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On certain groups with involutive generators

Dedicated to Professor S. Iyanaga for his 60th birthday

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A group W, to be studied in this note, is supposed to have some special subset R, i. e. (1) R generates W, each element of R is involutive, (i. e. of order two), (2) R satisfies a certain condition ($\boldsymbol{C}$) given in our Definition 1. Such an R will be called a *good system of involutive generators* of W. For example take the *Weyl group* of a semi-simple Lie algebra as W, and take the set of *fundamental reflexions* as R, then our requirements (1) and (2) are fulfilled by them. Indeed, as H. Matsumoto [**1**] has shown, *a good system of involutive generators* is a natural generalization of a set of *fundamental reflexions in a Weyl group* in the following sense.

(I) If W is a Weyl group (in the generalized sense) associated to a BN-pair, and R be the set of canonical generators of W (see Tits [3]), then R is a good system of involutive generators of W.

(II) All the group theoretical properties of W follow from ($\boldsymbol{C}$). Indeed we can write down the fundamental relations among the elements of R.

Now let Γ be a group of automorphisms of W, and assume that each element of Γ induces a permutation of R. The purpose of this note is to study the structure of the group W^{Γ} of the set of all Γ-fixed points of W. Let R_j ($j \in \boldsymbol{J}'$) be Γ-orbits of R, and W_j be the group generated by R_j. Our theorems say;

W_j^{Γ} is either of order one or of order two (Theorem 1).

Take the generator s_j from each non-trivial W_j^{Γ}, then $\{s_j\}$ is a good system of involutive generators of W^{Γ} (Theorem 2 and 3).

Such phenomena for a Weyl group (in the ordinary sense) was recognized by R. Steinberg [**2**], and used in his construction of new simple groups. Generalized version treated in this note has of course similar application to the theory of descent of BN-pairs (cf. [**4**]).

* After the manuscript was submitted, the author has learned from Prof. N. Iwahori that the result of this paper (including appendix) was known by R. Steinberg (yet unpublished) by using a geometrical realization of W.

§ **1.** Let W be a group generated by involutive elements, and R be a set of involutive generators indexed by a set $\boldsymbol{I}$, $R=\{r_i\,;\,i\in \boldsymbol{I}\}$ (we always assume $r_i\neq 1$ and $r_i\neq r_j$ if $i\neq j$). Since W is generated by R, any $w\in W$ has an expression like $w=r_{i(1)}r_{i(2)}\cdots r_{i(l)}$ where l is called the length of the expression. Such an expression is called a reduced expression of w if it has the smallest length among all the expressions of w by R. For any $w\in W$, let $l(w)$ denote the length of a reduced expression of w. A finite sequence of indices $(i(1), i(2), \cdots, i(l))$ is called reduced if $l(w)=l$ for $w=r_{i(1)}r_{i(2)}\cdots r_{i(l)}$. Let Σ denote the set of all the reduced sequences of indices. If $(i(1), i(2), \cdots, i(l))\in\Sigma$, then it is obvious that $(i(t), i(t+1), \cdots, i(s))\in\Sigma$, for any pair of integers t, s $1\leqq t<s\leqq 1$. For a pair of sequences $(i(1), i(2), \cdots, i(l))$ and $(j(1), j(2), \cdots, j(m))$, if $r_{i(1)}r_{i(2)}\cdots r_{i(l)}=r_{j(1)}r_{j(2)}\cdots r_{j(m)}$, we call they are equivalent, and write as $(i(1), i(2), \cdots, i(l))\sim(j(1), j(2), \cdots, j(m))$.

DEFINITION 1. A system of involutive generators R is called a good system of involutive generators of W, if it satisfies the following condition $(\boldsymbol{C})$.

$(\boldsymbol{C})$ For $i(0)\in\boldsymbol{I}$ and $(i(1), i(2), \cdots, i(l))\in\Sigma$, if $(i(0), i(1), \cdots, i(l))\notin\Sigma$, then there exists an integer m, $1\leqq m\leqq l$, s. t. $(i(1), i(2), \cdots, i(m))\sim(i(0), i(1), \cdots, i(m-1))$.

