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On the generalized decomposition numbers of the symmetric group

Dedicated to Professor Iyanaga on his 60th birthday

By Masaru OSIMA

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Introduction

Let G be a group of finite order and let p be a fixed prime number. We consider the representations of G in the field Ω of the g-th roots of unity. Then every absolutely irreducible representation of G can be written with coefficients in Ω. Let $\mathfrak{p}$ be a prime ideal divisor of p in Ω and let $\mathfrak{o}_{\mathfrak{p}}$ be the ring of all $\mathfrak{p}$-integers of Ω, and Ω^* the residue class field of $\mathfrak{o}_{\mathfrak{p}}$ (mod $\mathfrak{p}$). We denote by α^* the residue class of $\alpha \in \mathfrak{o}_{\mathfrak{p}}$.

Let $\zeta_0 = 1, \zeta_1, \cdots, \zeta_{m-1}$ be the (absolutely) irreducible characters of G and let $\varphi_0 = 1, \varphi_1, \cdots, \varphi_{n-1}$ be the modular irreducible characters of G for p. Then we have for a p-regular element y in G

$$\zeta_i(y) = \sum_{\kappa} d_{i\kappa} \varphi_\kappa(y) \tag{1}$$

where the $d_{i\kappa}$ are non-negative rational integers and are called the decomposition numbers of G. The irreducible characters ζ_i and the modular irreducible characters φ_κ are distributed into a certain number of blocks $B_0, B_1, \cdots, B_{s-1}$ for p, each ζ_i and each φ_κ belonging to exactly one block B_σ. In (1) we have $d_{i\kappa} = 0$ for $\zeta_i \in B_\sigma$ if φ_κ is not contained in B_σ.

In the following we denote by x the p-element of G. Let $\varphi_0^x = 1, \varphi_1^x, \cdots, \varphi_{r-1}^x$ be the modular irreducible characters of the normalizer $N(x)$ of x in G. We have for a p-regular element y in $N(x)$

$$\zeta_i(xy) = \sum_{\kappa} d_{i\kappa}^x \varphi_\kappa^x(y) \tag{2}$$

where the $d_{i\kappa}^x$ are the algebraic integers and are called the generalized decomposition numbers of G. We have $d_{i\kappa} = d_{i\kappa}^1$ for $x = 1$. Let us denote by $B^{(\sigma)}$ the collection of all blocks $\tilde{B}_\tau$ of $N(x)$ which determine a given block B_σ of G. In (2) we have $d_{i\kappa}^x = 0$ for $\zeta_i \in B_\sigma$ if φ_κ^x is not contained in $B^{(\sigma)}$ ([1], [3]).

Recently A. Kerber [5] proved the following

THEOREM 1. *The generalized decomposition numbers of the symmetric group* *are rational integers.*

He also determined the generalized decomposition numbers of the symmetric group S_n for $p=2$ and $n\leqq 9$. In section 1 we shall give a simpler proof of Theorem 1. By our method we can determine directly the generalized decomposition numbers of S_n. In section 2 we shall obtain the necessary and sufficient condition that two irreducible characters ζ_i^x and ζ_j^x of $N(x)$ belong to the same block. As is well known, the block of S_n is determined by its p-core ([4], [6], [7], [9]). Similarly, we shall prove that the block of $N(x)$ is determined by its p-core. The aim of section 3 is to find the block of S_n which is determined by a given block of $N(x)$. We obtain the following

THEOREM 2. *Let Young diagram* $[\alpha_0]$ *be the* p*-core of the block* $\tilde{B}_\tau$ *of* $N(x)$. *Then* $\tilde{B}_\tau$ *determines the block of* S_n *with the same* p*-core* $[\alpha_0]$.

Let $B^{(\sigma)}$ be the collection of all blocks $\tilde{B}_\tau$ which determine the block B_σ of S_n. Then Theorem 2 implies that every $B^{(\sigma)}$ consists of one block of $N(x)$.

