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Some classes of semi-groups of nonlinear transformations and their generators[1]

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1. Introduction. Suppose the linear space X is a Banach space under the norm $\| \|$, X_0 is a linear subspace of X, not necessarily $\| \|$-closed, K is a positive cone in X_0, and $\| \|_0$ is a semi-norm on X_0. We do not suppose that $\| \|_0$ is $\| \|$-continuous. Let $S=\{x \in K : \|x\|_0 \leqq 1\}$, and $\rho(x, y)=\|x-y\|$ for x, y in S. We suppose that (S, ρ) is a complete metric space and consider semi-groups of transformations from S into S. This covers a variety of settings. If $K=X_0$, and $\| \|_0$ is a norm on X_0, then (S, ρ) is a *Saks space*, see [8]. If $K=X_0=X$, and $\| \|_0=\| \|$, then S is just the closed unit ball in X. If $K=X_0=X$, and $\| \|_0 \equiv 0$, then $S=X$. Some examples are given in Section 3 to show why we take S in this generality. For instance, no simpler setting seems sufficient to cover the case of a semi-group of transformations giving the solutions of a quasi-linear partial differential equation.

The theory of semi-groups of linear transformations in a Banach space is developed quite thoroughly in the treatise [5] of Hille and Phillips. Semi-groups in topological vector spaces are treated by Yosida in [**10**] and Komatsu in [**6**]. It turns out that many points of the linear theory hold true for semi-groups of nonlinear transformations, which have been dealt with by Browder [**1**], Neuberger [7], Segal [9], and the author [3]. The purpose here is to continue the development of a non-linear analogue to the Hille and Phillips theory. We now state enough definitions to enable us to describe the results of Section 2.

After this paper was submitted, and shortly before publication, the papers [6], [8], and [**10**] of Kato, Kōmura, and Oharu, respectively, came to the author's attention. These, together with Neuberger's paper [**9**], all consider problems similar to those considered here. This is especially true of [**9**] and [**10**]. All of these papers, however, restrict themselves to the case $S=X$.

Let Φ denote the collection of all transformations from S into S. A *semi-group of transformations in* S means a function G from $[0, \infty)$ into Φ such

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that $G(0)=I$, the identity transformation on S, and $G(s)G(t)=G(s+t)$ for $s, t \geqq 0$. If G is such a semi-group, and x is in S, then G_x denotes the function $G(\cdot)x$ from $[0, \infty)$ into S. A semi-group G of transformations in S is said to be of *class* (C, S) if

$$\lim_{t\to 0} \|G(t)x-x\|=0 \tag{1}$$

for each x in S, and

$$\|G(t)y-G(t)x\| \leqq \|y-x\| \tag{2}$$

for $t \geqq 0$ and x, y in S. Observe that (1) and (2) imply that for each x in S, the function G_x is $\| \|$-continuous on $[0, \infty)$. This condition is called *strong continuity* in the linear case. All told, the class (C, S) is a nonlinear analogue to the class of strongly continuous semi-groups of (linear) contraction operators.

A semi-group G of class (C, S) is said to be of *class* $(C, S)'$ if for each x in some dense subset of S, the function G_x is a continuously differentiable function from $[0, \infty)$ into $(X, \| \|)$. In the linear analogue, $(C, S)'=(C, S)$, but an example is given in Section 3 to show that does not follow here. If G is a semi-group of class $(C, S)'$ then the *infinitesimal generator* of G is the transformation A whose domain $D(A)$ is the set of all x in S such that G_x is a continuously differentiable function from $[0, \infty)$ into $(X, \| \|)$, and whose value at such an x is given by $Ax=G_x'(0)$. Thus

$$A_x=\lim_{h\to 0} A_h x$$

for x in $D(A)$, where

$$A_h=(1/h)[G(h)-I]$$

for $h>0$.

We prove that two distinct semi-groups of class $(C, S)'$ cannot have the same infinitesimal generator. If A is the infinitesimal generator of a semi-group of class $(C, S)'$, and $\lambda>0$, then

$$\|(\lambda I-A)y-(\lambda I-A)x\| \geqq \lambda\|y-x\| \tag{3}$$

for all x, y in $D(A)$. The condition (3) is used to give a characterization for the transformations which are the infinitesimal generators of semi-groups of class $(C, S)'$.

We define two other class of semi-groups, (UC, S) and $(UC, S)'$, which are subclasses of (C, S) and $(C, S)'$, respectively; they are analogues of the class of *uniformly continuous* semi-groups of (linear) contraction operators. The infinitesimal generators of semi-groups of class $(UC, S)'$ are characterized. It is then shown that if G is a semi-group of class $(C, S)'$ and $A_h=(1/h)[G(h)-I]$ for $h>0$, then A_h is the infinitesimal generator of a semi-group G^h of class $(UC, S)'$, and that for each x in S,

$$G(t)x=\lim_{h\to 0} G^h(t)x\,;$$

where the convergence is uniform for t in bounded intervals. Also, the semi-groups G^h are constructable in an elementary way from their infinitesimal generators.

Section 3 contains examples and remarks. Section 4 deals with questions concerning the "resolvent" of the infinitesimal generator. One of the results we get is an exponential formula for the semi-group as was obtained by Oharu in [**10**] in the case $S=X$. Oharu replaces the condition that G_x be continuously differentiable for all x in some dense subset by the condition that G_x have a continuous right derivative for all x in some dense subset, but in general, a continuous vector valued function having a continuous right derivative is continuously differentiable, see [**13**, page 239]. The papers [**6**] and [**8**], which we saw just prior to publication of this paper, give interesting sufficient conditions on a transformation A in order that it be the infinitesimal generator of a semi-group of class $(C, X)'$. It seems likely that these conditions would have some sort of extension to the case $S\neq X$.

