so $n(\alpha)=7$ or 27 by Lemma 5. If $n(\alpha)=27$, then $|N_{G_1}(S)|=2^9\cdot 3^4$ and $21\leqq(N_{G_1}(S):C_{G_1}(\pi_1)\cap N_{G_1}(S))=3^3$ or 3^4. This is impossible because of Table I and Lemma 8. Thus $n(\alpha)=7$ and $|N_{G_1}(S)|=2^9\cdot 3\cdot 7$. This implies that $(N_{G_1}(S):C_{G_1}(\pi_1)\cap N_{G_1}(S))=21$ and $(N_{G_1}(S):C_{G_1}(\pi_1\pi_2)\cap N_{G_1}(S))=3$, 7 or 21. Since $Z(S\langle\mu\rangle)\sim Z(S\langle\tau\rangle)$ we have $\pi_1\pi_2\sim\alpha\alpha'$ or $\pi_1\pi_2\sim\alpha'$ and $\pi_1\pi_3'\sim\alpha\pi_1'\pi_2'$ by the following table which is selfexplanatory.

$Z(S\langle\mu\rangle$	$\sim$	$Z(S\langle\tau\rangle)$
π_1 π_2 π_3 π_3' $\pi_3\pi_3'$	$\sim$	π_3 π_3' $\pi_3\pi_3'$ $\pi_1'\pi_2'$ $\pi_1\pi_2\pi_1'\pi_2'$
α $\alpha\pi_3'$ $\pi_1\pi_2\pi_3'$	$\sim$	α $\alpha\pi_3'$ $\pi_1\pi_2\pi_3'$
$\pi_1\pi_2$		$\pi_1\pi_2$
$\pi_2\pi_3$ $\pi_1\pi_3$		$\alpha\alpha'$ $\alpha'\pi_3$
$\pi_1\pi_3'$ $\pi_2\pi_3'$		α' $\alpha'\pi_1\pi_2$
$\pi_1\pi_3\pi_3'$ $\pi_2\pi_3\pi_3'$		$\pi_1'\pi_2'\pi_3$ $\alpha\pi_1'\pi_2'$

In the table the first and third column give respectively, the involutions in $Z(S\langle\mu\rangle)$ and in $Z(S\langle\tau\rangle)$. Similarly as in Lemma 7 we may assume that $\pi_1\pi_2 \sim \alpha\alpha'$ and therefore $\alpha' \sim \pi_1\pi_3' \sim \pi_1'\pi_2 \sim \alpha\pi_1'\pi_2'$. The proof is complete.

REMARK. By Lemma 5 if $n(\alpha)=7$ then $N_{G_1}(S)/S \cong PSL(2, 7)$. In the case (ii) the fusion of the conjugacy classes of involutions are completely determined.

LEMMA 10. *If* $S\langle\tau\rangle \not\sim S\langle\mu\rangle$ *then* $\alpha \sim \alpha\pi_1'\pi_2'$ *and* $\pi_1\pi_2 \sim \pi_1'\pi_2'$.

PROOF. Put $W=C_{D_1}(\alpha')=S\langle\tau\rangle$. It is $Z(W)=\langle\pi_1\pi_2, \pi_1'\pi_2', \pi_3, \pi_3'\rangle$ and $W'=\langle\pi_1\pi_2, \pi_1'\pi_2'\rangle$. Let D be a group of order 2^8 with $W \subset D \subset C_{G_1}(\alpha')$ and $Z(D) \cong Z_2 \times Z_2 \times Z_2$. Put $N=\langle W\langle\mu\rangle, D\rangle$. By the same way as Lemma 9 N is a $\{2, 3\}$-group and $\alpha \notin Z(N)$. Assume that N is a 2-group. N must be a Sylow 2-subgroup of G_1. D and $W\langle\mu\rangle$ are the only two maximal subgroups of N with center $Z_2 \times Z_2 \times Z_2$ and containing S. Thus they are conjugate to $S\langle\mu, \mu'\rangle$ or $S\langle\tau, \mu\rangle$ in G_1. Since $Z(S\langle\mu, \mu'\rangle)=\langle\pi_1, \pi_2, \pi_3\rangle$ and $Z(S\langle\tau, \mu\rangle)=Z(W\langle\mu\rangle)=\langle\pi_1\pi_2, \pi_3, \pi_3'\rangle$, D must be conjugate to $S\langle\mu, \mu'\rangle$. On the other hand since $\alpha \notin Z(D)$ and $\alpha' \in Z(D)$, we have $Z(D)=\langle\alpha', \pi_1\pi_2, \pi_1'\pi_2'\rangle$, $\langle\alpha', \pi_1'\pi_2', \pi_3\rangle$, $\langle\alpha', \pi_1\pi_2, \pi_1'\pi_2'\pi_3\rangle$ or $\langle\alpha', \pi_1\pi_2, \pi_1'\pi_2'\pi_3\rangle$. Because $\langle\alpha\rangle$ is weakly closed in $Z(S\langle\mu, \mu'\rangle)$ with respect to G_1, this contradicts Lemma 9. Hence N is not a 2-group and N contains an element x of order 3. If $[x, W']=1$, $[x, Z(W)]\neq 1$ implies that twelve involutions in $Z(W)-W'$ are divided into four associated classes by x. Since $\pi_3 \sim \pi_3' \sim \pi_3\pi_3'$ and $\alpha \sim \alpha' \sim \alpha\alpha' \sim \alpha\pi_3' \sim \alpha'\pi_3 \sim \alpha'\pi_1\pi_2 \sim \pi_1\pi_2\pi_3'$, $\alpha\pi_1'\pi_2' \sim \pi_1'\pi_2'\pi_3$, we must have $\alpha \sim \alpha\pi_1'\pi_2'$. If $[x, W']=W'$ then $Z(N) \cap W'=1$ and so $Z(W\langle\mu\rangle)=\langle\pi_1\pi_2\rangle \times Z(N)$ because of $Z(D) \cong Z_2 \times Z_2 \times Z_2$. It follows from $\alpha \notin Z(N)$ that $Z(N)=\langle\pi_3, \pi_3'\rangle$ or $\langle\pi_1\pi_2\pi_3', \pi_3\rangle$ and then $[x, \pi_3]=1$ in both cases. Therefore we get $(\pi_1\pi_2\pi_3)^x=(\pi_1\pi_2)^x\pi_3=\pi_1'\pi_2'\pi_3$ or $\pi_1\pi_2\pi_1'\pi_2'\pi_3$, that is, $\alpha \sim \pi_1'\pi_2'\pi_3$ or $\alpha\pi_1'\pi_2'$. $\alpha\pi_1'\pi_2' \sim \pi_1'\pi_2'\pi_3$ in H_1 implies that $\alpha \sim \alpha\pi_1'\pi_2'$. Thus we have proved that $\alpha \sim \alpha\pi_1'\pi_2'$. Then we must have $n(\alpha)=27$ and $|N_{G_1}(S)|=2^9 \cdot 3^4$ because $n(\alpha)=15$ or 27 by Lemma 9 (i) and Lemma 5. Since $15<(N_{G_1}(S) : C_{G_1}(\pi_1\pi_2) \cap N_{G_1}(S))$ divides 3^4, we get $(N_{G_1}(S) : C_{G_1}(\pi_1\pi_2) \cap N_{G_1}(S))=3^3$ by table I and so $\pi_1\pi_2 \sim \pi_1'\pi_2'$. The lemma is proved.

By preceding lemmas and by a theorem of Kondo [**12**] we get the following lemma for groups G_1, G_2 and G_3 with properties (1), (2) of our theorem.

LEMMA 11. *The group* G_i *possesses precisely three or four conjugacy classes of involutions. If notation is chosen suitably, the possibilities for the fusion of involutions of* G_i *are*

Case I $\quad \pi_1 \sim \pi_1'$

$\pi_1\pi_2 \sim \pi_1\pi_2' \sim \pi_1'\pi_2'$

$\alpha \sim \alpha\pi_1' \sim \alpha\pi_1'\pi_2' \sim \alpha\alpha' \sim \alpha'$

Case II $\quad \pi_1 \sim \pi_1' \sim \pi_1'\pi_2'$

$\pi_1\pi_2 \sim \alpha\alpha'$

$$\alpha \sim \alpha\pi_1'$$
$$\pi_1\pi_2' \sim \alpha\pi_1'\pi_2' \sim \alpha'$$

Case III $\pi_1 \sim \pi_1' \sim \lambda$
$$\pi_1\pi_2 \sim \pi_1\pi_2' \sim \pi_1'\pi_2' \sim \lambda\pi_1 \sim \lambda\pi_1'$$
$$\alpha \sim \alpha\pi_1' \sim \alpha\pi_1'\pi_2' \sim \alpha\alpha' \sim \alpha' \sim \lambda\pi_1\pi_2 \sim \lambda\pi_1\pi_2' \sim \lambda\pi_1'\pi_2'.$$

REMARK 1. The Case I and II are occupied only in G_1 and the Case III only in G_2, G_3.