1. Proof of Theorem 1.

Let x be a p-element of S_n which consists of a_i cycles of length p^i ($0\leqq i\leqq k$, $a_i\geqq 0$). The normalizer $N(x)$ of x in S_n is the direct product of its subgroups $S(a_i, p^i)$:

$$N(x)=S(a_0, 1)\times S(a_1, p)\times \cdots \times S(a_k, p^k) \tag{3}$$

where the $S(a_i, p^i)$ are called the generalized symmetric groups ([8]). $S(a_i, p^i)$ is the semi-direct product of the normal subgroup Q_i of order $(p^i)^{a_i}$ and the subgroup S_{ai}^* which is isomorphic with the symmetric group S_{a_i}:

$$S(a_i, p^i)=S_{ai}^* Q_i,\quad S_{ai}^*\cap Q_i=1,\quad S_{ai}^*\cong S_{a_i}\,. \tag{4}$$

Evidently we have $S(a_0, 1)=S_{a_0}$. Since $S(a_i, p^i)/Q_i\cong S_{ai}^*$, (4) implies that every modular irreducible character of $S(a_i, p^i)$ is given by the modular irreducible character of S_{a_i}. Let us denote by Φ_n and Φ^x the matrices of the modular irreducible characters of S_n and $N(x)$ respectively. Since the modular irreducible character φ^x of $N(x)$ is the product of the modular irreducible characters φ^i of S_{a_i}:

$$\varphi^x=\varphi^0\varphi^1\cdots\varphi^k\,, \tag{5}$$

we see that Φ^x is the Kronecker product of Φ_{a_i}:

$$\Phi^x=\Phi_{a_0}\times\Phi_{a_1}\times\cdots\times\Phi_{a_k}\,. \tag{6}$$

LEMMA 1. *Let* x *be a* p*-element of* S_n. *Then the modular irreducible characters* $\varphi^x(y)$ *of* $N(x)$ *are rational integers.*

PROOF. As is well known, the irreducible characters $\zeta_i(g)$ of S_n are rational integers. Since the modular irreducible characters $\varphi_\kappa(y)$ of S_n can be expressed by the irreducible characters $\zeta_i(y)$ of S_n (restricted to p-regular elements) with integral coefficients, $\varphi_\kappa(y)$ are rational integers. This, combining with (5), yields the proof of Lemma 1.

Let g be an element of S_n. We then have $g=xy=yx$ where x is a p-element and y is a p-regular element. The p-element x is called the p-factor of g. Let $y_0=1, y_1, \cdots, y_{t-1}$ be a complete system of representatives for the p-regular elements in $N(x)$ such that they all lie in different classes of $N(x)$ but that every p-regular element in $N(x)$ is conjugate to one of them. Then the xy_i $(i=0, 1, \cdots, t-1)$ consist of a complete system of representatives for the classes of G which contain an element whose p-factor is conjugate to x in G. We set

$$Z^x=(\zeta_i(xy_j))\,. \tag{7}$$

We then have from (2)

$$Z^x=D^x\Phi^x \tag{8}$$

where $D^x=(d^x_{i\kappa})$. Hence

$$D^x=Z^x(\Phi^x)^{-1}\,. \tag{9}$$

This, combining with Lemma 1, shows that the $d^x_{i\kappa}$ are rational numbers. Since the $d^x_{i\kappa}$ are algebraic integers, we see readily that the $d^x_{i\kappa}$ are rational integers. This completes the proof of Theorem 1.

As an example we shall calculate the $d^x_{i\kappa}$ of S_6 for $p=2$ and $x=(12)\,(34)\,(56)$ (see [5] p. 45). Since $N(x)=S(3, 2)$, we have by (6)

$$\Phi^x=\Phi_3=\begin{bmatrix}1 & 1\\ 2 & -1\end{bmatrix}.$$

We have for $y_0=1$ and $y_1=(135)(246)$

$$Z^x=\begin{bmatrix}1 & 1\\ -1 & -1\\ 3 & 0\\ -2 & 1\\ -3 & 0\\ 0 & 0\\ 3 & 0\\ 2 & -1\\ -3 & 0\\ 1 & 1\\ -1 & -1\end{bmatrix}.$$

Hence we can obtain from (9)

$$D^x = \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 1 & 1 \\ 0 & -1 \\ -1 & -1 \\ 0 & 0 \\ 1 & 1 \\ 0 & 1 \\ -1 & -1 \\ 1 & 0 \\ -1 & 0 \end{bmatrix}.$$

2. The blocks of characters of the normalizer $N(x)$.

First we shall mention the following

LEMMA 2. *Two irreducible characters of S_n belong to the same block if and only if they have the same p-core.*

This fact was first conjectured by Nakayama [**6**] and was proved by Brauer and Robinson jointly [**4**].

Let ζ^x be an irreducible character of $N(x)$. According to (3), we have

$$\zeta^x = \zeta^0 \zeta^1 \cdots \zeta^k \tag{10}$$

where ζ^i denotes the irreducible character of $S(a_i, p^i)$. In particular, ζ^0 may be considered as the irreducible character of S_{a_0}.

