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On a certain invariant of the groups of type E_6 and E_7

Dedicated to Professor S. Iyanaga on his 60th birthday

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In my recent paper [**9**], I have introduced an invariant $\gamma(G)$ for a connected semi-simple algebraic group G, which generalizes the classical invariants of Hasse and of Minkowski-Hasse, and have shown that, for a classical simple group G, $\gamma(G)$ can actually be determined explicitly in terms of these classical invariants[1]. For exceptional groups, however, I gave only a very brief indication for the case where the ground field is a local field or an algebraic number field ([**9**], 250-251). The purpose of this note[2] is to give a more comprehensive account for a more general case, establishing a principle which enables us to reduce the determination of $\gamma(G)$ for an exceptional group G to that for a suitably chosen *classical* subgroup G' of G defined over the same ground field. The existence of such a subgroup G' will be ascertained for the groups of type E_6 and E_7 constructed recently by Tits [**12**].

1. Throughout this paper, k is a field of characteristic zero, (though it seems likely that most of our results remain true over any perfect field of characteristic different from 2 and 3). $\bar{k}$ is a fixed algebraic closure of k and $\mathfrak{g}=\mathrm{Gal}(\bar{k}/k)$ is the Galois group of $\bar{k}/k$ operating on $\bar{k}$ from the right. For an algebraic group G defined over k, we write the Galois cohomology set or group $H^i(\mathfrak{g}, G_{\bar{k}})$ $(i=1, 2)$ as $H^i(k, G)$. $\mathbf{E}_n=\{\zeta_n\}$ is the group of all n-th roots of unity contained in $\bar{k}$. In principle, we follow the notation in [**9**].

Let G_1 be an algebraic group defined over k. By an *inner k-form* of G_1,

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1) Taking this opportunity, I would like to correct some of the misprints in the relevant part of [**9**]. On page 246, line 10, for " $\mathfrak{K}^{\Sigma m_i}$ " read " $\mathfrak{K}^{\Sigma i m_i}$ "; similar corrections are also necessary for the formulas (28), (28′) in page 250. On page 249, line 9, for " $k(\sqrt{(-1)^{1/2\,nr}}\det(S))$ " read " $k(\sqrt{(-1)^{1/2\,nr}\det(S)})$ "

2) By a communication from Professor Tits, the author learnt after completion of the paper that similar topics had also been treated by him in a series of lectures delivered at Yale University in the winter of 1967.

Added in proof: By a communication with Tits, it appeared that in 8 the relation $\mathfrak{C}_2\sim\mathfrak{D}'$ and so (11) is always true without any assumption.

we understand a pair (G, f) formed of an algebraic group G defined over k and a $\bar{k}$-isomorphism f of G onto G_1 such that $f^{\sigma}\circ f^{-1}$ is an inner automorphism of G_1 for every $\sigma\in\mathfrak{G}$. To such a pair (G, f), we associate an element $\gamma(G, f)$ in $H^2(k, Z_1)$, where Z_1 is the center of G_1, as follows. Put

$$f^{\sigma}\circ f^{-1}=I_{g_{\sigma}} \quad \text{and} \quad \delta(g_{\sigma})=g_{\sigma}^{\tau}g_{\tau}g_{\sigma\tau}^{-1}=c_{\sigma,\tau},$$

where $g_{\sigma}\in(G_1)_{\bar{k}}$ and $I_{g_{\sigma}}$ denotes the inner automorphism of G_1 defined by $I_{g_{\sigma}}(g)=g_{\sigma}gg_{\sigma}^{-1}$ for $g\in G_1$. Then it is clear that $(c_{\sigma,\tau})$ is a 2-cocycle of $\mathfrak{G}$ in $(Z_1)_{\bar{k}}$, whose cohomology class is uniquely determined, independently of the choice of the 1-cochain (g_{σ}). (We always take it implicitly that all cochains we consider are $\bar{k}$-rational and continuous in the sense of Krull topology on $\mathfrak{G}$.) We denote the cohomology class of $(c_{\sigma,\tau})$ by $\gamma_k(G, f)$ or simply by $\gamma(G, f)$ whenever k is tacitly fixed.

Two inner k-forms (G, f) and (G', f') of G_1 are said to be *i-equivalent* if there exists a k-isomorphism φ of G onto G' such that $f'\circ\varphi\circ f^{-1}$ is an inner automorphism of G_1. It is immediate that the cohomology class $\gamma(G, f)$ depends only on the i-equivalenc class of (G, f).

In the case where G_1 is a connected reductive algebraic group, the number of i-equivalence classes of inner k-forms of G_1 contained in a k-isomorphism class of k-forms of G_1 (in the ordinary sense) is finite. Moreover, it is known ([**9**], p. 242) that, for any connected semi-simple algebraic group G defined over k, there exists an inner k-form (G_1, f^{-1}) of G such that G_1 is of Steinberg type, and the i-equivalence class of such (G_1, f^{-1}) is uniquely determined by G. Hence, in this case, we define the inveriant $\gamma(G)$ by setting $\gamma(G)=\gamma(G_1, f^{-1})\in H^2(k, Z)$, Z denoting the center of G. If one denotes by f^* the isomorphism of $H^2(k, Z)$ onto $H^2(k, Z_1)$ induced in a natural way by f, then one has

$$\gamma(G)=f^{*-1}(\gamma(G, f)). \tag{1}$$

(Note that f induces on $Z_{\bar{k}}$ a $\mathfrak{G}$-isomorphism $Z_{\bar{k}}\to(Z_1)_{\bar{k}}$.)

EXAMPLE. $G=SL(m, \mathfrak{K}_r)$, where $\mathfrak{K}_r$ is a normal division algebra of degree r over k. Let f be a $\bar{k}$-isomorphism of G onto $G_1=SL(mr)$ determined by the (unique) irreducible representation of $\mathfrak{K}_r$ (as an associative algebra). Then (G_1, f^{-1}) is an inner k-form of G as described above, and through the natural identification $Z\cong Z_1=\mathbf{E}_{mr}$ (induced by f), one has $\gamma(G)=c(\mathfrak{K}_r)\in H^2(k, \mathbf{E}_{mr})$ (where $c(\mathfrak{K}_r)$ denotes the "Hasse invariant" of $\mathfrak{K}_r$).