REMARK 2. In Kondo's notations [**12**], λ, μ, μ' correspond to λ_3, $\lambda_1\lambda_2$, $\lambda_1\lambda_3$ respectively and π_1, π_1', π_2, π_2', π_3, π_3' are the same as his notations.

In the following we study the Case I and the Case III

DEFINITION. We call the representatives π_1, $\pi_1\pi_2$, α, *canonical representatives* of the conjugacy classes of involutions.

Since the extension of D_i over S splits, the extension of $N_{G_i}(S)$ over S splits by a theorem of Gaschüts [**3**]. Let K_i be a complement of S in $N_{G_i}(S)$. Denote by P_i a Sylow 3-subgroup of K_i, by $\langle \tilde{\tau}, \tilde{\mu}, \tilde{\mu}'; \tilde{\tau}^2 = \tilde{\mu}'^2 = \tilde{\mu}^2 = 1, \tilde{\tau}\tilde{\mu}'\tilde{\tau} = \tilde{\mu}\tilde{\mu}', \tilde{\tau}\tilde{\mu} = \tilde{\mu}\tilde{\tau}\rangle$ a Sylow 2-subgroup of K_1 and by $\langle \tilde{\tau}, \tilde{\mu}, \tilde{\mu}'\rangle \times \langle \tilde{\lambda}; \tilde{\lambda}^2 = 1\rangle$ a Sylow 2-subgroup of K_2 and K_3. It follows from the structure of H_i that we may assume $\xi \in P_i$ for $i = 1, 2, 3$, $\nu \in P_3$ and $\langle \tilde{\mu}, \tilde{\mu}', \xi, \tilde{\tau}\rangle \cong S_4$. Now we determine the structure of $N_{G_i}(S)$ and we prove the existence of the complement K_i which contains $\langle \mu, \mu', \xi, \tau\rangle$.

LEMMA 12. *There exist elements x_1, x_2, x_3 of order 3 in K_i with the following properties:*

$$x_1 \in C_{K_i}(\langle \pi_2, \pi_2', \pi_3, \pi_3'\rangle),\ \pi_1^{x_1} = \pi_1',\ \pi_1'^{x_1} = \pi_1\pi_1',$$
$$x_2 \in C_{K_i}(\langle \pi_1, \pi_1', \pi_3, \pi_3'\rangle),\ \pi_2^{x_2} = \pi_2',\ \pi_2'^{x_2} = \pi_2\pi_2',$$
$$x_3 \in C_{K_i}(\langle \pi_1, \pi_1', \pi_2, \pi_2'\rangle),\ \pi_3^{x_3} = \pi_3',\ \pi_3'^{x_3} = \pi_3\pi_3'.$$

Moreover we may assume that the complement K_i is the following groups:

$$K_1 = \langle x_1, x_2, x_3\rangle\langle \mu, \mu', \xi, \tau\rangle$$
$$K_2 = \langle x_1, x_2, x_3\rangle\langle \mu, \mu', \xi, \tau, \lambda\rangle$$
$$K_3 = \langle \nu\rangle\langle x_1, x_2, x_3\rangle\langle \mu, \mu', \xi, \tau, \lambda\rangle.$$

PROOF. Since $|C_{K_1}(\pi_3)| = 2^3 \cdot 3^2$, $|C_{K_2}(\pi_3)| = 2^4 \cdot 3^2$, $|C_{K_3}(\pi_3)| = 2^4 \cdot 3^3$, and $|C_{G_1}(\pi_2\pi_3) \cap N_{G_1}(S)| = 2^9 \cdot 3$, $|C_{G_2}(\pi_2\pi_3) \cap N_{G_2}(S)| = 2^{10} \cdot 3$, $|C_{G_3}(\pi_2\pi_3) \cap N_{G_3}(S)| = 2^{10} \cdot 3^2$ we have $(C_{K_i}(\pi_3) : C_{K_i}(\langle \pi_2, \pi_3\rangle)) \equiv 0 \pmod{3}$. On the other hand $(C_{K_i}(\pi_3) : C_{K_i}(\langle \pi_2, \pi_3\rangle)) = |ccl_{C(\pi_3) \cap K_i}(\pi_2)| < |ccl_{N(S) \cap G_i}(\pi_2)| = 9$ implies that $|C_{K_i}(\langle \pi_2, \pi_3\rangle)| \equiv 0 \pmod{3}$. Let x_1 be an element of order powers of 3 in $C_{K_i}(\langle \pi_2, \pi_3\rangle)$. We may assume that x_1 acts on S as an element of order 3. Thus using Lemma 11, the following table implies that $x_1 \in C_{K_i}(\langle \pi_2, \pi_2', \pi_3, \pi_3'\rangle)$.

$\pi_3'^{x_1}$	$(\pi_3\pi_3')^{x_1}$	
π_1	$\pi_1\pi_3$	$\pi_1\pi_2$
π_2	$\pi_2\pi_3$	$\pi_1\pi_2$
π_1'	$\pi_1'\pi_3$	$\pi_1\pi_2$
π_2'	$\pi_2'\pi_3$	$\pi_1\pi_2$
$\pi_1\pi_1'$	$\pi_1\pi_1'\pi_3$	$\pi_1\pi_2$
$\pi_2\pi_2'$	$\pi_2\pi_2'\pi_3$	$\pi_1\pi_2$
$\pi_3\pi_3'$	π_3'	π_1
π_3'	$\pi_3\pi_3'$	π_1
π_3	1	

In the table the first, second and third column give respectively, $\pi_3^{x_1}$, $(\pi_3\pi_3')^{x_1}$ and the canonical representatives of $ccl_{G_i}((\pi_3\pi_3')^{x_1})$.

Since $ccl_{N(S)\cap G_i}(\pi_1) - ccl_{N(S)\cap G_i}(\pi_1) \cap \langle \pi_2, \pi_2', \pi_3, \pi_3' \rangle = \{\pi_1, \pi_1', \pi_1\pi_1'\}$, we get $[x_1, \langle \pi_1, \pi_1' \rangle] \subset \langle \pi_1, \pi_1' \rangle$ and we may assume that $\pi_1^{x_1} = \pi_1'$. $\pi_1'^{x_1} = \pi_1\pi_1'$. Similarly we have elements x_2, x_3 of order powers of 3 with desired properties. Representing K_i on the vector space S over $GF(2)$ we get in terms of the basis π_1, π_1', π_2, π_2', π_3, π_3':

$$x_1 \longrightarrow \left[\begin{array}{cc|cc|cc} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \qquad x_2 \longrightarrow \left[\begin{array}{cc|cc|cc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$x_3 \longrightarrow \left[\begin{array}{cc|cc|cc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right] \qquad \xi \longrightarrow \left[\begin{array}{cc|cc|cc} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\tilde{\tau} \longrightarrow \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad \tilde{\mu} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\tilde{\mu}' \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \qquad \tilde{\lambda} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Therefore the following relations hold modulo $C_{G_i}(S) \cap K_i$.