2. Basic properties of semi-groups of class (C, S).

2.1. THEOREM. *Suppose G is a semi-group of class $(C, S)'$, A is the infinitesimal generator of G, and x is in $D(A)$. Then $G(t)x$ is in $D(A)$ for $t\geqq 0$,*

$$G'_x(t)=AG(t)x$$

for $t\geqq 0$, and

$$\|G'_x(t)\|\leqq\|Ax\|$$

for $t\geqq 0$.

PROOF. If $t\geqq 0$, then

$$G_{G(t)x}(s)=G_x(s+t)\,,$$

for $s\geqq 0$, so that $G(t)x$ is in $D(A)$, and

$$G'_{G(t)x}(s)=G'_x(s+t)\,,$$

$$G'_x(t)=G'_{G(t)x}(0)=AG(t)x\,.$$

Also, for $t\geqq 0$ and $h>0$,

$$(1/h)[G_x(t+h)-G_x(t)]=(1/h)[G(t)G(h)x-G(t)x]\,,$$

$$(1/h)\|G_x(t+h)-G_x(t)\|\leqq(1/h)\|G(h)x-x\|\,,$$

$$\|G'_x(t)\|\leqq\|Ax\|\,.$$

2.2. LEMMA. *Suppose $[a, b]$ is a number interval, and f is a continuously differentiable function from $[a, b]$ into $(X, \|\ \|)$. Define g on $[a, b]$ by $g(t)=\|f(t)\|$. Then g is nonincreasing on $[a, b]$ if and only if for each $\lambda>0$,*

$$\|\lambda f(t)-f'(t)\| \geqq \lambda \|f(t)\| \tag{4}$$

for $a \leqq t \leqq b$.

PROOF. Suppose g is nonincreasing on $[a, b]$, $\lambda > 0$, and (4) fails to hold for some t in $[a, b]$. Then by continuity, it fails to hold on some closed interval $[c, d]$ with $a \leqq c < d \leqq b$. Thus,

$$\|\lambda f(t)-f'(t)\| < \lambda \|f(t)\| \leqq \lambda \|f(c)\|$$

for $c \leqq t \leqq d$,

$$\left\| \frac{d}{dt} e^{-\lambda t} f(t) \right\| < \lambda e^{-\lambda t} \|f(c)\|$$

for $c \leqq t \leqq d$, and

$$\|e^{-\lambda d} f(d) - e^{-\lambda c} f(c)\| < (e^{-\lambda c} - e^{-\lambda d}) \|f(c)\| ,$$

$$e^{-\lambda c} g(c) - e^{-\lambda d} g(d) < e^{-\lambda c} g(c) - e^{-\lambda d} g(c) ,$$

$$e^{-\lambda d} g(c) < e^{-\lambda d} g(d) ,$$

a contradiction.

Now suppose that (4) holds, that $a \leqq c < b$, and $0 < h < b-c$. Then,

$$\|\lambda f(c+h)-f'(c+h)\| - \lambda \|f(c)\| \geqq \lambda [g(c+h)-g(c)] ,$$

and

$$\|\lambda [f(c+h)-f(c)]-f'(c+h)\| \geqq \lambda [g(c+h)-g(c)]$$

for $\lambda > 0$. Substituting $\lambda = 1/h$, we obtain

$$\|(1/h)[f(c+h)-f(c)]-f'(c+h)\| \geqq (1/h)[g(c+h)-g(c)] .$$

Thus, $[D^+g](t) \leqq 0$ for $a \leqq t < b$, where D^+g indicates the upper derivative of g from the right. It follows that g is nonincreasing on $[a, b]$.

2.3. THEOREM. *If A is the infinitesimal generator of a semi-group of class $(C, S)'$, and $\lambda > 0$, then*

$$\|(\lambda I - A)y - (\lambda I - A)x\| \geqq \lambda \|y-x\|$$

for x and y in $D(A)$.

PROOF. Suppose A is the infinitesimal generator of the semi-group G of class $(C, S)'$, and that x and y are in $D(A)$. Let

$$f(t) = G(t)y - G(t)x, \ g(t) = \|f(t)\|$$

for $t \geqq 0$. It follows from the inequality (2) of Section 1 and the semi-group property of G that g is nonincreasing.

$$\|\lambda f(t)-f'(t)\| \geqq \lambda \|f(t)\| ,$$

$$\|\lambda [G(t)y-G(t)x]-[AG(t)y-AG(t)x]\| \geqq \lambda \|G(t)y-G(t)x\|$$

$$\|(\lambda I-A)G(t)y-(\lambda I-A)G(t)x\| \geqq \lambda \|G(t)y-G(t)x\|$$

for $t \geqq 0$. Setting $t=0$ yields the desired result.

2.4. THEOREM. *Two distinct semi-groups of class $(C, S)'$ cannot have the* *same infinitesimal generator.*

PROOF. Suppose that G and H are semi-groups of class $(C, S)'$, and that A is the infinitesimal generator of G and of H. If x is in $D(A)$,

$$f(t)=G(t)x-H(t)x$$

for $t \geqq 0$, and $\lambda>0$, then

$$\lambda f(t)-f'(t)=(\lambda I-A)G(t)x-(\lambda I-A)H(t)x$$

for $t \geqq 0$, so that

$$\|\lambda f(t)-f'(t)\| \geqq \lambda \|f(t)\|$$

for $t \geqq 0$, and

$$\|G(t)x-H(t)x\| \leqq \|G(0)x-H(0)x\|=0$$

for $t \geqq 0$.

If y is in S, x is in $D(A)$, and $t \geqq 0$, then

$$\|G(t)y-H(t)y\| \leqq \|G(t)y-G(t)x\|+$$
$$\|G(t)x-H(t)x\|+\|H(t)x-H(t)y\| \leqq 2\|y-x\|,$$

so that $G=H$.

2.5. THEOREM. *Suppose A is a transformation from a dense subset $D(A)$ of S into X. If A satisfies the properties* (i) *and* (ii) *given below, then A has an extension which is the infinitesimal generator of a semi-group of class* $(C, S)'$. *A is the infinitesimal generator of a semi-group of class $(C, S)'$ if and only if A is maximal with respect to the properties* (i) *and* (ii).