2. The following lemma is fundamental.

LEMMA 1. *Let G_1 and G_1' be algebraic groups defined over k, and let φ_1 be a k-morphism of G_1' into G_1. Suppose there is a k-closed subgroup G_1'' of G_1 such that, denoting by Z_1, Z_1', Z_1'' the center of G_1, G_1', G_1'', respectively, one has*

(i) $$Z_{G_1}(\varphi_1(G_1'))=\varphi_1(Z_1')\cdot G_1'',$$

$Z_{G_1}(\cdots)$ denoting the centralizer of $\cdots$ in G_1;

(ii) $$\varphi_1(Z_1') = Z_1 \times Z_1'' \quad \textit{(direct product)};$$

(iii) *the natural map $H^1(k, G_1''/Z_1'') \xrightarrow{\Delta} H^2(k, Z_1'')$ is bejective.*

Let further (G', f') be an inner k-form of G_1'. Then:

1) *There exist an inner k-form (G, f) of G_1 and a k-morphism φ of G' into G such that one has $f \circ \varphi = \varphi_1 \circ f'$.*

2) *If $(\bar{G}, \bar{f}, \bar{\varphi})$ is another triple satisfying the same condition as (G, f, φ), then there is a $\bar{k}$-isomorphism ψ of G onto $\bar{G}$ such that $\bar{\varphi} = \psi \circ \varphi$, $\bar{f} \circ \psi \circ f^{-1}$ is an inner automorphism of G_1, and $\psi^\sigma \circ \psi^{-1} = I_{d_\sigma''}$ where (d_σ'') is a 1-cocycle of $\mathfrak{G}$ in $\bar{f}^{-1}(Z_1'')_{\bar{k}}$.*

3) *For any inner k-form (G, f) of G_1 satisfying the condition in 1), $\gamma(G, f)$ coincides with the Z_1-part of $\varphi_1^*(\gamma(G', f'))$ in the direct decomposition (ii), where φ_1^* denotes the natural homomorphism of $H^2(k, Z_1')$ into $H^2(k, \varphi_1(Z_1'))$ induced by φ_1.*

PROOF. 1) Put $f'^\sigma \circ f'^{-1} = I_{g_\sigma'}$, $g_\sigma' \in (G_1')_{\bar{k}}$, and $\delta(g_\sigma') = c_{\sigma,\tau}' \in Z_1'$. By (ii) one has

$$\varphi_1(c_{\sigma,\tau}') = c_{\sigma,\tau} \cdot c_{\sigma,\tau}''^{-1}, \tag{2}$$

where $(c_{\sigma,\tau})$ and $(c_{\sigma,\tau}'')$ are (uniquely determined) 2-cocycles of $\mathfrak{G}$ in Z_1 and Z_1'', respectively. By (iii) (the surjectivity), there exists $g_\sigma'' \in (G_1'')_{\bar{k}}$ such that $\delta(g_\sigma'') = c_{\sigma,\tau}''$. Put

$$g_\sigma = \varphi_1(g_\sigma') \cdot g_\sigma'';$$

then by (i) one has $\delta(g_\sigma) = c_{\sigma,\tau}$. Hence there is an inner k-form (G, f) of G_1 such that $f^\sigma \circ f^{-1} = I_{g_\sigma}$. Put $\varphi = f^{-1} \circ \varphi_1 \circ f'$. Then, for every $\sigma \in \mathfrak{G}$, one has

$$\varphi^\sigma = f^{-\sigma} \circ \varphi_1 \circ f'^\sigma = f^{-1} \circ I_{g_\sigma}^{-1} \circ \varphi_1 \circ I_{g_\sigma'} \circ f' = f^{-1} \circ I_{g_\sigma^{-1} \cdot \varphi_1(g_\sigma')} \circ \varphi_1 \circ f'.$$

Since by (i) one has $g_\sigma^{-1} \cdot \varphi_1(g_\sigma') \in G_1'' \subset Z_{G_1}(\varphi_1(G_1'))$, one has $\varphi^\sigma = \varphi$, i.e. φ is defined over k. (Note that the converse of this is also true).

2) Let $(\bar{G}, \bar{f}, \bar{\varphi})$ be another triple satisfying the conditions stated in 1), and put $\bar{f}^\sigma \circ \bar{f}^{-1} = I_{\bar{g}_\sigma}$, $\delta(\bar{g}_\sigma) = \bar{c}_{\sigma,\tau}$ with $\bar{g}_\sigma \in (G_1)_{\bar{k}}$, $\bar{c}_{\sigma,\tau} \in Z_1$. As we have just noted above, $\bar{\varphi}^\sigma = \bar{\varphi}$ $(\sigma \in \mathfrak{G})$ implies that $\bar{g}_\sigma^{-1} \cdot \varphi_1(g_\sigma') \in Z_{G_1}(\varphi_1(G_1'))$. Hence, by (i), one may put

$$\bar{g}_\sigma^{-1} \cdot \varphi_1(g_\sigma') = \varphi_1(c_\sigma') \cdot \bar{g}_\sigma''^{-1} \quad \text{or} \quad \bar{g}_\sigma = \varphi_1(c_\sigma'^{-1} g_\sigma') \cdot \bar{g}_\sigma''$$

with $c_\sigma' \in (Z_1')_{\bar{k}}$ and $\bar{g}_\sigma'' \in (G_1'')_{\bar{k}}$. Then one has

$$\bar{c}_{\sigma,\tau} = \delta(\varphi_1(c_\sigma'))^{-1} \cdot \varphi_1(c_{\sigma,\tau}') \cdot \delta(\bar{g}_\sigma''),$$

which, by (i), (ii), implies that $\delta(\bar{g}_\sigma'') \in G_1'' \cap \varphi_1(Z_1') = Z_1''$. Writing $\varphi_1(c_\sigma') = c_\sigma \cdot c_\sigma''^{-1}$ with $c_\sigma \in Z_1$ and $c_\sigma'' \in Z_1''$ and comparing the Z-parts and Z''-parts in the above equality, one obtains in view of (2)

$$\text{(2a)} \qquad \begin{cases} \bar{c}_{\sigma,\tau} = \delta(c_\sigma)^{-1} c_{\sigma,\tau}\,, \\ \delta(\bar{g}''_\sigma) = \delta(c''_\sigma)^{-1} \cdot c''_{\sigma,\tau} = \delta(c''^{-1}_\sigma g''_\sigma)\,. \end{cases}$$

By (iii) (the injectivity), the second equality of (2a) implies that there is $h \in (G''_1)_{\bar{k}}$ and a 1-cocycle (a''_σ) of $\mathfrak{G}$ in $(Z''_1)_{\bar{k}}$ such that one has

$$\bar{g}''_\sigma = a''_\sigma c''^{-1}_\sigma h^\sigma g''_\sigma h^{-1}\,;$$

then one has also $\bar{g}_\sigma = c_\sigma^{-1} h^\sigma g_\sigma h^{-1} \cdot a''_\sigma$. Now put $\psi = \bar{f}^{-1} \circ I_h \circ f$. Then, since $h \in Z_{G_1}(\varphi_1(G'_1))$, one has

$$\psi \circ \varphi = \bar{f}^{-1} \circ I_h \circ f \circ \varphi = \bar{f}^{-1} \circ I_h \circ \varphi_1 \circ f' = \bar{f}^{-1} \circ \varphi_1 \circ f' = \bar{\varphi}$$

and, for every $\sigma \in \mathfrak{G}$,

$$\begin{aligned} \psi^\sigma &= \bar{f}^{-\sigma} \circ I_{h^\sigma} \circ f^\sigma = \bar{f}^{-1} \circ I_{\bar{g}_\sigma}^{-1} \circ I_{h^\sigma} \circ I_{g_\sigma} \circ f = \bar{f}^{-1} \circ I_{a''^{-1}_\sigma h} \circ f \\ &= I_{\bar{f}^{-1}(a''^{-1}_\sigma)} \circ \psi\,, \end{aligned}$$

i.e., one has $\psi^\sigma \circ \psi^{-1} = I_{d''_\sigma}$ with $d''_\sigma = \bar{f}^{-1}(a''^{-1}_\sigma) \in \bar{f}^{-1}(Z''_1)$.