$$\tilde{\tau}x_1\tilde{\tau} = x_2, \quad \tilde{\mu}x_1\tilde{\mu} = x_1^{-1}, \quad \tilde{\mu}'x_1\tilde{\mu}'_1 = x_1^{-1}, \quad \tilde{\mu}x_2\tilde{\mu} = x_2^{-1}, \quad \tilde{\mu}'x_2 = x_2\tilde{\mu}',$$

$$\tilde{\tau}x_3 = x_3\tilde{\tau}, \quad \tilde{\mu}x_3 = x_3\tilde{\mu}, \quad \tilde{\mu}'x_3\tilde{\mu}' = x_3^{-1}, \quad x_1x_3 = x_3x_1, \quad x_1x_2 = x_2x_1,$$

$$x_2x_3 = x_3x_2, \quad \xi x_1\xi^{-1} = x_2, \quad \xi x_2\xi^{-1} = x_3, \quad \tilde{\lambda}x_1 = x_1\tilde{\lambda}, \quad \tilde{\lambda}x_2 = x_2\tilde{\lambda},$$

$$\tilde{\lambda}x_3\tilde{\lambda} = x_3^{-1}, \quad \tilde{\lambda}\nu\tilde{\lambda} = \nu^{-1}, \quad \nu x_1 = x_1\nu, \quad \nu x_2 = x_2\nu, \quad \nu x_3 = x_3\nu,$$

$$x_1^3 = x_2^3 = x_3^3 = 1\,.$$

Since S is a selfcentralizing subgroup in G_1 and G_2 these relations hold in G_1 and G_2. Since $C_{G_3}(S) = S \times \langle \nu \rangle$ these relations hold in G_3 modulo $\langle \nu \rangle$. But we can prove that these relations hold in G_3 except $\xi x_1\xi^{-1} = x_2$ and $\xi x_2\xi^{-1} = x_3$. $[S, \nu] = 1$ implies that $\tilde{\lambda}\nu\tilde{\lambda} = \nu^{-1}$ in G_3. Assume that $x_1x_2 = x_2x_1\nu^k$, $\tilde{\lambda}x_1\tilde{\lambda} = x_1\nu^i$, $\tilde{\lambda}x_2\tilde{\lambda} = x_2\nu^j$ for $0 \leqq i, j, k \leqq 2$. It is $x_1x_2\nu^{i+j} = (x_1x_2)^{\tilde{\lambda}} = (x_2x_1\nu^k)^{\tilde{\lambda}} = x_2x_1\nu^{i+j-k}$ and so $x_1x_2 = x_2x_1\nu^{-k} = x_2x_1\nu^k$. Thus we have $k = 0$ and then $x_1x_2 = x_2x_1$. Similarly we have $x_1x_3 = x_3x_1$ and $x_2x_3 = x_3x_2$. Assume that $x_1^3 = \nu^m$ for $m = 0, 1, 2$. If $m \neq 0$, then it is $\Omega_1(\langle x_1 \rangle) = \langle \nu \rangle$, $[\tilde{\mu}, \langle x_1 \rangle] \subset \langle x_1 \rangle$ and $[\tilde{\mu}, \Omega_1(\langle x_1 \rangle)] = 1$. Hence $[\tilde{\mu}, x_1] = 1$ by a theorem of Huppert [**9**]. This is impossible. Thus $m = 0$ and $x_1^3 = 1$. Similarly we have $x_2^3 = x_3^3 = 1$ and so the group $\langle x_1, x_2, x_3, \nu \rangle$ is an elementary abelian group of order 3^4. Since the action of $\tilde{\lambda}$ on $\langle x_1, x_2, x_3, \nu \rangle$ is completely reducible, it follows from $\tilde{\lambda}\nu\tilde{\lambda} = \nu^{-1}$ that $[\tilde{\lambda}, \langle x_1, x_2 \rangle] = 1$ and $\tilde{\lambda}x_3\tilde{\lambda} = x_3^{-1}$ in G_3. Assume that $\tilde{\mu}x_1\tilde{\mu} = x_1^{-1}\nu^l$ for $l = 0, 1, 2$. It is $x_1\tilde{\mu}x_1^{-1} = \tilde{\mu}x_1^{-1}\nu^l x_1^{-1} = \tilde{\mu}x_1^{-2}\nu^l$. $[\tilde{\mu}x_1^{-2}, \nu^l] = 1$ implies that $l = 0$. Similarly we can prove $\tilde{\tau}x_1\tilde{\tau} = x_2$, $\tilde{\mu}x_1\tilde{\mu} = x_1^{-1}$, $\tilde{\mu}'x_1\tilde{\mu}' = x_1^{-1}$, $\tilde{\mu}x_2\tilde{\mu} = x_2^{-1}$, $\tilde{\mu}'x_2 = x_2\tilde{\mu}'$, $\tilde{\tau}x_3 = x_3\tilde{\tau}$, $\tilde{\mu}x_3 = x_3\tilde{\mu}$, $\tilde{\mu}'x_3\tilde{\mu}' = x_3^{-1}$ in G_3. Therefore the structure of $N_{G_i}(S)$ is almost determined except the action of ξ on $\langle x_1, x_2, x_3, \nu \rangle$ in G_3. Since $K_1 \cap H_1 \cong S_4$ and so $\xi\tilde{\mu}\xi^{-1} = \tilde{\mu}\tilde{\mu}'$,

$\tilde{\xi}\tilde{\mu}\tilde{\mu}'\xi^{-1} = \tilde{\mu}'$, $\tilde{\tau}\xi\tilde{\tau} = \xi^{-1}$, $\tilde{\tau}\tilde{\mu}'\tilde{\tau} = \tilde{\mu}\tilde{\mu}'$ it is easily verified that the complement of S in $N_{H_1}(S)$ is conjugate to one of the following groups.

$$\langle \mu, \mu', \xi, \tau \rangle$$

$$\langle \mu, \mu', \xi, \tau\alpha \rangle$$

$$\langle \mu\pi_1\pi_3, \mu'\pi_2\pi_3, \xi, \tau\alpha' \rangle$$

$$\langle \mu\pi_1\pi_3, \mu'\pi_2\pi_3, \xi, \tau\alpha\alpha' \rangle .$$

If $\tilde{\tau} = \tau\alpha$, then $\tau\alpha = (\tau\alpha)^{x_1 \cdot x_2 \cdot x_3} = \tau^{x_1 \cdot x_2 \cdot x_3} \cdot \alpha'$ and so $\tau^{x_1 \cdot x_2 \cdot x_3} = \tau\alpha\alpha'$ which is impossible by Lemma 11. Similarly we have $\tilde{\tau} \neq \tau\alpha'$ and $\tilde{\tau} \neq \tau\alpha\alpha'$. Therefore we may assume that $\tilde{\tau} = \tau$, $\tilde{\mu} = \mu$ and $\tilde{\mu}' = \mu'$. This proves that $K_1 = \langle x_1, x_2, x_3 \rangle \langle \mu, \mu', \xi, \tau \rangle$. $[\lambda, \langle \mu, \mu', \tau \rangle] = 1$ implies that $\tilde{\lambda} = \lambda$, $\lambda\pi_1\pi_2$, $\lambda\pi_3$ or $\lambda\alpha$. On the other hand it is $\tilde{\lambda}x_3\tilde{\lambda} = x_3^{-1}$ and $1 = [\tilde{\lambda}, x_1] = [\tilde{\lambda}, x_2]$. Hence we must have $\tilde{\lambda} = \lambda$. This implies that $K_2 = \langle x_1, x_2, x_3 \rangle \langle \mu, \mu', \xi, \tau, \lambda \rangle$ and $K_3 = \langle \nu \rangle \langle x_1, x_2, x_3 \rangle \langle \mu, \mu', \xi, \tau, \lambda \rangle$. The proof is complete.

Throughout the present paper the meanings of x_1, x_2, x_3 in this lemma will be preserved.

§ 4. The structure of the group $C_{G_i}(\pi_1)$ and $C_{G_i}(\pi_1\pi_2)$.