(i) *If $\lambda>0$, then*

$$\|(\lambda I-A)y-(\lambda I-A)x\| \geqq \lambda\|y-x\|$$

for x and y in $D(A)$.

(ii) *If x is in $D(A)$, then there is a function α from $[0, \infty)$ into S such that α is a continuously differentiable function from $[0, \infty)$ into $(X, \|\ \|)$, $\alpha(0)=x$, and $\alpha'(t)=A\alpha(t)$ for $t \geqq 0$.*

PROOF. We will prove that if A satisfies (i) and (ii) then A has an extension which is the infinitesimal generator of a semi-group of class $(C, S)'$. The rest of the conclusion will then follow readily.

First we show that if x is in $D(A)$, then there are not two distinct functions α satisfying the conditions of (ii). Indeed, if α and β are two such functions,

$$f(t)=\alpha(t)-\beta(t)$$

for $t \geqq 0$, and $\lambda>0$, then

$$\lambda f(t)-f'(t)=(\lambda I-A)\alpha(t)-(\lambda I-A)\beta(t),$$
$$\|\lambda f(t)-f'(t)\| \geqq \lambda\|f(t)\|$$

for $t \geqq 0$, so that $\|f(t)\|=0$ for $t \geqq 0$.

For each x in $D(A)$, let α_x denote the unique function having the properties given in (ii). For each $t \geqq 0$, define $H(t)$ on $D(A)$ by $H(t)x = \alpha_x(t)$. If x is in $D(A)$, $t \geqq 0$,

$$\varphi(s) = H(t+s)x$$

for $s \geqq 0$, and

$$\theta(s) = H(s)\ H(t)x$$

for $s \geqq 0$, then

$$\varphi(0) = \theta(0) = H(t)x\,,$$

$$\varphi'(s) = \alpha'_x(t+s) = A\alpha_x(t+s) = A\varphi(s)$$

for $s \geqq 0$, and

$$\theta'(s) = \alpha'_{H(t)x}(s) = A\alpha_{H(t)x}(s) = A\theta(s)$$

for $s \geqq 0$, so that $\varphi(s) = \theta(s)$ for $s \geqq 0$. Thus, H does have the semi-group property.

Also, for x and y in $D(A)$, an application of Lemma 2.2 with

$$f(t) = H(t)y - H(t)x$$

shows that

$$\|H(t)y - H(t)x\| \leqq \|y - x\|$$

for $t \geqq 0$. For each $t \geqq 0$, let $G(t)$ denote the unique continuous extension of $H(t)$ to S. Then G is a semi-group of class $(C, S)'$ whose infinitesimal generator is an extension of A.

2.6. DEFINITION. A semi-group G of class (C, S) is said to be of *class* (UC, S) if there is a real function η on $[0, \infty)$ such that $\eta(0+) = \eta(0) = 0$, and

$$\|[G(t)y - G(t)x] - [y - x]\| \leqq \eta(t)\|y - x\|$$

for x, y in S and $t \geqq 0$. A semi-group G of class $(UC, S)'$ is said to be of *class* $(UC, S)'$ if it is of class $(C, S)'$, the domain of its infinitesimal generator A is all of S, and A satisfies a *Lipschitz condition*; i.e., there is a number $L > 0$ such that

$$\|Ay - Ax\| \leqq L\|y - x\|$$

for x and y in S.

2.7. THEOREM. *Suppose A is a transformation from S into X. Then A is the infinitesimal generator of a semi-group of class $(UC, S)'$ if and only if A satisfies a Lipschitz condition as well as the conditions* (i) *and* (ii) *of Theorem* 2.5.

PROOF. If A is the infinitesimal generator of a semi-group of class $(UC, S)'$, then clearly A satisfies a Lipschitz condition and conditions (i) and (ii). If A satisfies conditions (i) and (ii), then A is the infinitesimal generator of a semi-group G of class $(C, S)'$. We need only show that the Lipschitz condition on A implies that G is of class (UC, S). This is true, because if x and y are in S, and $t \geqq 0$, then

$$[G(t)y - G(t)x] - [y - x] = \int_0^t [AG(s)y - AG(s)x]ds\,,$$

so that

$$\|[G(t)y-G(t)x]-[y-x]\| \leqq tL\|y-x\|,$$

where L is a Lipschitz constant for A.

2.8. THEOREM. *Suppose F is a transformation from S into S,*

$$\|Fy-Fx\| \leqq \|y-x\|$$

for x and y in S, $k>0$, and

$$A=k[F-I].$$

Then A is the infinitesimal generator of a semi-group of class $(UC, S)'$. In particular, A_h is such a generator if $A_h=(1/h)[G(h)-I]$ for some $h>0$ and some semi-group G of class (C, S).

PROOF. Obviously, A satisfies a Lipschitz condition with Lipschitz constant not exceeding $2k$. Also, A satisfies the condition (i) of Theorem 2.5, for if $\lambda>0$, then

$$\lambda I-A=(\lambda+k)I-kF,$$

and for x and y in S,

$$\begin{aligned}&\|(\lambda I-A)y-(\lambda I-A)x\| \\ &\quad \geqq (\lambda+k)\|y-x\|-k\|Fy-Fx\| \geqq \lambda\|y-x\|.\end{aligned}$$