3) is clear from the definitions and (2), (2a), q.e.d.

REMARK 1. The conditions (i), (ii) imply (i)′ $Z_{G_1}(\varphi_1(G'_1)) = Z \times G''_1$ (direct product); and (i)′ in turn implies (ii)′ $\varphi_1(Z'_1) \subset Z_1 \times Z''_1$. As is seen from the above proof, the conditions (i), (ii) in Lemma 1 can be replaced by a weaker condition (i)′.

REMARK 2. The condition (iii) is satisfied if G''_1 is k-isomorphic to $SL(n)$ and if the ground field k has the following property: (P_n) *For any normal division algebra* $\mathfrak{K}$ *over* k *such that* $\mathfrak{K}^n \sim 1$ *one has* $\deg \mathfrak{K} | n$.

In fact, it is well-known that the canonical map $\Delta : H^1(k, SL(n)/\mathbf{E}_n) \rightarrow H^2(k, \mathbf{E}_n)$ is injective, and also there is a canonical monomorphism of $H^2(k, \mathbf{E}_n)$ into the Brauer group $\mathcal{B}(k)$ of k (see Example in 1). If the algebra class of a normal division algebra $\mathfrak{K}$ over k belongs to the image of this monomorphism, then one has clearly $\mathfrak{K}^n \sim 1$. On the other hand, the algebra class of $\mathfrak{K}$ comes from an element of $H^1(k, SL(n)/\mathbf{E}_n)$ if and only if it contains a k-form of $\mathcal{M}_n$ (the total matric algebra of degree n), or, in other words, the degree of $\mathfrak{K}$ divides n. Hence, under the condition (P_n), Δ is bijective. It should also be noted that for the proofs of 2) and 3) we needed only the injectivity of Δ, which holds whenever G''_1 is k-isomorphic to $SL(n)$, without the assumption (P_n) for k.

3. We shall now apply Lemma 1 to the following situation. Let G_1 and G'_1 be (connected) simply connected (absolutely simple) Steinberg groups over

k of one of the types listed below:

G_1	${}^1E_6$	${}^2E_6$	E_7	${}^3D_4$	${}^6D_4$
G_1'	${}^1A_5$	${}^2A_5$	${}^1D_6$	${}^3(3A_1)$	${}^6(3A_1)$

(For the meaning of the notation, see [11].) Then the centers of G_1 and G_1' are as follows:

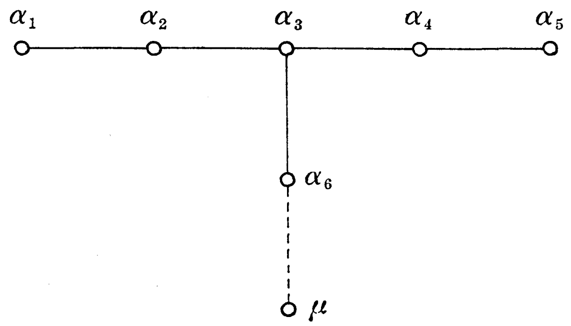
$Z_1 \cong$	$\mathbf{E}_3$	$\mathbf{E}_2$	$\mathbf{E}_2 \times \mathbf{E}_2$
$Z_1' \cong$	$\mathbf{E}_6$	$\mathbf{E}_2 \times \mathbf{E}_2$	$\mathbf{E}_2 \times \mathbf{E}_2 \times \mathbf{E}_2$

The isomorphism in this list is a $\mathfrak{G}$-isomorphism, if and only if the group G_1 or G_1' is of Chevalley type. In general, the corresponding G_1 and G_1' will have a common splitting field k', and the action of $\mathfrak{G}$ on Z_1 and Z_1' will be determined uniquely by k'. In each case, we shall construct a k-morphism φ_1 of G_1' into G_1 (which will turn out to be a monomorphism) in such a way that $\varphi_1(G_1')$ is a "regular" k-closed subgroup of G_1[3]. (By a regular closed subgroup of G_1, we mean a closed subgroup corresponding to a "regular" subalgebra of the Lie algebra of G_1 in the sense of Dynkin [4].) For all cases, G_1'' will be a k-closed subgroup of G_1 which is a simply connected Chevalley group of type A_1 and so Z_1'' is $\cong \mathbf{E}_2$. Thus, by the Remark 2 in 2, the condition (iii) of Lemma 1 is satisfied, provided k satisfies the condition (P_2).

4. *The case* ${}^1E_6$. Let G_1 and G_1' be simply connected Chevalley groups over k of type E_6 and A_5, respectively. Then, one has $\mathfrak{G}$-isomorphisms

$$Z_1 \cong \mathbf{E}_3\,, \qquad Z_1' \cong \mathbf{E}_6\,. \tag{3}$$

Let T_1 and T_1' be k-trivial maximal tori in G_1 and G_1', respectively. Let further $\mathfrak{r}$ be the root system of G_1 relative to T_1, $\Delta = \{\alpha_1, \cdots, \alpha_6\}$ a fundamental system



3) It can be proven directly that, if G_1 is a simply connected semi-simple algebraic group and if H_1 is a regular closed subgroup corresponding to a subset of a fundamental system of G_1, then H_1 is also simply connected.

of $\mathfrak{r}$, and μ the lowest root (i.e., $-\mu=\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4+\alpha_5+2\alpha_6$) (see the figure). Then it is clear that there is a k-isogeny φ_1 of G_1' onto a regular k-closed subgroup $G_1(\{\alpha_1, \cdots, \alpha_5\})$ such that $\varphi_1(T_1')\subset T_1$. (In general, for any subset Γ of $\mathfrak{r}$, one denotes by $G_1(\Gamma)$ the regular closed subgroup of G_1 corresponding to the (closed) subsystem $\mathfrak{r}\cap\{\Gamma\}_Z$ of $\mathfrak{r}$.) One puts also $G_1''=G_1(\{\mu\})$.