LEMMA 13. *$C_{G_i}(\pi_1) = (\langle \pi_1, \pi_1' \rangle \times F_i)\langle \mu \rangle$ where $F_1 \cong A_8$ or A_9, $F_2 \cong A_{10}$, $F_3 \cong A_{11}$ and $F_1\langle \mu \rangle \cong S_8$ or S_9, $F_2\langle \mu \rangle \cong S_{10}$, $F_3\langle \mu \rangle \cong S_{11}$. Moreover $[x_1, F_i] = 1$.*

PROOF. Put $\mathfrak{G}_i = C_{G_i}(\pi_3)$ and $\bar{\mathfrak{G}}_i = C_{G_i}(\pi_3)/\langle \pi_3 \rangle$. In the epimorphism $\mathfrak{G}_i \to \bar{\mathfrak{G}}_i$ put $\alpha \to \bar{\alpha}$, $\pi_2 \to \bar{\pi}_2$, $\pi_1' \to \bar{\pi}_1'$, $\pi_2' \to \bar{\pi}_2'$, $\pi_3' \to \bar{\pi}_3'$, $\mu \to \bar{\mu}$, $\mu' \to \bar{\mu}'$, $\tau \to \bar{\tau}$, $\lambda \to \bar{\lambda}$, $\rho \to \bar{\rho}$, $x_1 \to \bar{x}_1$, $x_2 \to \bar{x}_2$, and $\nu \to \bar{\nu}$. Let T_i be a Sylow 2-subgroup of $\bar{\mathfrak{G}}_i$. Then $T_1 = \langle \bar{\alpha}, \bar{\pi}_1', \bar{\pi}_2, \bar{\pi}_2', \bar{\pi}_3', \bar{\mu}, \bar{\mu}', \bar{\tau} \rangle$ and $T_2 = T_3 = \langle T_1, \bar{\lambda} \rangle$. It is $T_1' = \langle \bar{\pi}_1, \bar{\pi}_2, \bar{\pi}_1'\bar{\pi}_2', \bar{\mu} \rangle$, $Z(T_1) = \langle \bar{\alpha}, \bar{\pi}_3' \rangle$, $Z(T_2) = Z(T_3) = \langle \bar{\alpha}, \bar{\pi}_3', \bar{\lambda} \rangle$ and $C(\bar{\alpha}) \cap \bar{\mathfrak{G}}_1 = \langle T_1, \bar{\rho} \rangle$, $C(\bar{\alpha}) \cap \bar{\mathfrak{G}}_2 = \langle T_2, \bar{\rho} \rangle$, $C(\bar{\alpha}) \cap \bar{\mathfrak{G}}_3 = \langle T_3, \bar{\rho}, \bar{\nu} \rangle$. By Lemma 11 $N(Z(T_i)) \cap \bar{\mathfrak{G}}_i = C(Z(T_i)) \cap \bar{\mathfrak{G}}_i$ and so $N(T_i) \cap \bar{\mathfrak{G}}_i = T_i$. Thus it follows from a transfer theorem that $\bar{\mathfrak{G}}_i/\bar{\mathfrak{G}}_i'(2) \cong T_i/\langle T_i \cap T_i'^g ; g \in \bar{\mathfrak{G}}_i \rangle$. Since $\bar{\mathfrak{G}}_i'(2) \ni \bar{x}_1, \bar{x}_2, \bar{\rho}$, and $\bar{\mathfrak{G}}_i'(2) \supset T_i'$ we have $\langle T_i \cap T_i'^g ; g \in \bar{\mathfrak{G}}_i \rangle \supset \langle \bar{\pi}_1, \bar{\pi}_2, \bar{\pi}_1', \bar{\pi}_2', \bar{\mu}, \bar{\tau} \rangle$ for $i = 1, 2, 3$. Every element of T_i' is conjugate to $\bar{\pi}_1$, $\bar{\pi}_1\bar{\pi}_2$, or $\bar{\mu}\bar{\pi}_1'\bar{\pi}_2'$ in $\bar{\mathfrak{G}}_i'(2)$. Since $(\bar{\mu}\bar{\pi}_1'\bar{\pi}_2')^2 = \bar{\pi}_1\bar{\pi}_2$, every element of order 4 in $T_i - \langle \bar{\pi}_1, \bar{\pi}_2, \bar{\pi}_1', \bar{\pi}_2', \bar{\mu}, \bar{\tau} \rangle$ is not conjugate to $\bar{\mu}\bar{\pi}_1'\bar{\pi}_2'$ in $\bar{\mathfrak{G}}_i$ by Lemma 11. By the following table and Lemma 11, we get

$$\langle T_1 \cap T_1'^g ; g \in \bar{\mathfrak{G}}_1' \rangle = \langle \bar{\pi}_1, \bar{\pi}_2, \bar{\pi}_1', \bar{\pi}_2', \bar{\mu}, \bar{\tau} \rangle .$$

In the table x is some element of $\mathfrak{G}_1 = C_{G_1}(\pi_3)$. The first, second, third and fourth column give respectively, π_1^x, $(\pi_1\pi_2)^x$, α^x and $(\pi_1\pi_3)^x$.

π_1^x	$(\pi_1\pi_2)^x$	α^x	$(\pi_1\pi_3)^x$	conclusion
μ'			$\mu'\pi_3$	$\bar{\pi}_1 \not\sim \bar{\mu}'$
$\mu'\pi_3$			μ'	
π_3'			$\pi_3\pi_3'$	$\bar{\pi}_1 \not\sim \bar{\pi}_3'$
$\pi_3\pi_3'$			π_3'	
	$\mu'\pi_2$	$\mu'\pi_2\pi_3$		$\bar{\alpha} \not\sim \bar{\mu}'\bar{\pi}_2$
	$\mu'\pi_2\pi_3$	$\mu'\pi_2$		
	$\pi_1\pi_3'$	$\pi_1\pi_3\pi_3'$		$\bar{\alpha} \not\sim \bar{\pi}_1\bar{\pi}_3'$
	$\pi_1\pi_3\pi_3'$	$\pi_1\pi_3'$		
	$\mu'\pi_2'$	$\mu'\pi_2'\pi_3$		$\bar{\alpha} \not\sim \bar{\mu}'\bar{\pi}_2'$
	$\mu'\pi_2'\pi_3$	$\mu'\pi_2'$		
	$\mu\pi_3'$	$\mu\pi_3\pi_3'$		$\bar{\alpha} \not\sim \bar{\mu}\bar{\pi}_3'$
	$\mu\pi_3\pi_3'$	$\mu\pi_3'$		
	$\pi_1\pi_3'$	$\pi_1'\pi_3\pi_3'$		$\bar{\alpha} \not\sim \bar{\pi}_1'\bar{\pi}_3'$
	$\pi_1'\pi_3\pi_3'$	$\pi_1'\pi_3'$		
	$\pi_1\pi_2\pi_3'$	$\alpha\pi_3'$		$\bar{\alpha} \not\sim \bar{\pi}_1\bar{\pi}_2\bar{\pi}_3'$
	$\alpha\pi_3'$	$\pi_1\pi_2\pi_3'$		
	$\pi_1\pi_2'\pi_3'$	$\pi_1\pi_2'\pi_3\pi_3'$		$\bar{\alpha} \not\sim \bar{\pi}_1\bar{\pi}_2'\bar{\pi}_3'$
	$\pi_1\pi_2'\pi_3\pi_3'$	$\pi_1\pi_2'\pi_3'$		
	$\alpha'\pi_1$	$\alpha'\pi_1\pi_3$		$\bar{\alpha} \not\sim \bar{\alpha}'\bar{\pi}_1$
	$\alpha'\pi_1\pi_3$	$\alpha'\pi_1$		
	$\tau\pi_3\pi_3'$	$\tau\pi_3'$		$\bar{\alpha} \not\sim \bar{\tau}\bar{\pi}_3'$
	$\tau\pi_3'$	$\tau\pi_3\pi_3'$		
	$\alpha'\pi_3$	α'		$\bar{\alpha} \not\sim \bar{\alpha}'$
	α'	$\alpha'\pi_3$		

Let u be an element of order 3 with $\pi_3^u=\lambda$, $\lambda^u=\lambda\pi_3$ and $\mu'^u=\mu'\lambda$ (see Kondo [12]). Let y be an element of order 2 in H_2 with $\pi_1^y=\mu'$, $\pi_3^y=\mu'\pi_1\pi_3$ and $\lambda^y=\mu'\lambda\pi_1$. It is $(\pi_1)^{yuy^{-1}}=\mu'\lambda$. On the other hand $(\pi_3)^{yuy^{-1}}=\mu'\pi_1$ and for some element $w\in H_2$, $(\mu'\pi_1)^w=\pi_3$, $[w, \mu'\lambda]=1$. Therefore we have $[(\pi_1)^{yuy^{-1}}]^w=\mu'\lambda$ and $[yuy^{-1}, \pi_3]=1$. Thus $\bar{\pi}_1\sim\bar{\mu}'\bar{\lambda}$ in $\bar{\mathfrak{G}}_2$ and $\bar{\mathfrak{G}}_3$. Hence similar argument shows that

$$\langle T_j\cap T_j'^g\,;\, g\in\bar{\mathfrak{G}}_j\rangle=\langle\bar{\pi}_1, \bar{\pi}_2, \bar{\pi}_1', \bar{\pi}_2', \bar{\mu}, \bar{\tau}, \bar{\mu}'\bar{\lambda}\rangle \qquad \text{for } j=2, 3\,.$$