We show that A satisfies the condition (ii) of Theorem 2.5. Take x in S, and consider the integral equation

$$g(t)=x+k\int_0^t e^{ks}F(e^{-ks}g(s))ds.$$

We show that the method of repeated integrations will apply in this setting. Suppose g is a $\|\ \|$-continuous function from $[0, \infty)$ into K such that $g(0)=x$, and

$$\|g(t)\|_0 \leqq e^{kt}$$

for $t \geqq 0$. Then the function $t \to e^{-kt}g(t)$ is a $\|\ \|$-continuous function from $[0, \infty)$ into S. Define γ on $[0, \infty)$ by

$$\gamma(t)=x+k\int_0^t e^{ks}F(e^{-ks}g(s))ds.$$

Then γ is a $\|\ \|$-continuous function from $[0, \infty)$ into the $\|\ \|$-closure of K. Also, the left Riemann sums for the integral are norm convergent, they all lie in K and thus in X_0, and if R is one of these left sums, then

$$\|x+kR\|_0 \leqq 1+[e^{kt}-1],$$

and

$$\|e^{-kt}[x+kR]\|_0 \leqq 1,$$

so that $e^{-kt}\gamma(t)$ is in S. That is, γ is a $\|\ \|$-continuous function from $[0, \infty)$ into K, $\gamma(0)=x$, and $\|\gamma(t)\|_0 \leqq e^{kt}$ for $t \geqq 0$. Thus, the method of repeated integrations applies to yield a $\| \|$-continuous function g from $[0, \infty)$ into K such that $e^{-kt}g(t)$ is in S for $t \geqq 0$, and

$$g(t) = x + k\int_0^t e^{ks}F(e^{-ks}g(s))ds$$

for $t \geqq 0$. If we let $\alpha(t) = e^{-kt}g(t)$ for $t \geqq 0$, then α is a function from $[0, \infty)$ into S such that α is a continuously differentiable function from $[0, \infty)$ into $(X, \| \|)$, $\alpha(0) = x$, and $\alpha'(t) = A\alpha(t)$ for $t \geqq 0$. By Theorem 2.7, A is the infinitesimal generator of a semi-group of class $(UC, S)'$.

2.9. THEOREM. *Suppose G is a semi-group of class $(C, S)'$, and for each $h > 0$, let $A_h = (1/h)[G(h) - I]$, and let G^h denote the class $(UC, S)'$ semi-group with infinitesimal generator A_h. Then if x is in S, and $t \geqq 0$, then*

$$G(t)x = \lim_{h \to 0} G^h(t)x\,,$$

and the convergence is uniform for t in bounded intervals.

PROOF. We denote the infinitesimal generator of G by A and begin by establishing some inequalities.

If $h > 0$, $\delta > 0$, and x is in S, then

$$\|[G^h(\delta) - I]x - \delta A_h x\| \leqq (2\delta^2/h)\|A_h x\|\,. \tag{5}$$

If $\delta > 0$, and x is in $D(A)$, then

$$\|[G(\delta) - I]x - \delta Ax\| \leqq \delta \sup_{0 \leqq t \leqq \delta} \|G'_x(t) - Ax\|\,. \tag{6}$$

If $h > 0$, and x is in $D(A)$, then

$$\|A_h G(t)x - AG(t)x\| \leqq \sup_{t \leqq s \leqq t+h} \|G'_x(s) - G'_x(t)\| \tag{7}$$

for all $t \geqq 0$.

To prove (5), observe that

$$[G^h(\delta) - I]x - \delta A_h x = \int_0^\delta [A_h G^h(t)x - A_h x]dt\,,$$

$$\|[G^h(\delta) - I]x - \delta A_h x\| \leqq (2\delta/h) \sup_{0 \leqq t \leqq \delta} \|G^h(t)x - x\| \leqq (2\delta^2/h)\|A_h x\|\,.$$

To prove (6), observe that

$$[G(\delta) - I]x - \delta Ax = \int_0^\delta [G'_x(t) - Ax]dt\,.$$

To prove (7), observe that

$$A_h G(t)x - AG(t)x = (1/h)\int_t^{t+h} [G'_x(s) - G'_x(t)]ds\,.$$

We now prove that if x is in $D(A)$, and $b > 0$, then G^h_x converges uniformly to G_x on $[0, b]$. Suppose $\varepsilon > 0$, and choose $r > 0$ so that

$$\|A_h x - Ax\| < \varepsilon/3b$$

for $0 < h < r$, and

$$\|G_x'(u)-G_x'(s)\| < \varepsilon/3b$$

for s, u in $[0, b]$ and $|s-u|<r$. Suppose t is in $(0, b]$, and n is a positive integer such that $(t/n)<r$. Let $\delta=t/n$. Then for $0<h<r$,

$$G^h(t)x-G(t)x=\sum_{j=1}^{n}[G^h(j\delta)G(t-j\delta)x-G^h(j\delta-\delta)G(t-j\delta+\delta)x]$$

$$=\sum_{j=1}^{n}[G^h(j\delta-\delta)G^h(\delta)G(t-j\delta)x-G^h(j\delta-\delta)G(t-j\delta+\delta)x].$$

Thus,

$$\|G^h(t)x-G(t)x\| \leqq \sum_{j=1}^{n}\|G^h(\delta)G(t-j\delta)x-G(t-j\delta+\delta)x\|.$$

But for $1\leqq j\leqq n$,

$$\begin{aligned} &G^h(\delta)G(t-j\delta)x-G(t-j\delta+\delta)x \\ &\quad=[G^h(\delta)-I]G(t-j\delta)x-[G(\delta)-I]G(t-j\delta)x \\ &\quad=[G^h(\delta)-I]G(t-j\delta)x-\delta A_hG(t-j\delta)x \\ &\qquad+\delta A_hG(t-j\delta)x-\delta AG(t-j\delta)x \\ &\qquad+\delta AG(t-j\delta)x-[G(\delta)-I]G(t-j\delta)x. \end{aligned}$$

Thus, by inequalities (5), (6), and (7),

$$\begin{aligned} &\|G^h(\delta)G(t-j\delta)x-G(t-j\delta+\delta)x\| \\ &\quad\leqq(2\delta^2/h)\|A_hG(t-j\delta)x\|+(\delta\varepsilon/3b)+(\delta\varepsilon/3b) \\ &\quad\leqq(2\delta\varepsilon/3b)+(2\delta^2/h)[\|Ax\|+(\varepsilon/3b)], \end{aligned}$$

so that

$$\|G^h(t)x-G(t)x\| \leqq (2\varepsilon/3)+(2t^2/nh)[\|Ax\|+(\varepsilon/3b)].$$

But this is true for all $n>t/r$, and therefore,

$$\|G^h(t)x-G(t)x\| \leqq 2\varepsilon/3<\varepsilon.$$

Now for y in S, $b>0$, and $\varepsilon>0$, choose x in $D(A)$ so that $\|x-y\|<\varepsilon/4$. Choose $r>0$ so that

$$\|G^h(t)x-G(t)x\|<\varepsilon/2$$

for $0<h<r$ and $0\leqq t\leqq b$. Then

$$\|G^h(t)y-G(t)y\|<\varepsilon$$

for $0<h<r$ and $0\leqq t\leqq b$.