In order to see that the conditions (i), (ii) of Lemma 1 are satisfied, we need the following

LEMMA 2. *Let ρ_1 be an irreducible representation of G_1 of dimension 27 with the highest weight $\lambda_1=\frac{1}{3}(4\alpha_1+5\alpha_2+6\alpha_3+4\alpha_4+2\alpha_5+3\alpha_6)$. Then one has*

$$\rho_1\circ\varphi_1\sim\rho_1'+\rho_1'+\rho_4',$$

where ρ_i' stands for the i-th skew-symmetric tensor representation of G_1' in the standard numbering.

(Cf. [2], pp. 142-143; [3], pp. 20-23. In Cartan's notation, one has $\alpha_i=\omega_{i,i+1}=\bar{\omega}_i-\bar{\omega}_{i+1}(1\leqq i\leqq 5)$, $\alpha_6=\omega_{567}=\bar{\omega}_5+\bar{\omega}_6+\bar{\omega}_7$, $\mu=\omega_{000}=3\bar{\omega}_0$. The weights of ρ_1 are given by $\bar{\omega}_i-\bar{\omega}_0$, $\bar{\omega}_i+2\bar{\omega}_0$, $-\bar{\omega}_i-\bar{\omega}_j-\bar{\omega}_0$ $(1\leqq i, j\leqq 6, i\neq j)$. It is then easy to see that $(-\bar{\omega}_i-\bar{\omega}_j-\bar{\omega}_0)\circ(\varphi_1|T_1')$ (resp. $(\bar{\omega}_i-\bar{\omega}_0)\circ(\varphi_1|T_1')$, resp. $(\bar{\omega}_i+2\bar{\omega}_0)\circ(\varphi_1|T_1')$) constitute the set of weights of ρ_4' (resp. ρ_1', resp. ρ_1') relative to T_1'.)

It follows that one can find a generator z of Z_1' such that

$$\rho_1(\varphi_1(z))=\text{diag.}(\zeta_6 1_{12}, \zeta_6^4 1_{15}),$$

where ζ_r is the primitive r-th root of unity (in $\bar{k}$) and 1_r is the unit matrix of degree r. This shows that both ρ_1 and φ_1 are faithful and $\varphi_1(z^2)$ is a generator of Z_1. On the other hand, it is clear that G_1'' is contained in the centralizer $Z_{G_1}(\varphi_1(G_1'))$. By Schur's lemma, the matrices of degree 27 which commute elementwise with $\rho_1(\varphi_1(G_1'))$ are of the form diag.$(x\otimes 1_6, \eta 1_{15})$, where $x\in GL(2)$ and η is a scalar. Hence, in order to complete the proof of (i), it is enough to show that, if a matrix of the form diag.$(\xi 1_{12}, \eta 1_{15})$ is in $\rho_1(G_1)$, then it is in $\rho_1(\varphi_1(Z_1'))$. From the fact that $\rho_1(G_1)$ leaves a certain cubic form $(\sum_{i\neq k} x_i y_k z_{ik}-\sum z_{\lambda\mu}z_{\nu\rho}z_{\sigma\tau}$ in the notation of [2] loc. cit.) invariant, it follows that $\xi^2\eta=\eta^3=1$, whence $\xi^6=1$, $\eta=\xi^4$, which proves our assertion. At the same time, one sees that G_1'' is k-isomorphic to $SL(2)$ and $\varphi_1(z^3)$ is the generator of Z_1''. Thus we have also (ii).

When k satisfies the condition (P_2), the condition (iii) of Lemma 1 is also satisfied. Therefore, applying Lemma 1, one concludes that to every i-equivalence class of inner k-form (G', f') of G_1' there corresponds a certain number of i-equivalence classes of inner k-forms (G, f) of G_1, for whith one has

$$(4)\qquad \gamma(G)=Z\text{-part of }\varphi^*(\gamma(G'))$$
$$=\varphi^*(\gamma(G'))^4,$$

where Z (resp. Z') is the center of G (resp. G'), which is also $\mathfrak{G}$-isomorphic to $\mathbf{E}_3$ (resp. $\mathbf{E}_6$). More specifically, when G' is k-isomorphic to $SL(6/r, \mathfrak{K}_r)$, one may identify Z' with $\mathbf{E}_6$ through the irreducible representation of $SL(6/r, \mathfrak{K}_r)$ (defined over $\bar{k}$) which comes from the (unique) irreducible representation of $\mathfrak{K}_r$ (as an associative algebra). Then, by what we have proved above, this identification gives rise to the corresponding identification of Z with $\mathbf{E}_3$, and in this sense one has

$$\gamma(G) = c(\mathfrak{K}_r)^4 , \tag{4'}$$

where $c(\mathfrak{K}_r) \in H^2(k, E_6)$ is the Hasse invariant of $\mathfrak{K}_r$.

We may reformulate our result in the following form, which also gives a characterization of the k-forms G obtained by our method.

THEOREM 1. *Let G be a simply connected absolutely simple algebraic group of type E_6 defined over k. Suppose there exists a regular k-closed subgroup G' of type ${}^1A_5$. Then G is of type ${}^1E_6$. If G' is k-isomorphic to $SL(6/r, \mathfrak{K}_r)$, then through the natural identification mentioned above one has*

$$\gamma(G) = c(\mathfrak{K}_r)^4 .$$

PROOF. Since there is only one class of regular closed subgroups of type A_5 in G with respect to the inner automorphisms ([4], p. 149, Table 11), one may suppose that G' is of the form $G(\{\alpha_1, \cdots, \alpha_5\})$ with respect to a maximal torus T defined over $\bar{k}$ and a fundamental system $\{\alpha_1, \cdots, \alpha_6\}$. Let G_1 be a simply connected Chevalley group of type E_6 over k and let T_1 be a k-trivial maximal torus in G_1. Then one can find a $\bar{k}$-isomorphism $f: G \to G_1$ such that $f(T) = T_1$. Let $\varphi: G' \to G$ be the inclusion monomorphism (defined over k), and put $f' = f|G'$, $G_1' = f'(G')$, and $\varphi_1 = f \circ \varphi \circ f'^{-1}$. Then $G_1' = G_1(\{\alpha_1, \cdots, \alpha_5\})$ (with respect to T_1), so that G_1' is a k-closed subgroup of G_1, which is a simply connected Chevalley group of type A_5 over k, and φ_1 is also defined over k. Since G' is of type ${}^1A_5$, the isomorphism $Z' \cong \mathbf{E}_6$ is a $\mathfrak{G}$-isomorphism. Therefore the same is also true for $Z \cong \mathbf{E}_3$, which means that G is of type ${}^1E_6$. It follows that $f^\sigma \circ f^{-1}$ (resp. $f'^\sigma \circ f'^{-1}$) is an inner automorphism of G_1 (resp. G_1'). Thus one restores the situation considered above (except for the condition (P_2) on k, which we do not need), and the last statement of the Theorem follows.