This implies that $(\bar{\mathfrak{G}}_i:\bar{\mathfrak{G}}_i'(2))=4$ for $i=1, 2, 3$. It is $\bar{\mathfrak{G}}_i'(2)\ni\bar{x}_1, \bar{x}_2, \bar{\rho}$ and then focal group of $T_i\cap\bar{\mathfrak{G}}_i'(2)$ in $\bar{\mathfrak{G}}_i'(2)$ contains $\langle\bar{\pi}_1, \bar{\pi}_2, \bar{\pi}_1', \bar{\pi}_2', \bar{\mu}, \bar{\tau}\rangle$. Because $\langle\bar{\pi}_3', \bar{\mu}\bar{\mu}', \bar{\mu}'\bar{\lambda}\rangle$ is an abelian group, Higman's theorem [8] implies that $O^2(\bar{\mathfrak{G}}_i)=\bar{\mathfrak{G}}_i'(2)$. Moreover we have

$$C(\bar{\alpha})\cap\bar{\mathfrak{G}}_1'(2)=\langle\bar{\pi}_1, \bar{\pi}_2, \bar{\pi}_1', \bar{\pi}_2', \bar{\mu}, \bar{\tau}, \bar{\rho}\rangle$$

$$C(\bar{\alpha})\cap\bar{\mathfrak{G}}_2'(2)=\langle C(\bar{\alpha})\cap\bar{\mathfrak{G}}_1'(2), \bar{\mu}'\bar{\lambda}\rangle$$

$$C(\bar{\alpha})\cap\bar{\mathfrak{G}}_3'(2)=\langle C(\bar{\alpha})\cap\bar{\mathfrak{G}}_2'(2), \bar{\nu}\rangle\,.$$

We establish the isomorphism from $C(\bar{\alpha})\cap\bar{\mathfrak{G}}_3'(2)$ onto $C_{A_{11}}((1, 2)(3, 4)(5, 6)(7, 8))$ by mapping the generators $\bar{\alpha}, \bar{\pi}_1, \bar{\mu}, \bar{\alpha}\bar{\pi}_1'\bar{\pi}_2', \bar{\tau}, \bar{\rho}, \bar{\pi}_1\bar{\pi}_1', \bar{\mu}'\bar{\lambda}, \bar{\nu}$ of $C(\bar{\alpha})\cap\bar{\mathfrak{G}}_3'(2)$ onto the generators (1, 2)(3, 4)(5, 6)(7, 8), (1, 2)(3, 4), (1, 2)(5, 6), (1, 3)(2, 4)(5, 7)(6, 8), (1, 5)(3, 7)(2, 6)(4, 8), (1, 3, 5)(2, 4, 6), (1, 3)(2, 4), (1, 2)(13, 14), (13, 14, 15) of $C_{A_{11}}$ ((1, 2)(3, 4)(5, 6)(7, 8)) in this order and then verifying that the same relations are satisfied by both systems of generators. Hence the result of Kondo [10] implies that $\bar{\mathfrak{G}}_3'(2)\cong A_{11}$. Similarly we get $\bar{\mathfrak{G}}_2'(2)\cong A_{10}$, $\bar{\mathfrak{G}}_1'(2)\cong A_8$ or A_9 by a theorem of Held [6], [7]. A Sylow 2-subgroup of $\mathfrak{G}_1'(2)$ is $\langle\pi_3\rangle\times\langle\pi_1, \pi_1', \pi_2, \pi_2', \mu, \tau\rangle$ and that of $\mathfrak{G}_2'(2)$, $\mathfrak{G}_3'(2)$ is $\langle\pi_3\rangle\times\langle\pi_1, \pi_1', \pi_2, \pi_2', \mu, \tau, \mu'\lambda\rangle$. Thus it follows from Gaschütz's theorem [3] that $\mathfrak{G}_i'(2)=\langle\pi_3\rangle\times K_i$ where $K_1\cong A_8$ or A_9, $K_2\cong A_{10}$ and $K_3\cong A_{11}$. Since $K_i\ni\pi_1, \pi_2, \pi_1'\pi_2', x_1, x_2, \rho$ and $\pi_1\not\sim\mu'\lambda\pi_3$, a Sylow 2-subgroup of K_1 is $\langle\pi_1, \pi_2, \pi_1', \pi_2', \mu, \tau\rangle$ and that of K_2, K_3 is $\langle\pi_1, \pi_2, \pi_1', \pi_2', \mu, \tau, \mu'\lambda\rangle$.

We shall consider now $K_i\langle\mu\mu'\rangle=X$. Assume that $C_X(K_i)=\langle y\mu\mu'\rangle$ is of order 2 for some $y\in K_i$. It is $1=[\pi_1\pi_2, y\mu\mu']=[\pi_1\pi_2, y]=[y, \mu\mu']$ and $y^2=1$. Since $1=[\rho^{\pi_2\pi_2'}, y\mu\mu']=[\rho^{\pi_2\pi_2'}, y]$ we have $y\in\langle\pi_1\pi_2\cdot(\rho)^{\pi_1\pi_2'}\rangle$ and then $y=\pi_1\pi_2$ which is impossible because $\pi_2'^{y\mu\mu'}=\pi_2\pi_2'$. It follows $C_X(K_i)=1$ and $K_1\langle\mu\mu'\rangle\cong S_8$ or S_9, $K_2\langle\mu\mu'\rangle\cong S_{10}$, $K_3\langle\mu\mu'\rangle\cong S_{11}$. $\mathfrak{G}_i\rhd K_i$ implies that $\mathfrak{G}_i=(C_{\mathfrak{G}_i}(K_i)\times K_i)\langle\mu\mu'\rangle$. Assume that $\pi_3'=ca\mu\mu'$ where $c\in C_{\mathfrak{G}_i}(K_i)$ and $a\in K_i$. By Lemma 12 we have $[a, x_1]=[a, \pi_1]=[a, \pi_1']=[a, \pi_2]=1$ which is impossible because $K_1\cong A_8$ or A_9, $K_2\cong A_{10}$ and $K_3\cong A_{11}$. Thus we have $\pi_3'=ac$. Since $[a, \langle\pi_1, \pi_1', \pi_2, \pi_2'\rangle]=1$ and $a^2=[a, x_1]=[a, x_2]=1$, it follows from the structure of K_i that $a=1$ and so $\pi_3'=c\in C_{\mathfrak{G}_i}(K_i)$. $|C_{\mathfrak{G}_i}(K_i)|=4$ implies that $C_{\mathfrak{G}_i}(K_i)=\langle\pi_3, \pi_3'\rangle$. By the conjugation in H_1 we get $C_{G_1}(\pi_1)=(\langle\pi_1, \pi_1'\rangle\times F_i)\langle\mu\rangle$ where $F_1\cong A_8$ or

A_9, $F_2 \cong A_{10}$, $F_3 \cong A_{11}$ and $F_1\langle\mu\rangle \cong S_8$ or S_9, $F_2\langle\mu\rangle \cong S_{10}$, $F_3\langle\mu\rangle \cong S_{11}$. Since $[x_1, \langle\pi_1, \pi_1'\rangle] \subset \langle\pi_1, \pi_1'\rangle$ and $C_{G_i}(\langle\pi_1, \pi_1'\rangle) = \langle\pi_1, \pi_1'\rangle \times F_i$, we have $[x_1, F_i] \subset F_i$. The element x_1 is of order 3 and so x_1 induces an inner automorphism on F_i. Because $[x_1, \mu\mu'] = 1$ and $\langle\pi_2, \pi_2', \pi_3, \pi_3'\rangle$ is a self-centralizing subgroup of F_1 and F_2 we get $[x_1, F_1] = [x_1, F_2] = 1$. $[\lambda, \nu] \neq 1$ and $[\lambda, x_1] = 1$ imply that $[x_1, F_3] = 1$. The proof is complete.

LEMMA 14. (Dickson [2]) *The symmetric group S_l is generated by $l-1$ elements $z_1, z_2, \cdots, z_{l-1}$ satisfying the following relations:*

$$z_1^2 = \cdots = z_{l-1}^2 = (z_i z_{i+1})^3 = (z_j z_k)^2 = 1$$
$$(1 \leqq i \leqq l-1,\ 1 \leqq j < k \leqq l-1).$$

The alternating group A_l is generated by $l-2$ elements $y_1, y_2, \cdots, y_{l-2}$ satisfying the following relations:

$$y_1^3 = y_2^2 = \cdots = y_{l-2}^2 = (y_i y_{i+1})^3 = (y_j y_k)^2 = 1$$
$$(1 \leqq i \leqq l-3,\ 1 \leqq j < k \leqq l-2).$$

DEFINITION. We call a set of such generators of S_l *a set of canonical generators* of S_l and that of A_l *a set of canonical generators* of A_l.