3. Examples of semi-groups.

3.1. Example. Let X denote the space of all bounded continuous real valued functions on R, the set of real numbers, and let $\|\ \|$ denote the supremum norm on X. Let X_0 denote the collection of all functions in X which satisfy a Lipschitz condition, and for x in X_0, let $\|x\|_0$ denote the smallest Lipschitz constant for x. Suppose F is a real valued function on R, $L>0$, and $|F(s)-F(u)|\leqq L|s-u|$ for all s and u in R. For each $t>0$, let

$$B_t=\{x\in X_0:\|x\|_0<1/tL\}\,,$$

and let $B_0=X$.

We will construct a function G with domain $[0,\infty)$ such that for each $t\geqq 0$, $G(t)$ is a transformation from B_t into B_0. Moreover, for $t,u\geqq 0$, $G(t)B_{t+u}\subset B_u$, and $G(t)G(u)\supset G(t+u)$. Thus, G will be a semi-group of transformations, although not of class (C,S) for any S. There are two reasons for including this example. A little specialization yields the semi-group of Example 2.2, which is of class $(C,S)'$ for a certain set S. Also, this semi-group is an interesting one, and it points out the need for a more general theory. The most interesting property of this semi-group is as follows. If $u>0$, x is in B_u, x, F continuously differentiable, and z is defined on $R\times[0,u)$ by

$$z(s,t)=[G(t)x](s)\,,$$

then $z(s,0)=x(s)$ for all s in R, and

$$\partial z/\partial t=F(z)\partial z/\partial s \tag{8}$$

for all s in R and $t\geqq 0$.

For $t\geqq 0$ and x in B_t, we define $G(t)x$ to be the solution z of the functional equation

$$z(s)=x(s+tF(z(s)))\,. \tag{9}$$

To show that (9) has a solution, consider the mapping P from X into X defined by

$$[Pw](s)=x(s+tF(w(s)))\,.$$

For each z and w in X,

$$\|Pz-Pw\|\leqq t\|x\|_0L\|z-w\|\,,$$

and since x is in B_t, P is a contraction mapping in the complete metric space $(X,\|\ \|)$, and thus has a unique fixed point.

Notice we have

$$\|G(t)x\|_0\leqq\|x\|_0/(1-tL\|x\|_0)\,.$$

Thus $G(t)x$ is in B_u for x in B_{t+u}. Also, letting $z=G(t)x$, $w=G(u)z$, we have

$$\begin{aligned}z(s)&=x(s+tF(z(s)))\,,\\ w(s)&=z(s+uF(w(s)))=z(\sigma)\\ &=x(\sigma+tF(z(\sigma)))=x(s+(t+u)F(w(s)))\,,\end{aligned}$$

so that $w=G(t+u)x=G(t)G(u)x$. This establishes the semi-group property for G.

G also has several continuity properties.

$$\|G(t)x\| \leqq \|x\| ,$$

$$\|G(t)x-x\| \leqq \|x\|_0 t(L\|x\|+|F(0)|)$$

and

$$\|G(t)y-G(t)x\| \leqq \|y-x\|/(1-tL\|x\|_0) .$$

If x is in X, and x has a bounded uniformly continuous derivative, then x is in B_t for sufficiently small t, and

$$\|(1/t)[G(t)x-x]-F(x)x'\| \to 0$$

as $t \to 0$. Application of the implicit function theorem will establish the fact that G has the property, claimed with respect to the partial differential equation (8), and that if F has a uniformly continuous derivative, then G has an "infinitesimal generator", $Ax=F(x)x'$, whose domain includes the functions in X with bounded uniformly continuous derivatives. If x is such a function, then $G_x=G(\cdot)x$ has domain $[0, 1/L\|x\|_0)$, G_x is a continuously differentiable function from $[0, 1/L\|x\|_0)$ into $(X, \| \|)$, and $G'_x(t)=AG(t)x$ for $0 \leqq t < 1/L\|x\|_0$.

3.2. Example. Take X, X_0, and $\| \|_0$ as in Example 3.1, let K denote the set of all nonnegative, nondecreasing functions in X_0, and let

$$S=\{x \in K : \|x\|_0 \leqq 1\} .$$

We specialize on the preceding example by taking $F(s)=-s$. We still define the transformations $G(t)$ by means of the functional equation (9), but this time we obtain a semi-group G of class $(C, S)'$. Moreover, if x is a function in S which has a continuous derivative, and z is defined on $R \times [0, \infty)$ by $z(s, t) =[G(t)x](s)$, then $z(s, 0)=x(s)$ for all s in R, and

$$\partial z/\partial t=-z\ \partial z/\partial s \tag{10}$$

for all s in R, and all $t \geqq 0$. Also, the domain $D(A)$ of the infinitesimal generator, $Ax=-xx'$, of G includes all x in S which have a uniformly continuous derivative.

We use a different technique to solve the functional equation (9), because the mapping P considered in Example 2.1 is not a contraction mapping on $(X, \| \|)$ if $t>1$. Instead, take x in S, $t \geqq 0$, s in R, and define g on $[0, \infty)$ by

$$g(y)=x(s-ty) .$$

Then g is continuous and nonincreasing, and $g(0) \geqq 0$, so the equation $g(y)=y$ has exactly one solution. If for each s in R, we let $z(s)$ denote this solution, then

$$z(s)=x(s-tz(s))$$

for all s in R. For each $t \geqq 0$, we define $G(t)$ on S by defining $G(t)x$ to be the solution z of this functional equation.