5. *The case ${}^2E_6$.* Let G_1 and G_1' be simply connected Steinberg groups over k of type ${}^2E_6$ and ${}^2A_5$, respectively. Then there exists a quadratic extension k' of k over which G_1 splits (i.e., becomes of Chevalley type). For any fixed isomorphism $Z_1 \cong \mathbf{E}_3$, the 'splitting field' k' can be characterized by the action of the Galois group as follows:

$$Z_1 \ni z \leftrightarrow \zeta \in \mathbf{E}_3$$
$$\Longrightarrow \begin{cases} z^\sigma \leftrightarrow \zeta^\sigma & \text{if} \quad \sigma \in \mathrm{Gal}(\bar{k}/k'), \\ z^\sigma \leftrightarrow \zeta^{-\sigma} & \text{if} \quad \sigma \notin \mathrm{Gal}(\bar{k}/k'). \end{cases}$$

The situation is quite similar for G_1'. Hence, if there is a k-morphism φ_1: $G_1' \to G_1$ as described in Lemma 1, then the injection: $Z_1 \to \varphi_1(Z_1')$ will induce a $\mathcal{G}$-monomorphism of Z_1 into Z_1', and so the splitting fields for G_1 and G_1' should coincide. Conversely, if G_1 and G_1' have a common splitting field k', then one can find a k-morphism φ_1 as follows. Let T_1 and T_1' be maximal tori defined over k in G_1 and G_1', respectively, containing a maximal k-trivial torus in the respective groups, and take a $\mathcal{G}$-fundamental system $\Delta = \{\alpha_1, \cdots, \alpha_6\}$ in the sense of [8]. (These imply that T_1 and T_1' are k'-trivial and, if σ_0 denotes the generator of $\mathrm{Gal}(k'/k)$, one has $\alpha_1^{\sigma_0} = \alpha_5$, $\alpha_2^{\sigma_0} = \alpha_4$, $\alpha_3^{\sigma_0} = \alpha_3$, $\alpha_6^{\sigma_0} = \alpha_6$.) It is then clear that $G_1(\{\alpha_1, \cdots, \alpha_5\})$ is a k-closed subgroup of G_1, which is also a Steinberg group with the same splitting field k', and $T_1 \cap G_1(\{\alpha_1, \cdots, \alpha_5\})$ contains a maximal k-trivial torus in $G_1(\{\alpha_1, \cdots, \alpha_5\})$. Therefore, there exists a k-isogeny φ_1 of G_1' onto $G_1(\{\alpha_1, \cdots, \alpha_5\})$ such that $\varphi_1(T_1') \subset T_1$ ([8], p. 233).

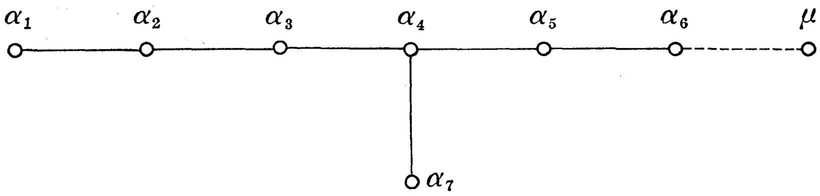
Since the conditions (i), (ii) of Lemma 1 have nothing to do with the ground field k, the proofs given in 4 remain valid in the present case. Also one has $G_1'' = G_1(\{\mu\}) \cong SL(2)$ (over k). Hence one can apply Lemma 1 to obtain a quite similar result as in 4. In particular, if (G, f) is an inner k-form of G_1 corresponding to an inner k-form (G', f') of G_1' in the sense of Lemma 1, then $\gamma(G)$ is given by the Z-part of $\varphi^*(\gamma(G'))$. Also, by a similar argument, one obtains the following

THEOREM 1′. *Let G be a simply connected absolutely simple algebraic group of type E_6 defined over k. Suppose there exists a regular k-closed subgroup G' of type ${}^2A_5$. Then, G is of type ${}^2E_6$ (belonging to the same quadratic extension k'/k) and $\gamma(G)$ is given by the Z-part of $\gamma(G')$.*

6. *The case E_7.* Let G_1 and G_1' be simply connected Chevalley groups over k of type E_7 and D_6, respectively. Then one has

(5) $$Z_1 \cong \mathbf{E}_2, \qquad Z_1' \cong \mathbf{E}_2 \times \mathbf{E}_2.$$

(This time the operations of the Galois group are all trivial.) Let T_1 and T_1' be k-trivial maximal tori in G_1 and G_1', respectively, and let $\{\alpha_1, \cdots, \alpha_7\}$ be a



fundamental system of G_1 relative to T_1, and μ the lowest root (i.e., $-\mu=\alpha_1+2\alpha_2+3\alpha_3+4\alpha_4+3\alpha_5+2\alpha_6+2\alpha_7$) (see the figure). Then one has a k-isogeny φ_1 of G_1' onto $G_1(\{\alpha_1,\cdots,\alpha_5,\alpha_7\})$ such that $\varphi_1(T_1')\subset T_1$. One puts also $G_1''=G_1(\{\mu\})$. Then one has the following

LEMMA 3. *Let ρ_1 be an irreducible representation of G_1 of dimension 56 with the highest weight $\lambda_1=\frac{3}{2}\alpha_1+2\alpha_2+\frac{5}{2}\alpha_3+3\alpha_4+2\alpha_5+\alpha_6+\frac{3}{2}\alpha_7$. Then, one has*

$$\rho_1\circ\varphi_1\sim\rho_1'+\rho_1'+\rho_6',$$

where ρ_1' and ρ_6' are the irreducible representations of G_1' corresponding to the fundamental weights λ_1' and λ_6', respectively. (The λ_i''s are numerated in such a way that $\frac{2\langle\alpha_i',\lambda_j'\rangle}{\langle\alpha_i',\alpha_i'\rangle}=\delta_{ij}$, where $\alpha_i'=\alpha_i\circ(\varphi_1|T_1')$ for $1\leqq i\leqq 5$ and $\alpha_6'=\alpha_7\circ(\varphi_1|T_1')$. In particular, ρ_6' is the "second spin representation" in this numbering.)

(Cf. [**2**], pp. 143-144; [**3**], pp. 24-27. Note that in this case $\rho_1(G_1)$ leaves an alternating form invariant.)