LEMMA 3. *Two irreducible characters*

$$\zeta_i^x = \zeta_{i_0}^0 \zeta_{i_1}^1 \cdots \zeta_{i_k}^k$$

$$\zeta_j^x = \zeta_{j_0}^0 \zeta_{j_1}^1 \cdots \zeta_{j_k}^k$$

of $N(x)$ belong to the same block if and only if two characters $\zeta_{i_0}^0$ and $\zeta_{j_0}^0$ of S_{a_0} belong to the same block of S_{a_0}.

PROOF. For $i > 0$, $S(a_i, p^i)$ has only one block ([**11**], Lemma 10). Hence we readily obtain the proof of Lemma 3.

We shall denote by B_τ^0 the block of S_{a_0} which contains $\zeta_{i_0}^0$. Then the block of $N(x)$ which contains ζ_i^x is completely determined by B_τ^0. Hence we shall denote by $\tilde{B}_\tau$ this block of $N(x)$.

Let Young diagram $[\alpha_0]$ be the p-core of the irreducible character $\zeta_{i_0}^0 \in B_\tau^0$. Then we shall call $[\alpha_0]$ the p-core of the irreducible character $\zeta_i^x \in \tilde{B}_\tau$. Then Lemma 2, combining with Lemma 3, yields

THEOREM 3. *Two irreducible characters of $N(x)$ belong to the same block if and only if they have the same p-core.*

Theorem 3 is reduced to Lemma 2 for $x = 1$. We have (cf. [**5**], p. 49).

COROLLARY 1. *$N(x)$ has only one block if $a_0 \leqq 1$ for $p \neq 2$ and $a_0 \leqq 2$ for $p = 2$.*

COROLLARY 2. *Let B_0 be the first block of S_n, that is, the block which contains the principal character $\zeta_0=1$. Then $\zeta_i(xy)=0$ for $\zeta_i \notin B_0$ if $a_0 \leqq 1$ for $p \neq 2$ and $a_0 \leqq 2$ for $p=2$.*

We can also obtain Corollary 2 by using the Murnaghan-Nakayama recursion formula.

3. Proof of Theorem 2.

Let G be a group of finite order, and let $\Gamma=\Gamma(G)$ denote the group ring of G over Ω. We denote by $\Lambda=\Lambda(G)$ the center of Γ. Let K_α be a class of conjugate elements in G. If necessary, we denote by the same notation K_α the sum of all elements in K_α. Then $K_1, K_2, \cdots, K_m$ form a basis of Λ and we have

$$K_\alpha K_\beta = \sum_\gamma a_{\alpha\beta\gamma} K_\gamma \tag{11}$$

where the $a_{\alpha\beta\gamma}$ are non-negative rational integers.

Let H be a subgroup of G of an order p^h, $h \geqq 0$, and let $C(H)$ be the centralizer of H in G. We consider the subgroup $N=HC(H)$. If we set $K_\alpha^0 = K_\alpha \cap C(H)$, then either $K_\alpha^0=0$ or K_α^0 is a sum of complete classes of N. We obtain from (11)

$$K_\alpha^0 K_\beta^0 = \sum_\gamma a_{\alpha\beta\gamma} K_\gamma^0 \qquad (\mathrm{mod}\ p)\,. \tag{12}$$

The classes K_α with $K_\alpha^0=0$ form the basis of an ideal T^* of the center Λ^* of the modular group ring Γ^*. The $K_\alpha^0 \neq 0$ can be considered as the basis of a subring R^* of the center $\Lambda^*(N)$ of the modular group ring $\Gamma^*(N)$. According to (12) we have ([**2**])

$$\Lambda^*(G)/T^* \cong R^*\,. \tag{13}$$

Let B be a block of G. We set

$$\eta = \sum_{\alpha=1}^m b_\alpha K_\alpha \tag{14}$$

where

$$b_\alpha = \sum_{\zeta_i \in B} \zeta_i(1)\bar{\zeta}_i(g_\alpha)/g(G)\,. \tag{15}$$