Then each transformation $G(t)$ carries S into S. The fact that G is a semi-group of transformations follows as in Example 3.1. Some careful work with the appropriate inequalities shows that G is of class (C, S). The implicit function theorem shows that G is of class $(C, S)'$, has the infinitesimal generator $Ax = -xx'$, with $D(A)$ including all x in S having a uniformly continuous derivative, and that G has the property claimed with respect to the partial differential equation (10).

3.3. EXAMPLE. We omit the details here, but indicate how Examples 3.1 and 3.2 generalize to the partial differential equation

$$\partial z/\partial t = \sum p_i(s, z(s))\partial z/\partial s_i . \tag{11}$$

Take X to be the space of all bounded continuous real-valued functions on real Euclidean n-space E^n, and let $\| \ \|$ denote the supremum norm on X. Let X_0 denote the subspace of X consisting of those functions in X which satisfy a Lipschitz condition, and for each x in X_0, let $\|x\|_0$ denote the smallest Lipschitz constant for x. For each $i = 1, 2, \cdots, n$, take p_i to be a real valued function on $E^n \times R$ which satisfies a Lipschitz condition. Let P denote the vector function

$$P = (p_1, p_2, \cdots, p_n) .$$

For each s in E^n and z in R, let $J(s, z)$ denote the unique continuous function y from $[0, \infty)$ into E^n such that $y(0) = s$, and

$$y'(t) = P(y(t), z)$$

for $t \geqq 0$. For each s in E^n, z in R, and $t \geqq 0$, let

$$F(s, t, z) = [J(s, z)](t) .$$

For suitable $t \geqq 0$ and x in X_0, let $G(t)x$ denote the solution z of the functional equation

$$z(s) = x(F(s, t, z(s))) .$$

The details go through as in Example 3.1. In general, G is a semi-group of transformations giving solutions to the partial differential equation (11). By choosing suitable functions p_i and restricting the transformations $G(t)$ to a suitable set S, one obtains a semi-group of class $(C, S)'$.

3.4. EXAMPLE. In the linear analogue, $(C, S)' = (C, S)$, but this example shows that is not true here. Let $X = X_0 = K = R$, the set of all real numbers. For x in X, let $\|x\| = |x|$, and let $\|x\|_0 = 0$. Thus, $S = X$. Let F denote a strictly increasing continuous function from R onto R which is concave downward, but which is differentiable on no open interval. For each $t \geqq 0$, define the transformation $G(t)$ on S by

$$G(t)x=F(t+F^{-1}(x)) .$$

Then G is of class (C, S) but not of class $(C, S)'$.

3.5. REMARKS. The way in which the semi-group G of Example 3.4 fails to be of class $(C, S)'$ is not too bad. For each x in S, G_x is differentiable except at a countable set, G'_x has right and left hand limits everywhere and $(0, \infty)$, and G'_x is continuous wherever it is defined.

The semi-group of Example 3.1 (and its generalization in Example 3.2) is much worse in a way, because the transformations $G(t)$ do not all have the same domain. It would seem desirable to have a theory of semi-groups of transformations which would cover Examples 3.1, 3.3, and 3.4.

An interesting question in connection with the characterization theorem, Theorem 2.5, is whether every infinitesimal generator of a class $(C, S)'$ semi-group has a closed graph. It is easy to see that every such generator A has the following closure property. If $\{x_n\}\subset D(A)$, $x_n\rightarrow x$, $Ax_n\rightarrow y$, and there is positive number δ such that $\{AG_{x_n}\}$ is quasi-uniformly convergent (see [**4**, p. 268]) on $[0, \delta]$, then x is in $D(A)$, and $Ax=y$. To prove this statement, one needs merely to remember that by Theorem 2.1, $\|AG_{x_n}(t)\|\leqq\|Ax_n\|$, and apply the Lebesgue dominated convergence theorem to get

$$G(t)x-x=\int_0^t f(s)ds$$

for $0\leqq t\leqq\delta$, where f is the (continuous) pointwise limit of $\{AG_{x_n}\}$. Perhaps there is some additional (and unobjectionable) restrictions that could be placed on the class $(C, S)'$ to insure that the generators would have closed graphs.

It was mentioned in the introduction that the setting was chosen so as to be sufficiently general to cover Example 3.2. On the other hand, the semi-group G of example 3.2 has many properties that were not used in Section 2, as it was felt that these properties were too special to fit in with what is supposed to a collection of basic facts about a general class of semi-groups. We will list some of these properties here, however, because it seems that they might offer a clue for obtaining a theory which would still be fairly general and yet have more and stronger results than those in Section 2. The papers [**1**] and [**9**] of Browder and Segal certainly have a great deal to offer in that direction.

In Komatsu's paper [**6**], there are two topologies which play a role. In Section 2, the semi-norm $\| \|_0$ served only in determining the domain of the transformations $G(t)$, but the semi-group G of Example 3.2 has several continuity properties with respect to $\| \|_0$. For instance,

$$\|G(t)x\|_0\leqq\|x\|_0$$

for $t \geqq 0$ and x in S, and

$$\lim_{t \to 0} \|G(t)x - x\|_0 = 0 \tag{12}$$

if x is in S and has a uniformly continuous derivative. Also, for each x in S, there is a sequence $\{x_n\}$ in $D(A)$ such that $\{x_n\}$ converges to x in the norm $\| \|$, and $\{\|x_n\|_0\}$ converges upward to $\|x\|_0$.