In virtue of this Lemma, it can be proved exactly as in 4 that the conditions (i), (ii) of Lemma 1 are satisfied. Moreover, one can find generators z_1 and z_2 of Z_1' such that

$$\rho_1(\varphi_1(z_1))=\text{diag.}(-1_{24},1_{32}),$$

$$\rho_1(\varphi_1(z_2))=-1_{56}.$$

Thus $\varphi_1(z_1)$ and $\varphi_1(z_2)$ are the generators of Z_1'' and Z_1, respectively. In the following, we shall fix once and for all the isomorphisms (5) given by this choice of the generators.

One concludes from Lemma 1 that, if (G,f) is an inner k-form of G_1 corresponding to an inner k-form (G',f') of G_1', then $\gamma(G)$ is given by the Z-part of $\varphi^*(\gamma(G'))$. Through the identification of $Z'\cong Z_1'$ (resp. $Z\cong Z_1$) with $\mathbf{E}_2\times\mathbf{E}_2$ (resp. $\mathbf{E}_2$) mentioned above, one has

$$\gamma(G')=(c(\mathfrak{C}_1),\ c(\mathfrak{C}_2)),\qquad \gamma(G)=c(\mathfrak{C}_2),\tag{6}$$

where $\mathfrak{C}_1$ and $\mathfrak{C}_2$ denote the first and the second Clifford algebras (over k) associated with G' supplying the spin representations ρ_5' and ρ_6' respectively ([**9**], p. 249). From this, one obtains the following

THEOREM 2. *Let G be a simply connected absolutely simple algebraic group of type E_7 over k. Suppose there exists a regular k-closed subgroup G' of type D_6. Then, G' is of type ${}^1D_6$ and, if $\mathfrak{C}_2$ is the second Clifford algebra associated with G' (in the sense explained above), one has*

$$\gamma(G)=c(\mathfrak{C}_2).$$

In fact, since there is only one class of regular closed subgroups of type

D_6 in G ([4], loc. cit.), one may suppose that G' is of the form $G(\{\alpha_1, \cdots, \alpha_5, \alpha_7\})$. On the other hand, since the Galois group operates trivially on $Z'=Z\times Z''$, G' is of type ${}^1D_6$. The rest of the proof runs exactly in the same way as for Theorem 1.

7. Tits [12] gave recently a new method of constructing k-forms of (absolutely) simple Lie algebras of type E_6 and E_7 which contain in an obvious way simple Lie algebras of type A_5 and D_6, respectively. The invariant $\gamma(G)$ of the corresponding simply connected simple algebraic group G defined over k can therefore be determined by Theorems 1, 1′ and 2. Moreover, when k is a local field, all k-forms of E_6 and E_7 are obtained in this manner.

First, let us recall briefly the construction of Tits for the case E_6[4]. Let $\mathfrak{D}$ (resp. $\mathcal{C}$) be a quaternion (resp. octanion) algebra over k, and let $\mathcal{J}$ be a normal simple Jordan algebra of degree 3 and of dimension 9 over k (with the product $\circ$)[5]. Then one obtains simple Lie algebras of type E_6 and A_5 over k in the following form:

$$\text{(7)} \qquad \begin{cases} \mathfrak{g}=D(\mathcal{C})+\mathcal{C}_0\otimes\mathcal{J}_0+D(\mathcal{J}), \\ \mathfrak{g}'=D(\mathfrak{D})+\mathfrak{D}_0\otimes\mathcal{J}_0+D(\mathcal{J}), \end{cases}$$

where $D(\cdots)$ denotes the derivation algebra of $\cdots$ and $(\cdots)_0$ is the subspace of $\cdots$ formed of all elements of (reduced) trace zero. The product $[\ ]$ in $\mathfrak{g}$ is defined by the following rule: (i) $D(\mathcal{C})$ and $D(\mathcal{J})$ are Lie subalgebras of $\mathfrak{g}$ satisfying $[D(\mathcal{C}), D(\mathcal{J})]=0$; (ii) for $D\in D(\mathcal{C})$, $D'\in D(\mathcal{J})$, and $a\otimes u\in\mathcal{C}_0\otimes\mathcal{J}_0$, one has

$$[D+D', a\otimes u]=(Da)\otimes u+a\otimes(D'u);$$

(iii) for $a\otimes u$, $b\otimes v\in\mathcal{C}_0\otimes\mathcal{J}_0$, one has

$$[a\otimes u, b\otimes v]=(u, v)\langle a, b\rangle+(a*b)\otimes(u*v)+(a, b)\langle u, v\rangle,$$

where $(a, b)=\frac{1}{2}tr(ab)$, $a*b=ab-(a, b)1\in\mathcal{C}_0$, and $\langle a, b\rangle$ is a derivation of $\mathcal{C}$ defined by

$$\langle a, b\rangle(x)=\frac{1}{4}[[a, b], x]-\frac{3}{4}[a, b, x] \qquad \text{for } x\in\mathcal{C},$$

and similarly $(u, v)=\frac{1}{3}tr(u\circ v)$, $u*v=u\circ v-(u, v)1$, and

$$\langle u, v\rangle(x)=u\circ(v\circ x)-v\circ(u\circ x) \qquad \text{for } x\in\mathcal{J}.$$

The product in $\mathfrak{g}'$ is defined similarly.

Now suppose $\mathfrak{D}\subset\mathcal{C}$. Then one may write $\mathcal{C}=\mathfrak{D}+\mathfrak{D}\varepsilon_4$ with $\varepsilon_4\in\mathcal{C}_0$, $\varepsilon_4^2=\lambda$

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4) Actually there are two different constructions of the Lie algebras of type E_6 and E_7, but for the sake of simplicity we consider here only one of them.

5) For the theory of Jordan algebras the reader is referred to [7], [10], [12], [13].

$\in k$, $\lambda \neq 0$, and one has

$$(a+b\varepsilon_4)(c+d\varepsilon_4)=(ac+\lambda\bar{d}b)+(da+b\bar{c})\varepsilon_4$$

for $a, b, c, d \in \mathfrak{D}$, where the bar denotes the canonical involution in $\mathfrak{D}$. We imbed $D(\mathfrak{D})$ into $D(C)$ as follows. One has $D(\mathfrak{D})=\{D_a\ (a\in\mathfrak{D}_0)\}$, where $D_a(x)=[a, x]$ for $x\in\mathfrak{D}$, and D_a can be extended to a derivation of C by setting

$$D_a(x+y\varepsilon_4)=[a, x]-(ya)\varepsilon_4 .$$

(Note that this extension of D_a is independent of the choice of ε_4.) The injection $D(\mathfrak{D})\to D(C)$ thus defined is clearly a monomorphism of Lie algebra, and gives rise in a natural way to a monomorphism of $\mathfrak{g}'$ into $\mathfrak{g}$. In this sense, we have the following

LEMMA 4. *When* $\mathfrak{D}\subset C$, $\mathfrak{g}'$ *is a regular subalgebra of* $\mathfrak{g}$.