The group F_3 contains $x_2, x_3, \nu, \pi_2, \pi_3, \mu\mu', \lambda$ and $\pi_2 \sim \mu\mu' \sim \mu\mu'\pi_3 \sim \mu\mu'\lambda$. Since π_2 is a non-central involution of a Sylow 2-subgroup of F_3, π_2 is a product of two transpositions in $F_3\langle\mu\rangle$. The elements $\pi_2, \mu\mu', (\mu\mu')^{x_3}, \mu\mu'\pi_3, \mu\mu'\lambda$, $(\mu\mu'\lambda)^\nu$ normalize $\langle x_2\rangle$. It follows from the structure of $F_3 \cong A_{11}$ that there exist two elements δ_3 and ζ_3 which are conjugate to π_2 such that

$$y_1 = x_2,\ y_2 = \pi_2,\ y_3 = \delta_3,\ y_4 = \mu\mu',\ y_5 = (\mu\mu')^{x_3},$$
$$y_6 = \mu\mu'\pi_3,\ y_7 = \zeta_3,\ y_8 = \mu\mu'\lambda,\ y_9 = (\mu\mu'\lambda)^\nu$$

is a set of canonical generators of F_3. Similarly we can find $\delta_2, \zeta_2, \delta_1, \zeta_1, \delta_1'$, ζ_1' and then the groups F_1, F_2, F_3 are given as follows:

$$F_1 = \langle x_2, \pi_2, \delta_1, \mu\mu', (\mu\mu')^{x_3}, \mu\mu'\pi_3\rangle \quad \text{or}$$
$$\langle x_2, \pi_2, \delta_1', \mu\mu', (\mu\mu')^{x_3}, \mu\mu'\pi_3, \zeta_1'\rangle$$
$$F_2 = \langle x_2, \pi_2, \delta_2, \mu\mu', (\mu\mu')^{x_3}, \mu\mu'\pi_3, \zeta_2, \mu\mu'\lambda\rangle$$
$$F_3 = \langle x_2, \pi_2, \delta_3, \mu\mu', (\mu\mu')^{x_3}, \mu\mu'\pi_3, \zeta_3, \mu\mu'\lambda, (\mu\mu'\lambda)^\nu\rangle.$$

LEMMA 15. $C_{G_i}(\pi_1) \cap C_{G_i}(\pi_2) = \{\langle\pi_1, \pi_1'\rangle \times (\langle\pi_2, \pi_2'\rangle \times B_i)\langle\mu\mu'\rangle\}\langle\mu\rangle$, *where $B_1 \cong A_4$ or A_5, $B_2 \cong A_6$, $B_3 \cong A_7$ and $B_1\langle\mu\mu'\rangle \cong S_4$ or S_5, $B_2\langle\mu\mu'\rangle \cong S_6$, $B_3\langle\mu\mu'\rangle \cong S_7$.*

PROOF. The result follows from Lemma 13.

LEMMA 16. $[\mu, B_i] = 1$.

PROOF. Since $[\mu, C_{F_i}(\pi_2)] \subset C_{F_i}(\pi_2)$ and B_i is a characteristic subgroup of $C_{F_i}(\pi_2)$, we have $[\mu, B_i] \subset B_i$. It is $N_{G_i}(B_i) = \langle\mu\mu'\rangle(B_i \times C_{G_i}(B_i))$. Assume that

$\mu=\mu\mu' bc$ where $b\in B_i$ and $c\in C_{G_i}(B_i)$. It is $[\pi_3, b]=1$, $x_3^b=x_3^{-1}$ and then $b\in C_{B_i}(\langle\pi_3, \pi_3'\rangle)\cap N_{B_i}(\langle x_3\rangle)$. This contradicts the structure of B_i. Thus we have $\mu=bc$ and $[b, \langle\pi_3, \pi_3', x_3\rangle]=1$ implies that $b=1$ and so $\mu=c\in C_{G_i}(B_i)$.

By Lemmas 15 and 16 we have

$$C_{G_i}(\pi_1)\cap C_{G_i}(\pi_2)=(\langle\pi_1, \pi_2, \mu\rangle\langle\pi_1', \pi_2'\rangle\times B_i)\langle\mu\mu'\rangle .$$

The group B_i contains π_3 and x_3. Since $[\zeta_j, \pi_2]=[\zeta_j, \pi_1]=[\zeta_j, \pi_1']=1$ and $[\zeta_j, \pi_2']\neq 1$, ζ_j is a transposition of $B_i\langle\mu\mu'\rangle$ and then $\zeta_j\mu\mu'\in B_i$ for $j=2, 3$ and $i=2, 3$. The same situation holds for ζ_1'. Therefore

$$y_1=x_3,\ y_2=\pi_3,\ y_3=\mu\mu'\zeta_3,\ y_4=\lambda,\ y_5=\lambda^\nu$$

is a set of canonical generators of B_3. The group B_1, B_2, and B_3 are as follows.

$$B_1=\langle x_3, \pi_3\rangle \quad \text{or} \quad \langle x_3, \pi_3, \mu\mu'\zeta_1'\rangle$$

$$B_2=\langle x_3, \pi_3, \mu\mu'\zeta_2, \lambda\rangle$$

$$B_3=\langle x_3, \pi_3, \mu\mu'\zeta_3, \lambda, \lambda^\nu\rangle .$$

LEMMA 17. *Let z be an element of order 2 in $S\langle\mu\mu', \lambda\rangle$.*

(i) *If $\pi_1\sim\pi_1'z$ in $C_{G_i}(\pi_1\pi_2)$, then $z=\pi_1\pi_1'$ or $\pi_1'\pi_2$.*

(ii) *If $\pi_1'\sim\pi_1'z$ in $C_{G_i}(\pi_1\pi_2)$, then $z=\pi_1'$, $\pi_1'\pi_2\pi_2'$, $\pi_1'\pi_2'$, $\mu\mu'\pi_1'$, $\mu\mu'\pi_1'\pi_2$, $\mu\mu'\pi_1'\pi_3$, $\mu\mu'\pi_1'\pi_2\pi_3$, $\mu\mu'\lambda\pi_1'$, or $\mu\mu'\lambda\pi_1'\pi_2$.*

PROOF. Since $\pi_2=\pi_1\pi_1\pi_2\sim\pi_1'\pi_1\pi_2 z$ or $\pi_1'\pi_1\pi_2\sim\pi_1'\pi_1\pi_2 z$ in $C_{G_i}(\pi_1\pi_2)$ for the case (i) or (ii) respectively, the following table yields our results by Lemma 11.

z	$\pi_1'\pi_1\pi_2 z$	
$\pi_1\pi_1'$	π_2	π_1
$\pi_1'\pi_2$	π_1	π_1
π_1	$\pi_1'\pi_2$	$\pi_1\pi_2$
$\pi_1'\pi_2\pi_2'$	$\pi_1\pi_2'$	$\pi_1\pi_2$
$\pi_1'\pi_2'$	$\pi_1\pi_2\pi_2'$	$\pi_1\pi_2$
$\mu\mu'\pi_1'$	$\mu\mu'\pi_1\pi_2$	$\pi_1\pi_2$
$\mu\mu'\pi_1'\pi_2$	$\mu\mu'\pi_1$	$\pi_1\pi_2$
$\mu\mu'\pi_1'\pi_3$	$\mu\mu'\alpha$	$\pi_1\pi_2$
$\mu\mu'\pi_1'\pi_2\pi_3$	$\mu\mu'\pi_1\pi_3$	$\pi_1\pi_2$
$\mu\mu'\lambda\pi_1'$	$\mu\mu'\lambda\pi_1\pi_2$	$\pi_1\pi_2$
$\mu\mu'\lambda\pi_1'\pi_2$	$\mu\mu'\lambda\pi_1$	$\pi_1\pi_2$

$\pi_1'\pi_3$	α	α
$\pi_1'\pi_3'$	$\pi_1\pi_2\pi_3'$	α
$\pi_1'\pi_3\pi_3'$	$\alpha\pi_3'$	α
$\lambda\pi_1'$	$\lambda\pi_1\pi_2$	α
$\lambda\pi_1'\pi_3$	$\lambda\alpha$	α

In the table the first, second and third column give respectively, the involution z in $S\langle\mu\mu', \lambda\rangle$ with $\pi_1 \sim \pi_1'z$, the involution $\pi_1'\pi_1\pi_2 z$ and the canonical representative of $ccl_{G_i}(\pi_1'\pi_1\pi_2 z)$.