Here $g_\alpha \in K_\alpha$ and $g(G)$ denotes the order of G. Then we see that $b_\alpha \in \mathfrak{o}_\mathfrak{p}$ and

$$\eta^* = \sum_{\alpha=1}^m b_\alpha^* K_\alpha \tag{16}$$

is a primitive idempotent of Λ^* corresponding to B([**10**]). We have $b_\alpha^*=0$ for any p-singular class K_α. Let $\mathfrak{D}$ be the defect group of B. We denote by $\mathfrak{H}_\alpha$ the defect group of K_α. If K_α is a p-regular class such that $\mathfrak{H}_\alpha$ is not conjugate to some subgroup of $\mathfrak{D}$, then we have $b_\alpha^* = 0$. On the other hand, there exists a p-regular class K_β with the defect group $\mathfrak{H}_\beta \cong \mathfrak{D}$ such that $b_\beta^* \neq 0$ and

$$w_i(K_\beta) = g(G)\zeta_i(g_\beta)/n_\beta \zeta_i(1) \not\equiv 0 \pmod{\mathfrak{p}} \tag{17}$$

where n_β denotes the order of the normalizer $N(g_\beta)$ of g_β in G.

In the following we denote by η_σ^* the primitive idempotent of Λ^* corresponding to B_σ. If $\eta_\sigma^* \notin T^*$, then the element $\tilde{\eta}_\sigma^*$ of R^* corresponding to η_σ^* in (13) is a sum of primitive idempotents of the center $\Lambda^*(N)$. Hence the collection $B^{(\sigma)}$ of the blocks $\tilde{B}_\tau$ of N corresponds to $\tilde{\eta}_\sigma^*$. If $\tilde{B}_\tau$ is contained in $B^{(\sigma)}$, then we shall say that B_σ is determined by $\tilde{B}_\tau$ of N([2]). If $w_i(K_\alpha)$ is formed by means of a character ζ_i of B_σ while $\tilde{w}_j(\tilde{K}_\beta)$ is formed in an analogous manner by means of a character of $\tilde{B}_\tau$, then we see by (13) that

$$w_i(K_\alpha) \equiv \sum_\beta \tilde{w}_j(\tilde{K}_\beta) \pmod{\mathfrak{p}}. \tag{18}$$

Here $\tilde{K}_\beta$ ranges over all classes of N which lie in K_α.

Let x be a p-element of S_n as in section 1. Let $\tilde{K}_\alpha$ be a p-regular class of S_{α_0}. Then we see by (3) that $\tilde{K}_\alpha$ is also a class of $N(x)$. Since $S(a_i, p^i)$, $i>0$ has only one block, if $\tilde{w}_i(\tilde{K}_\alpha)$ is formed by means of a character ζ_i^x while $\overline{w}_{i_0}(\tilde{K}_\alpha)$ is formed by means of a character $\zeta_{i_0}^0$ in Lemma 3, then

$$\tilde{w}_i(\tilde{K}_\alpha) \equiv \overline{w}_{i_0}(\tilde{K}_\alpha) \pmod{\mathfrak{p}}. \tag{19}$$

The defect group of B_σ of S_n is conjugate to the p-Sylow-subgroup of $S(\beta, p)$ for a suitable β where $n = a + \beta p$ ([4]). Hence we may denote by $\mathfrak{D}^{(\beta)}$ the defect group of B_σ. The defect of B_σ is given by

$$d_\beta = \beta + e(\beta\,!)\,. \tag{20}$$

Here $e(m)$ denotes the exponent of the highest power of p dividing an integer m. Let K_α be the p-regular classes with the defect group $\mathfrak{H}_\alpha \cong \mathfrak{D}^{(\beta)}$. Then we see easily that K_α contains the p-regular element g_α of S_a such that the order of the normalizer $N(g_\alpha)$ in S_a is prime to p.

Now we shall give the proof of Theorem 2. We have from (3)

$$n = \sum_{i=0}^k a_i p^i = a_0 + lp \tag{21}$$

where we set $l = \sum_{i=1}^k a_i p^{i-1}$. We shall first consider the block B_σ of defect d_β such that $\beta < l$. Let K_α be the p-regular classes such that $\mathfrak{H}_\alpha \cong \mathfrak{D}^{(\beta)}$. Then we see by above argument that $K_\alpha \cap N(x) = 0$. This implies that $K_\alpha \in T^*$ and hence $\eta_\sigma^* \in T^*$. Thus the block B_σ which satisfies $\beta < l$ can not be determined by any block of $N(x)$.

In what follows we may assume that $\beta \geqq l$. Let $\tilde{B}_\tau$ be a given block of

$N(x)$ and let B_τ^0 be the block of S_{a_0} corresponding to $\tilde{B}_\tau$. Let the defect of B_τ^0 be d_γ. Then $a_0 = b + \gamma p$. The p-core of B_τ^0 and hence that of $\tilde{B}_\tau$ consists of b nodes. If we set $l + \gamma = l'$, then $n = b + l'p$.