In fact, take any non-zero element a_1 in $\mathfrak{D}_0$. Then one can define another sort of derivation of C by setting

$$D'_{a_1}(x+y\varepsilon_4)=(a_1y)\varepsilon_4 .$$

It is easy to check that one has $[D'_{a_1}, X]=0$ for all $X\in\mathfrak{g}'$. Hence, if a_1 is semi-simple and if $\mathfrak{h}'$ is any Cartan subalgebra of $\mathfrak{g}'$, then $\mathfrak{h}=\{D'_{a_1}\}_k+\mathfrak{h}'$ is a Cartan subalgebra of $\mathfrak{g}$ such that $[\mathfrak{h}, \mathfrak{g}']\subset\mathfrak{g}'$. Therefore, $\mathfrak{g}'$ is a regular subalgebra of $\mathfrak{g}$ with respect to $\mathfrak{h}$.

Now we have the following two cases:

1°. $\mathcal{J}=\mathcal{J}(\mathfrak{A}_3)$, where $\mathfrak{A}_3$ is a normal simple (associative) algebra of degree 3 over k and $\mathcal{J}(\mathfrak{A}_3)$ denotes the Jordan algebra obtained from $\mathfrak{A}_3$ by endowing it with the Jordan product $x\circ y=\frac{1}{2}(xy+yx)$ for $x, y\in\mathfrak{A}_3$.

2°. $\mathcal{J}=\mathcal{H}(\mathfrak{A}'_3, \iota)$, where $\mathfrak{A}'_3$ is a normal simple (associative) algebra of degree 3 over a quadratic extension k' of k with an involution of the second kind ι, and $\mathcal{H}(\mathfrak{A}'_3, \iota)$ denotes the Jordan algebra formed of all 'ι-hermitian' element in $\mathfrak{A}'_3$ (i.e., all $x\in\mathfrak{A}'_3$ such that $x^\iota=x$) with the Jordan product as above. In particular, when $\mathfrak{A}'_3\sim 1$ (over k'), one may write

$$\mathcal{J}=\mathcal{H}_3(k'/k\,;\ \gamma_1, \gamma_2, \gamma_3)=\{X\in\mathcal{M}_3(k')\,|\,H^{-1t}\bar{X}H=X\},$$

where $\gamma_i\in k$, $\gamma_i\neq 0$ $(1\leqq i\leqq 3)$, and $H=\mathrm{diag.}(\gamma_1, \gamma_2, \gamma_3)$.

It is then easy to show that, in the case 1°, $\mathfrak{g}'$ is canonically identified with the Lie algebra $(\mathfrak{D}\otimes\mathfrak{A}_3)_0$ with the Lie product $[x, y]=xy-yx$; while, in the case 2°, $\mathfrak{g}'$ is canonically identified with the Lie algebra formed of all $x\in\mathfrak{D}\otimes_k\mathfrak{A}'_3$ such that $tr_{\mathfrak{D}\otimes\mathfrak{A}'_3/k'}(x)=0$ and $x^{\iota'}+x=0$, with the Lie product as above, where ι' denotes the involution of the second kind in $\mathfrak{D}\otimes_k\mathfrak{A}'_3$ defined by $(x\otimes y)^{\iota'}=\bar{x}\otimes y^\iota$ for $x\in\mathfrak{D}$, $y\in\mathfrak{A}'_3$. Let G and G' be the simply connected simple algebraic groups defined over k correspnding to $\mathfrak{g}$ and $\mathfrak{g}'$, respectively.

Then, in the case 1°, G' is of type ${}^1A_5$ and by Theorem 1 one has

$$\gamma(G)=c(\mathfrak{A}_3)\,. \tag{8}$$

In the case 2°, G' is of type ${}^2A_5$ and $\gamma(G)$ can be determined by Theorem 1' and by [9], p. 245, (14); in particular, if $\mathfrak{A}_3'\sim 1$ (over k'), one has

$$\gamma(G)=(c'_{\sigma,\tau})\,,$$

where

$$c'_{\sigma,\tau}=\begin{cases} 1 & \text{if} \quad \sigma\in \mathrm{Gal}\,(\bar{k}/k')\,, \\ \sqrt[3]{\gamma_1\gamma_2\gamma_3}^{\,\tau-1} & \text{if} \quad \sigma\notin \mathrm{Gal}\,(\bar{k}/k'),\ \tau\in \mathrm{Gal}\,(\bar{k}/k')\,, \\ \sqrt[3]{\gamma_1\gamma_2\gamma_3}^{\,1-\tau} & \text{if} \quad \sigma,\,\tau\notin \mathrm{Gal}\,(\bar{k}/k')\,, \end{cases}$$

whence it is easy to see that $(c'_{\sigma,\tau})\sim 1$ and so $\gamma(G)=1$.

8. The simple Lie algebras of type E_7 and D_6 constructed by Tits are of the following form:[4)]

$$\begin{cases} \mathfrak{g}=D(C)+C_0\otimes\mathcal{J}_0'+D(\mathcal{J}')\,, \\ \mathfrak{g}'=D(\mathfrak{D})+\mathfrak{D}_0\otimes\mathcal{J}_0'+D(\mathcal{J}')\,, \end{cases} \tag{9}$$

where $\mathfrak{D}$ and C are as before, but $\mathcal{J}'$ is a normal simple Jordan algebra of degree 3 and of dimension 15 over k. When k satisfies (P_2), one may assume

$$\mathcal{J}'=\mathcal{H}_3(\mathfrak{D}'\,;\,\gamma_1,\gamma_2,\gamma_3)\,, \tag{10}$$

where $\mathfrak{D}'$ is another quaternion algebra over k, $\gamma_i\in k$, $\gamma_i\neq 0$, and $\mathcal{H}_3(\mathfrak{D}'\,;\gamma_1,\gamma_2,\gamma_3)$ denotes the Jordan algebra formed of all $X\in\mathcal{M}_3(\mathfrak{D}')$ such that $H^{-1}{}^t\bar{X}H=X$ with $H=\mathrm{diag.}(\gamma_1,\gamma_2,\gamma_3)$. The products are defined quite similarly as in 7.