LEMMA 18. *Put* $X_i = ccl_{C(\pi_1\pi_2)\cap G_i}(\pi_1)$ *for* $i=1, 2, 3$. *Then* $X_1 \cap S\langle\mu, \mu'\rangle = X_2 \cap S\langle\mu, \mu', \lambda\rangle = X_3 \cap S\langle\mu, \mu', \lambda\rangle = \{\pi_1, \pi_2, \mu, \mu\pi_1, \mu\pi_2, \mu\pi_1\pi_2\}$.

PROOF. For every element $h \in G_{G_i}(\pi_1\pi_2)$ we have $\pi_2^h = \pi_1^h\pi_1\pi_2$ and then the following table implies our result.

π_1^h	$\pi_1^h \cdot \pi_1\pi_2$	
π_1	π_2	π_1
π_2	π_1	π_1
π_1'	$\pi_1'\pi_1\pi_2$	$\pi_1\pi_2$
π_2'	$\pi_1\pi_2\pi_2'$	$\pi_1\pi_2$
$\pi_1\pi_1'$	$\pi_1'\pi_2$	$\pi_1\pi_2$
$\pi_2\pi_2'$	$\pi_1\pi_2'$	$\pi_1\pi_2$
π_3	α	α
$\pi_3\pi_3'$	$\alpha\pi_3'$	α
π_3'	$\pi_1\pi_2\pi_3'$	α
μ	$\mu\pi_1\pi_2$	π_1
$\mu\pi_1$	$\mu\pi_2$	π_1
$\mu\pi_2$	$\mu\pi_1$	π_1
$\mu\pi_1\pi_2$	μ	π_1
μ'	$\mu'\pi_1\pi_2$	$\pi_1\pi_2$
$\mu'\pi_1$	$\mu'\pi_2$	$\pi_1\pi_2$
$\mu'\pi_3$	$\mu'\alpha$	$\pi_1\pi_2$

$\mu'\pi_1\pi_3$	$\mu'\pi_2\pi_3$	$\pi_1\pi_2$
$\mu\mu'$	$\mu\mu'\pi_1\pi_2$	$\pi_1\pi_2$
$\mu\mu'\pi_2$	$\mu\mu'\pi_1$	$\pi_1\pi_2$
$\mu\mu'\pi_3$	$\mu\mu'\alpha$	$\pi_1\pi_2$
$\mu\mu'\pi_2\pi_3$	$\mu\mu'\pi_1\pi_3$	$\pi_1\pi_2$
λ	$\lambda\pi_1\pi_2$	α
$\lambda\pi_3$	$\lambda\alpha$	α
$\mu'\lambda$	$\mu'\lambda\pi_1\pi_2$	$\pi_1\pi_2$
$\mu\mu'\lambda$	$\mu\mu'\lambda\pi_1\pi_2$	$\pi_1\pi_2$
$\mu'\lambda\pi_1$	$\mu'\lambda\pi_2$	$\pi_1\pi_2$
$\mu\mu'\lambda\pi_2$	$\mu\mu'\lambda\pi_1$	$\pi_1\pi_2$

In the table the first, second and third column give respectively, the involution in $S\langle\mu, \mu', \lambda\rangle$ which is conjugate to π_1, the product of the element in the first column and $\pi_1\pi_2$, and the canonical representative of $ccl_{G_i}(\pi_1^h\pi_1\pi_2)$.

LEMMA 19. $C_{G_i}(\pi_1\pi_2) \triangleright \langle\mu, \pi_1, \pi_2\rangle$.

PROOF. Since $\pi_1 \not\sim \pi_1'$ in $C_{G_i}(\pi_1\pi_2)$ by Lemma 18, it is well known that $\langle\pi_1^x, \pi_1'\rangle$ is a dihedral group with non-trivial center for all $x \in C_{G_i}(\pi_1\pi_2)$ (cf. Brauer and Fowler [1]). Put $\langle a(x)\rangle = Z(\langle\pi_1^x, \pi_1'\rangle)$. It follows from the structure of $\langle\pi_1^x, \pi_1'\rangle$ that $\pi_1^x \sim \pi_1'a(x)$ or $\pi_1' \sim \pi_1'a(x)$. It is $1=[a(x), \pi_1']=[a(x), \pi_1\pi_2]$ and by Lemma 13 we have

$$C_{G_i}(\pi_1') \cap C_{G_i}(\pi_1\pi_2) = \langle\pi_1, \pi_1'\rangle \times (\langle\pi_2, \pi_2'\rangle \times B_i)\langle\mu\mu'\rangle\,.$$

Since $S\langle\mu\mu'\rangle$ is a Sylow 2-subgroup of $C_{G_1}(\pi_1') \cap C_{G_1}(\pi_1\pi_2)$ and $S\langle\mu\mu', \lambda\rangle$ is a Sylow 2-subgroup of $C_{G_i}(\pi_1') \cap C_{G_i}(\pi_1\pi_2)$ for $i=2, 3$, there exists an element $b(x) \in B_i$ with $a(x)^{b(x)} \in S\langle\mu\mu'\rangle$ for $i=1$ or $a(x)^{b(x)} \in S\langle\mu\mu', \lambda\rangle$ for $i=2, 3$. $\langle\pi_1^x, \pi_1'\rangle^{b(x)} = \langle(\pi_1^x)^{b(x)}, \pi_1'\rangle$ implies that $(\pi_1^x)^{b(x)} \sim \pi_1'a(x)^{b(x)}$ or $\pi_1' \sim \pi_1'a(x)^{b(x)}$. Assume that $(\pi_1^x)^{b(x)} \sim \pi_1'a(x)^{b(x)}$. By Lemma 17 we have $a(x)^{b(x)} = \pi_1\pi_1'$ or $\pi_1'\pi_2$. $[b(x), \langle\pi_1\pi_1', \pi_1'\pi_2\rangle]=1$ implies that $a(x)=\pi_1\pi_1'$ or $\pi_1'\pi_2$. By our assumption $(\pi_1^x)^{y(x)} = \pi_1'a(x) = \pi_1$ or π_2 for some $y(x) \in \langle\pi_1^x, \pi_1'\rangle$. It is $1=[y(x), a(x)]=[y(x), \pi_1\pi_2]$ and then $1=[\pi_1^x, \pi_1\pi_1']=[y(x), \pi_1\pi_1']$. Since $\langle\pi_1, \pi_1'\rangle$ and $\langle\pi_2, \pi_2'\rangle$ are normal in $C_{G_i}(\pi_1\pi_1') \cap C_{G_i}(\pi_1\pi_2)$ we get $\pi_1^x \in \langle\pi_1, \pi_1'\rangle$ or $\pi_1^x \in \langle\pi_2, \pi_2'\rangle$. This implies that $\pi_1^x = \pi_1$ or $\pi_1^x = \pi_2$ by Lemma 18. Assume that $\pi_1' \sim \pi_1'a(x)^{b(x)}$. By Lemma 17 $a(x)^{b(x)} = \pi_1$, $\pi_1'\pi_2\pi_2'$, $\pi_1'\pi_2'$, $\mu\mu'\pi_1'$, $\mu\mu'\pi_1'\pi_2$, $\mu\mu'\pi_1'\pi_3$, $\mu\mu'\pi_1'\pi_2\pi_3$, $\mu\mu'\lambda\pi_1'$ or $\mu\mu'\lambda\pi_1'\pi_2$ and hence if $a(x)^{b(x)} \neq \pi_1$, then $a(x)^{b(x)} \sim \pi_1\pi_2$ or α in G_i. Since $(\pi_1^x)^{y(x)} = \pi_1^x a(x)$ for some $y(x) \in \langle\pi_1^x, \pi_1'\rangle$, it is $1=[\pi_1^x, (\pi_1^x)^{y(x)}]=[\pi_2^x, (\pi_1^x)^{y(x)}]$. Since $S^x\langle\mu, \mu'\rangle^x$

is a Sylow 2-subgroup of $C_{G_1}(\pi_1^x)\cap C_{G_1}(\pi_2^x)$ and $S^x\langle\mu, \mu', \lambda\rangle^x$ is a Sylow 2-subgroup of $C_{G_i}(\pi_1^x)\cap C_{G_i}(\pi_2^x)$ for $i=2, 3$, there exists an element $\tilde{b}(x)\in B_i^x$ with $[(\pi_1^x)^{y(x)}]^{\tilde{b}(x)}\in S^x\langle\mu, \mu'\rangle^x$ or $S^x\langle\mu, \mu', \lambda\rangle^x$. By Lemma 18, $[(\pi_1^x)^{y(x)}]^{\tilde{b}(x)}\in\langle\mu, \pi_1, \pi_2\rangle^x$. $[\tilde{b}(x), \langle\mu, \pi_1, \pi_2\rangle^x]=1$ implies that $[\tilde{b}(x), (\pi_1^x)^{y(x)}]=1$ and then $(\pi_1^x)^{y(x)}$ is one of the following elements:

$$\pi_1^x,\ \pi_2^x,\ \mu^x,\ (\mu\pi_1)^x,\ (\mu\pi_2)^x,\ (\mu\pi_1\pi_2)^x .$$