In [3], the author studied semi-groups of transformations where each transformation had a Fréchet derivative (see [2, Chapter VIII or [5, pp. 109-115]). The transformations $G(t)$ of Example 3.2 are not Fréchet differentiable, but there are linear transformations which are tangent in a sense. Suppose $t \geqq 0$, and x is in S and has a uniformly continuous derivative. Let $z = G(t)x$, and define the linear transformation T on X by

$$[T\varphi](s) = \varphi(s - tz(s))/[1 + tx'(s - tz(s))].$$

Then T is continuous from $(X, \| \|)$ into $(X, \| \|)$, and

$$(\|G(t)y - G(t)x - T(y-x)\| / \|y - x\|) \to 0$$

as $\|y-x\|$, $\|y-x\|_0 \to 0$. In view of the continuity property (12) of the preceding paragraph, we get

$$AG(t)x = TAx,$$

or, denoting T by $G(t)'x$,

$$AG(t)x = [G(t)'x]Ax.$$

4. Generators and resolvents.

4.1. DEFINITION. A semi-group G of class $(C, S)'$ is said to be of class $(RC, S)'$ if the range $R(I - \delta A)$ of $I - \delta A$ includes all of S for each $\delta > 0$, where A is the infinitesimal generator of G.

4.2. DEFINITION. If A is the infinitesimal generator of a semi-group of class $(C, S)'$, $\delta > 0$, and $S \subset R(I - \delta A)$, then $J(\delta, A)$ denotes the restriction to S of the inverse of $I - \delta A$. By Theorem 2.5, $I - \delta A$ is one-to-one, and

$$\|J(\delta, A)x - J(\delta, A)y\| \leqq \|x - y\|$$

for x and y in S.

4.3. THEOREM. *If G is a class $(UC, S)'$ semi-group of the type of Theorem 2.8, then G is of class $(RC, S)'$. In particular, the semi-groups G^h of Theorem 2.9 are of class $(RC, S)'$.*

PROOF. Let the generator of G be

$$A = k[F - I],$$

where $\|Fy - Fx\| \leqq \|x - y\|$ for x, y in S, and $k > 0$. Suppose $\delta > 0$ and y is in S. Then

$$x \to (1+\delta k)^{-1}y+\delta k(1+\delta k)^{-1}Fx$$

is a contraction mapping from S into S and thus has a unique fixed point x_0. But $(I-\delta A)x_0=y$.

4.4. THEOREM. *If G is of class $(RC, S)'$ with infinitesimal generator A, then*

$$x=\lim_{\delta\to 0} J(\delta, A)x$$

for all x in S.

PROOF. If $x \in D(A)$, then

$$\|J(\delta, A)x-x\| \leqq \|x-(I-\delta A)x\|=\delta\|Ax\| .$$

The functions $J(\delta, A)$ are equicontinuous, and $D(A)$ is dense in S.

4.5. THEOREM. *Suppose G is a semi-group of class $(RC, S)'$, and let A denote the infinitesimal generator of G. Then for each x in S and $t \geqq 0$,*

$$G(t)x=\lim_{n\to\infty} J(t/n, A)^n x ,$$

and the convergence is uniform for t in bounded intervals.

PROOF. If $x \in D(A)$ and $\delta>0$, then

$$\begin{aligned}\|J(\delta, A)x-G(\delta)x\| &\leqq \|x-G(\delta)x+\delta AG(\delta)x\| \\ &=\|\delta AG(\delta)x-\delta A_\delta x\| \leqq \delta\omega(G_x', [0, \delta]) \\ &=\delta \sup\{|G_x'(s)-G_x'(t)|\}(s, t \in [0, \delta]) ,\end{aligned}$$

where

$$A_\delta=(1/\delta)[G(\delta)-I] .$$

Take $b>0$, $x \in D(A)$, $t \in [0, b]$, n a positive integer, and $\delta=t/n$. Then

$$\begin{aligned}J(t/n, A)^n x-G(t)x &=J(\delta, A)^n x-G(n\delta)x \\ &=\sum_0^{n-1}[J(\delta, A)^{j+1}G(n\delta-j\delta-\delta)x-J(\delta, A)^j G(n\delta-j\delta)x] .\end{aligned}$$

Thus

$$\begin{aligned}&\|J(t/n, A)^n x-G(t)\| \\ &\leqq \sum_0^{n-1}\|[J(\delta, A)-G(\delta)]G(n\delta-j\delta-\delta)x\| \\ &\leqq \sum_0^{n-1}\delta\omega(G_x', [n\delta-j\delta-\delta, n\delta-j\delta]) \\ &\leqq t\omega(G_x', [0, b], t/n) \\ &=t\sup\{|G_x'(p)-G_x'(q)|\}(p, q \in [0, b], |p-q| \leqq t/n) .\end{aligned}$$

Since $D(A)$ is dense in S and the transformations $J(t/n, A)^n$ are equicontinuous, we have the desired result.

4.6. COROLLARY. *If G is of class $(C, S)'$, and we define* $A_h=(1/h)[G(h)-I]$

for $h>0$, *then for each* x *in* S *and* $t>0$,

$$G(t)x=\lim_{h\to 0}\lim_{n\to\infty}J(t/n,\ A_h)^n x\,,$$

and the convergence is uniform for t *in bounded intervals.*

PROOF. Apply Theorems 4.3, 4.5, and 2.9. This theorem resembles Neuberger's theorem in [**9**].