Now, analogously to Lemma 4, one sees that, when $\mathfrak{D}\subset C$, $\mathfrak{g}'$ is a regular subalgebra of $\mathfrak{g}$. Also, it is easy to see that $\mathfrak{g}'$ can be identified canonically with the Lie algebra formed of all $X\in\mathcal{M}_3(\mathfrak{D}\otimes\mathfrak{D}')$ such that $tr(X)=0$ and ${}^t\bar{X}H+HX=0$, where $\bar{X}$ is defined by means of the involution of the first kind in $\mathfrak{D}\otimes\mathfrak{D}'$ defined by $\overline{x\otimes y}=\bar{x}\otimes\bar{y}$ for $x\in\mathfrak{D}$, $y\in\mathfrak{D}'$. It follows that G' is of type ${}^1D_6$ and so by Theorem 2, denoting by $\mathfrak{C}_2$ the second Clifford algebra associated with G', one has

$$\gamma(G)=c(\mathfrak{C}_2)\,.$$

In the special cases, where $\mathfrak{D}'\subset C$ or $C\sim 1$, one can show that $\mathfrak{C}_2\sim\mathfrak{D}'$ and so

$$\gamma(G)=c(\mathfrak{D}')\,. \tag{11}$$

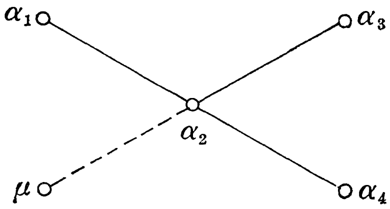
(This is always the case when k is a local field.)

In fact, if $\mathfrak{D}'\subset C$, one may take $\mathfrak{D}=\mathfrak{D}'=(\beta,\gamma)$. Then $\mathfrak{D}\otimes\mathfrak{D}'\sim 1$ and the 3-dimensional hermitian vector space over $\mathfrak{D}\otimes\mathfrak{D}'$ with the hermitian form H

reduces in an obvious manner to a 12-dimensional quadratic vector space over k with a symmetric bilinear form $S=\mathrm{diag}.(1, -\beta, -\gamma, \beta\gamma)\otimes H$. By an easy calculation, one then sees that the full Clifford algebra $C(S)$ is $\sim(\beta, \gamma)$ and so $\mathfrak{C}_1\sim\mathfrak{C}_2\sim(\beta, \gamma)$. Next, when $C'\sim 1$, one may take $\mathfrak{D}\sim 1$; put $\mathfrak{D}'=(\beta', \gamma')$. Then the 3-dimensional hermitian vector space over $\mathfrak{D}\otimes\mathfrak{D}'$ reduces to a 6-dimensional (right) vector space $\boldsymbol{V}'$ over $\mathfrak{D}'$ with a skew-hermitian form of index 3. Let $(\boldsymbol{e}_1, \cdots, \boldsymbol{e}_6)$ be any basis of $\boldsymbol{V}'$ over $\mathfrak{D}'$ for which the skew-hermitian form takes the form $\begin{pmatrix}0 & -1_3\\ 1_3 & 0\end{pmatrix}$ and put $e_i=\boldsymbol{e}_i\varepsilon_{11}'$ $(1\leqq i\leqq 6)$, where $\varepsilon_1'\in\mathfrak{D}_0'$, $\varepsilon_1'^2=\beta'$, $\varepsilon_{11}'=\frac{1}{2}(1+\sqrt{\beta'}^{-1}\varepsilon_1')$. Put further $K=k(\sqrt{\beta'})$. Then $W=\{e_1, \cdots, e_6\}_K$ is a maximal totally isotropic subspace of $\boldsymbol{V}_K'\varepsilon_{11}'$, which is now viewed as a 12-dimensional quadratic vector space over K. Let $W'=\{e_7, \cdots, e_{12}\}_K$ be a complementary totally isotropic subspace such that $S(e_i, e_{j+6})=\delta_{ij}$ $(1\leqq i, j\leqq 6)$, S denoting the symmetric bilinear form on $\boldsymbol{V}_K'\varepsilon_{11}'$. In terms of this basis, one can show that the second Clifford algebra $\mathfrak{C}_2$ (in the sense explained in 6) corresponds to the simple component of the even Clifford algebra $C^+(S)$ whose unit element is given by $\frac{1}{2}\left\{1+\prod_{i=1}^{6}(e_ie_{i+6}-e_{i+6}e_i)\right\}$. From this, one can conclude by a straightforward calculation that $\mathfrak{C}_2\sim(\beta', \gamma')$.

9. *The cases* ${}^3D_4$ *and* ${}^6D_4$. Let G_1 and $G_1'(=\prod_{i=1}^{3}G_{1i}')$ be simply connected Steinberg groups over k of type ${}^3D_4$ (or ${}^6D_4$) and ${}^3(3A_1)$ (or ${}^6(3A_1)$), respectively. Then, there is a cubic extension k_1' of k such that $G_1'=R_{k'_1/k}(G_{11}')$, and the splitting field k' for G_1' is the smallest Galois extension (of degree 3 or 6) of k containing k_1'. One has

$$(12)\qquad \begin{cases} Z_1\cong \mathbf{E}_2\times\mathbf{E}_2\,, \\ Z_1'\cong \mathbf{E}_2\times\mathbf{E}_2\times\mathbf{E}_2\ \ (=R_{k'_1/k}(\mathbf{E}_2))\,. \end{cases}$$



In view of the operations of the Galois group on Z_1 and Z_1', it is easy to see (as in 5) that one has a k-isogeny φ_1 of G_1' onto $G_1(\{\alpha_1, \alpha_3, \alpha_4\})$ if and only if G_1 has the same splitting field k'. One puts also $G_1''=G_1(\{\mu\})$, where μ is the lowest root. Then (as in 4) one can show that all the assumptions of Lemma 1 are satisfied, provided k satisfies (P_2). Moreover, if one calls z_i the generator of the center of G_{1i}' $(i=1, 2, 3)$, one sees that $\varphi_1(z_1z_2)$ and $\varphi_1(z_1z_3)$ are generators of Z_1 and $\varphi_1(z_1z_2z_3)$ is the generator of Z_1''. One fixes once and for all the isomorphisms (12) defined by this choice of the generators. Then, by the same argument as before one obtains the following

THEOREM 3. *Let G be a simply connected absolutely simple algebraic group of type D_4 defined over k. Suppose there exists a regular k-closed subgroup G' of type ${}^3(3A_1)$ or ${}^6(3A_1)$. Then, G is of type ${}^3D_4$ or ${}^6D_4$ (with the same 'nuclear' field k' [6]). If G' is k-isomorphic to $R_{k'_1/k}(SL(1, \mathfrak{D}'))$, where k_1' is a cubic extension of k and $\mathfrak{D}'$ is a quaternion algebra over k_1', then $\gamma(G)$ is given by the Z-part of $R^*_{k'_1/k}(c(\mathfrak{D}')) \in H^2(k, Z')$.*

In particular, if there is a quaternion algebra $\mathfrak{D}$ over k such that $\mathfrak{D}' = \mathfrak{D} \otimes_k k'$ (as is always the case when k is a local field), then it can easily be seen that $\gamma(G) = 1$.

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6) This terminology was borrowed from T. Ono, On algebraic groups and discrete groups, Nagoya Math. J., 27 (1966), 279–322.