On the other hand since $a(x)=\pi_1^x(\pi_1^x)^{y(x)}$, we get $a(x)=\pi_1\pi_2$ or $a(x)\sim\pi_1$. If $a(x)=\pi_1\pi_2$ then $a(x)^{b(x)}=\pi_1\pi_2$ which is impossible. Thus $a(x)\sim\pi_1$ and so $a(x)^{b(x)}=\pi_1$. $[b(x), \pi_1]=1$ yields $a(x)=\pi_1$. This implies that $[\pi_1^x, \pi_1]=1$. In both cases we proved that $[\pi_1^x, \pi_1]=1$ for all $x\in C_{G_i}(\pi_1\pi_2)$ and therefore $\pi_1^x\in C_{G_i}(\pi_1)\cap C_{G_i}(\pi_2)$. Again by Lemma 18 we get $\pi_1^x\in\langle\mu, \pi_1, \pi_2\rangle$. Since $\mu\sim\pi_1\sim\pi_2\sim\mu\pi_1\sim\mu\pi_1\pi_2\sim\mu\pi_2$ in $C_{G_i}(\pi_1\pi_2)$ we get $\langle\mu, \pi_1, \pi_2\rangle\lhd C_{G_i}(\pi_1\pi_2)$. The proof is complete.

LEMMA 20. $[\rho, B_i]=[\tau, B_i]=1$.

PROOF. Since $C_{G_i}(\langle\mu, \pi_1, \pi_2\rangle)=(\langle\mu, \pi_1, \pi_2\rangle\times B_i)\langle\mu\mu'\rangle$, B_i is a characteristic subgroup of $C_{G_i}(\langle\mu, \pi_1, \pi_2\rangle)$. Hence $[\rho, B_i]\subset B_i$ and $[\tau, B_i]\subset B_i$. Since $\rho^3=1$ we may assume that $\rho=bc$ where $b\in B_i$ and $c\in C_{G_i}(B_i)$. $\rho^3=b^3c^3=1$ implies that $b^3=c^3=1$. It is $\pi_3=\pi_3^\rho=\pi_3^b$, $\pi_3'=\pi_3'^\rho=\pi_3'^b$ and $\lambda=\lambda^\rho=\lambda^{bc}$. Thus we get $b\in C_{B_1}(\langle\pi_3, \pi_3'\rangle)$ or $b\in C_{B_2}(\langle\pi_3, \pi_3', \lambda\rangle)$. Since $b^3=1$, it follows from the structure of B_i that $b=1$ and so $\rho=c\in C_{G_i}(B_i)$. Similarly we get $\tau\in C_{G_i}(B_i)$.

LEMMA 21. $C_{G_i}(\pi_1\pi_2)=(\langle\mu, \pi_1, \pi_2\rangle\langle\pi_1', \pi_2', \tau, \rho\rangle\times B_i)\langle\mu\mu'\rangle$.

PROOF. Since $\pi_1\sim\pi_2\sim\mu\sim\mu\pi_1\sim\mu\pi_2\sim\mu\pi_1\pi_2\not\sim\pi_1\pi_2$ in G_i, $(N_{G_i}(\langle\mu, \pi_1, \pi_2\rangle): C_{G_i}(\langle\mu, \pi_1, \pi_2\rangle))$ divides $2^3\cdot 3$. It follows from the structure of H_i that $N_{G_i}(\langle\mu, \pi_1, \pi_2\rangle)=\langle\pi_1', \pi_2', \tau, \rho\rangle C_{G_i}(\langle\mu, \pi_1, \pi_2\rangle)$. Since $\langle\mu, \pi_1, \pi_2\rangle\lhd C_{G_i}(\pi_1\pi_2)$, the result follows from Lemmas 16 and 20.

§ 5. Final steps.

We are now in a position to apply Kondo's theorem [11]. By Lemmas 13, 21 and our assumption we get three isomorphisms

$$\theta_1:\ C_{G_3}(\alpha)\cong C_{A_{15}}(\hat{\alpha})$$

$$\theta_2:\ C_{G_3}(\pi_1\pi_2)\cong C_{A_{15}}(\hat{\pi}_1\hat{\pi}_2)$$

$$\theta_3:\ C_{G_3}(\pi_1)\cong C_{A_{15}}(\hat{\pi}_1)$$

defined as follows:

$$\pi_1\longrightarrow(1, 2)(3, 4) \qquad \pi_1'\longrightarrow(1, 3)(2, 4)$$

$$\pi_2\longrightarrow(5, 6)(7, 8) \qquad \pi_2'\longrightarrow(5, 7)(6, 8)$$

$$\pi_3\longrightarrow(9, 10)(11, 12) \qquad \pi_3'\longrightarrow(9, 11)(10, 12)$$

$$\mu \longrightarrow (1, 2)(5, 6) \qquad \sigma \longrightarrow (3, 5)(4, 6)$$
$$\mu' \longrightarrow (1, 2)(9, 10) \qquad \sigma' \longrightarrow (7, 9)(8, 10)$$
$$x_2 \longrightarrow (5, 7, 6) \qquad x_3 \longrightarrow (9, 11, 10)$$
$$\delta_3 \longrightarrow (5, 6)(8, 9) \qquad \zeta_3 \longrightarrow (5, 6)(12, 13)$$
$$\lambda \longrightarrow (9, 10)(13, 14) \qquad \nu \longrightarrow (13, 14, 15).$$

Put $\sigma''=(\mu\mu'\pi_2)^{\delta_3}$. Then σ'' is of order 2 and $\sigma'' \in C_{G_3}(\alpha)$.

LEMMA 22. *We may assume that* $\sigma'=\sigma''$.

PROOF. Since $\langle M, \pi_1', \pi_2', \pi_3' \rangle \subset C_{G_3}(\pi_1)$ and $\sigma'' \in C_{G_3}(\pi_1)$ it is easily verified that the action of σ'' on M by conjugation is the same as that of σ' and $(\pi_1'\sigma'')^2=(\pi_2'\sigma'')^3=(\sigma''\pi_3')^3=1$. On the other hand since $(\pi_1'\sigma)^3=(\sigma\pi_2')^3=1$, $[\sigma'', \mu]=[\sigma, \mu']=1$ implies that $[\sigma, \sigma'']=1$. Thus $\langle \pi_1', \sigma, \pi_2', \sigma'', \pi_3' \rangle \cong S_6$ and these elements form a set of canonical generators of S_6. Hence we may assume that $\sigma'=\sigma''$.

LEMMA 23. $\theta_1(\sigma')=\theta_3(\sigma')$.

PROOF. The result follows from Lemma 22.

Therefore we have proved that for all $1 \leqq i, j \leqq 3$, $\theta_i=\theta_j$ on $C_{G_3}(\alpha) \cap C_{G_3}(\pi_1\pi_2)$, $C_{G_3}(\alpha) \cap C_{G_3}(\pi_1)$ and $C_{G_3}(\pi_1\pi_2) \cap C_{G_3}(\pi_1)$. Thus the above correspondence satisfies the condition of a theorem of Kondo [**11**]. This implies that G_3 is isomorphic to A_{15}. Similarly G_1 is isomorphic to A_{12} or A_{13} and G_2 is isomorphic to A_{14}. The proof of our theorem is completed.

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