First we assume that $l' < \beta$. There exists a p-regular class $\tilde{K}_\alpha$ of S_{a_0} with the defect group $\mathfrak{H}_\alpha \cong \mathfrak{D}^{(\gamma)}$ such that $\overline{w}_{i_0}(\tilde{K}_\alpha) \not\equiv 0 \pmod{\mathfrak{p}}$ for $\zeta_{i_0}^0 \in B_\tau^0$. We then have by (19)

$$\tilde{w}_i(\tilde{K}_\alpha) \not\equiv 0 \pmod{\mathfrak{p}}. \tag{22}$$

The class $\tilde{K}_\alpha$ contains the p-regular element y_α of S_b such that the order of the normalizer $N(y_\alpha)$ in S_b is prime to p. Let K_α be the class of S_n containing y_α. Then we have $K_\alpha \cap N(x) = \tilde{K}_\alpha$. Since $l' < \beta$, we see that $h_\alpha < d_\beta$ where h_α denotes the defect of K_α. Hence we have for $\zeta_i \in B_\sigma$ ([**10**], Lemma 6)

$$w_i(K_\alpha) \equiv 0 \pmod{\mathfrak{p}}. \tag{23}$$

It follows from (18), (22) and (23) that if $l' < \beta$, then B_σ is not determined by $\tilde{B}_\tau$. By the similar argument we can see also that if $l \leqq \beta < l'$, then B_σ is not determined by $\tilde{B}_\tau$.

Finally we consider the case that $\beta = l'$. Since $n = b + l'p = a + \beta p$, we have $a = b$ and hence the p-cores of B_σ and $\tilde{B}_\tau$ consist of a nodes. Let K_α be a p-regular class of S_n with the defect group $\mathfrak{D}^{(\beta)}$. Then $K_\alpha \cap N(x) = \tilde{K}_\alpha$ is the p-regular class of S_{a_0} with the defect group $\mathfrak{D}^{(\gamma)}$. Now we assume that both B_σ and $\tilde{B}_\tau$ have the same p-core $[\alpha_0]$. Let χ_0 be the irreducible character of S_a determined by $[\alpha_0]$. Then χ_0 forms a block of its own. We see that $K_\alpha \cap S_a = K_\alpha^{(0)}$ is the p-regular class of S_a of defect 0.

Let g_γ be an element of S_n possessing β cycles of length p such that $K_\alpha^{(0)} \ni g_\alpha$ is obtained by removing those β cycles of length p. We then have for $\zeta_j \in B_\sigma$

$$\zeta_j(g_\alpha) \equiv \zeta_j(g_\gamma) \pmod{\mathfrak{p}}. \tag{24}$$

If we choose $B_\sigma \ni \zeta_j$ of height 0, then we see easily that

$$e(n_\alpha) = e(n_\gamma) = e(g(G)/\zeta_j(1)) = d_\beta$$

and

$$n_\alpha / n_\gamma = (\beta p)!/\beta!p^\beta \equiv (-1)^\beta \pmod{p}.$$

Hence we have by (24)

$$w_j(K_\alpha) \equiv (-1)^\beta w_j(K_\gamma) \pmod{\mathfrak{p}}. \tag{25}$$

Consequently, from (25) and ([7], (11))

$$w_j(K_\alpha) \equiv w_{\alpha_0}(K_\alpha^{(0)}) \pmod{\mathfrak{p}} \tag{26}$$

where $w_{\alpha_0}(K_\alpha^{(0)})$ is formed by means of χ_0. We obtain also by the same argument

(27) $$\bar{w}_{i_0}(\tilde{K}_\alpha) \equiv w_{\alpha_0}(K_\alpha^{(0)}) \pmod{\mathfrak{p}}$$

for $\zeta_{i_0}^0 \in B_\tau^0$.

It follows from (19), (26) and (27) that

(28) $$w_j(K_\alpha) \equiv \tilde{w}_i(\tilde{K}_\alpha) \pmod{\mathfrak{p}}$$

for $\zeta_i^x \in \tilde{B}_\tau$. Since we have (28) for any p-regular class K_α with the defect group $\mathfrak{D}^{(\beta)}$, we obtain the proof of Theorem 2 by (28) and ([**10**], Theorem 4, Corollary 2).

College of General Education
Osaka University

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