4.7. THEOREM. *If* $x\in D(A)$, *where* A *is the infinitesimal generator of a class* $(RC, S)'$ *semi-group* G, *then*

$$Ax=\lim_{\delta\to 0}(1/\delta)[J(\delta,\ A)x-x]=\lim_{\delta\to 0}AJ(\delta,\ A)x\,.$$

PROOF. By Theorem 4.5,

$$\|(1/\delta)[J(\delta,\ A)x-x]-(1/\delta)[G(\delta)x-x]\|\leqq\omega(G_x',\ [0,\ \delta])\,.$$

Also,

$$\|(1/\delta)[G(\delta)x-x]-Ax\|\leqq\omega(G_x',\ [0,\ \delta])\,.$$

4.8. THEOREM. *Suppose* G *is a semi-group of class* $(RC, S)'$ *with infinitesimal generator* A. *For each* $\varepsilon>0$, *let* $A^\varepsilon=AJ(\varepsilon, A)$, *and let* $G^{[\varepsilon]}$ *denote the class* $(UC, S)'$ *semi-group with infinitesimal generator* A^ε *(see Theorem 2.8). Then for each* x *in* S *and* $t>0$,

$$G(t)x=\lim_{\varepsilon\to 0}G^{[\varepsilon]}(t)x\,,$$

and the convergence is uniform for t *in bounded intervals.*

PROOF. Let $\varepsilon>0$, $h>0$, $0<t\leqq b$, $x\in D(A)$, n be a positive integer, and $\delta=t/n$. Then

$$\begin{aligned}\|G^{[\varepsilon]}(t)x-G(t)x\|\leqq&\ \|G^{[\varepsilon]}(t)x-J(t/n,\ A^\varepsilon)^n x\|\\&+\|J(t/n,\ A^\varepsilon)^n x-G(t)x\|\,.\end{aligned}$$

The first term approaches zero as $n\to\infty$, and the convergence is uniform for $0<t\leqq b$. Also

$$\begin{aligned}&\|J(\delta,\ A^\varepsilon)^n x-G(n\delta)x\|\\&\leqq\sum_0^{n-1}\|J(\delta,\ A^\varepsilon)^{j+1}G(n\delta-j\delta-\delta)x-J(\delta,\ A^\varepsilon)^j G(n\delta-j\delta)x\|\\&\leqq\sum_0^{n-1}\|[J(\delta,\ A^\varepsilon)-G(\delta)]G(n\delta-j\delta-\delta)x\|\\&\leqq\sum_0^{n-1}\|[I-G(\delta)+\delta A^\varepsilon G(\delta)]G(n\delta-j\delta-\delta)x\|\\&=\sum_0^{n-1}\delta\|[A^\varepsilon G(\delta)-A_\delta]G(n\delta-j\delta-\delta)x\|\end{aligned}$$

$$\leqq \sum_{0}^{n-1} \delta \| [A_\delta - AG(\delta)] G(n\delta - j\delta - \delta) x \|$$

$$+ \sum_{0}^{n-1} \delta \| [AG(\delta) - A^\varepsilon G(\delta)] G(n\delta - j\delta - \delta) x \|$$

$$\leqq t\omega(G'_x, [0, b], t/n) + t\omega(G'_x, [0, b], \varepsilon) .$$

Therefore

$$\| G^{[\varepsilon]}(t) x - G(t) x \| \leqq t\omega(G'_x, [0, b], \varepsilon) .$$

The rest follows from the denseness of $D(A)$ and the equicontinuity of the transformations $G^{[\varepsilon]}(t)$.

4.10. EXAMPLE. Here we show that the semi-group G of example 3.2 is of class $(RC, S)'$. Let $y \in S$ and $\delta > 0$. If y has a zero, let α denote the greatest zero of y. If y is nonvanishing, then let $\alpha = -\infty$. Let Ω denote the closure of the region between the curves y and $y/(1+\delta)$. Notice that if $x + \delta x x' = y$ on some open interval J, and $t \in J$, then

$$0 \leqq x'(t) \leqq 1$$

if and only if

$$y(t)/(1+\delta) \leqq x(t) \leqq y(t) .$$

If $t_0 > \alpha$, and

$$y(t_0)/(1+\delta) < x_0 < y(t_0) ,$$

then the usual existence and uniqueness theorem for ordinary differential equations asserts the existence of unique local solution x of

$$x + \delta x x' = y, \qquad x(t_0) = x_0 .$$

Elementary considerations show that such a local solution can be continued indefinitely to the right without leaving the region Ω. One can make no such assertion about continuation to the left, however.

Consider a sequence $\{(t_n, x_n)\}$ such that

$$\alpha < t_n, \ y(t_n)/(1+\delta) < x_n < y(t_n) ,$$

and $t_n \to \alpha$. Let f_n denote the unique function x defined on $[t_n, \infty)$ such that

$$x + \delta x x' = y, \qquad x(t_n) = x_n .$$

It follows that

$$y(t)/(1+\delta) \leqq f_n(t) \leqq y(t) ,$$

and

$$0 \leqq f'_n(t) \leqq 1$$

for $t \geqq t_n$. By uniform boundedness and equicontinuity, it follows that there is a subsequence $\{f_{m_n}\}$ of $\{f_n\}$ and a function f defined on (α, ∞) such that $\{f_{m_n}\}$ converges uniformly to f on each interval $[a, b]$ such that $\alpha < a < b$.

Moreover, f is nonnegative and nondecreasing, and

$$f+\delta ff'=y$$

on (α, ∞), so that $0\leqq f'(t)\leqq 1$ for $t>\alpha$.

If $\alpha=-\infty$, then we have constructed a solution of

$$x-\delta Ax=y\,,$$

where A is the infinitesimal generator of the semi-group G of Example 3.2. If $\alpha>-\infty$, then let $x(t)=f(t)$ for $t>\alpha$, and $x(t)=0$ for $t\leqq\alpha$. Then we still have $x-\delta Ax=y$, for letting

$$z(t, s)=[G(t)x](s)\,,$$

we have

$$z(t, s)=x(s-tz(t, s))\,,$$

so that $z(t, s)=0$ for $\delta\leqq\alpha$.

For $\delta>\alpha$

$$(1/t)[z(t, s)-x(s)]=-x'(\theta)z(t, s)\,,$$

where $s-tz(t, s)\leqq\theta\leqq s$. This may be seen to converge uniformly to $-x(s)x'(s)$ on $s>\alpha$. Thus $x\in D(A)$, and

$$[Ax](s)=\begin{cases} 0 & \text{for } s\leqq\alpha \\ -x(s)x'(s) & \text{for } s>\alpha\,. \end{cases}$$

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