We will write down some of the consequences of the condition $(\boldsymbol{C})$. Proofs are given in [**1**].

$(\boldsymbol{C}1)$ An integer m appeared in the statement of $(\boldsymbol{C})$ is uniquely determind by $i(0)$ and $(i(1), i(2), \cdots, i(l))$.

$(\boldsymbol{C}2)$ $l(r_iw)\neq l(w)$ for any $w\in W$ and $r_i\in R$.

$(\boldsymbol{C}3)$ Let S be a subset of R and W_S denote the set of elements which admits a reduced expression as a product of elements of S. Then W_S is a group and S is a good system of involutive generators of W_S.

$(\boldsymbol{C}4)$ R is a minimal set of generators of W.

$(\boldsymbol{C}5)$ If $l(w)$ is bounded for any $w\in W$, then R is finite and consequently W itself is of finite order.

LEMMA 1. *Let S be a subset of a good system of involutive generators R, and W_S be a group generated by S. If $w\in W$ satisfies the following conditions;*

$$(*)\qquad l(r_jw)<l(w)\quad \textit{for any}\quad r_j\in S$$

Then for any element s of W_S, $l(w)\geqq l(s)$ and $l(sw)=l(w)-l(s)$.

Furthermore let $s=r_{j(1)}r_{j(2)}\cdots r_{j(n)}$, $n=l(s)$ be a reduced expression of s by the elements of S (such an expression exists by $(\boldsymbol{C}3)$*). Then w has a reduced expression $w=r_{i(1)}r_{i(2)}\cdots r_{i(l)}$ s. t. $i(1)=j(1), i(2)=j(2), \cdots, i(n)=j(n)$.*

PROOF. We will prove the last statement by induction on n. All the rest of our claim follows immediately from it. Let $w=r_{i(1)}r_{i(2)}\cdots r_{i(l)}$ be a reduced expression of w. The assumption $l(r_jw)<l(w)$ implies $(j(1), i(1), i(2), \cdots, i(l))\notin\Sigma$. Since R is a good system of involutive generators, there exists m, $1\leqq m\leqq l$

s. t. $(j(1), i(1), \cdots, i(m-1)) \sim (i(1), i(2), \cdots, i(m))$, therefore $(i(1), i(2), \cdots, i(l)) \sim (j(1), i(1), \cdots, i(m-1), i(m+1), \cdots, i(l))$, i. e. $w = r_{j(1)} r_{i(1)} \cdots r_{i(m-1)} r_{i(m+1)} \cdots r_{i(l)}$. Thus we get a reduced expression of w starting with $r_{j(1)}$, i. e. Our claim is true for $n=1$. Now assume our claim is true for any positive integer less than n. Then w has a reduced expression $w = r_{i(1)} r_{i(2)} \cdots r_{i(l)}$ s. t. $i(1) = j(2), i(2) = j(3), \cdots, i(n-1) = j(n)$. The assumption of Lemma implies $(j(1), i(1), i(2), \cdots, i(l)) \notin \Sigma$. By the condition ($\boldsymbol{C}$), there exists m, $1 \leqq m \leqq l$ s. t. $(i(1), i(2), \cdots, i(m)) \sim (j(1), i(1), \cdots, i(m-1))$. If $m > n-1$, then $(i(1), i(2), \cdots, i(l)) \sim (j(1), j(2), \cdots, j(n), i(n), \cdots, i(m-1), i(m+1), \cdots, i(l))$, i. e. w has a reduced expression starting with $r_{j(1)}, r_{j(2)} \cdots r_{j(n)}$. Now we will show that $m > n-1$ is the only case possible and conclude the proof. Assume $m \leqq n-1$. Since $i(1) = j(2), i(2) = j(3), \cdots, i(m) = j(m+1)$, $m+1 \leqq n$, we have following equivalence; $(j(2), j(3), \cdots, j(m)) \sim (j(1), j(2), \cdots, j(m-1))$, i. e. $(j(1), j(2), \cdots, j(n)) \sim (j(2), j(3), \cdots, j(m-1), j(m+1), \cdots, j(n))$. This last equivalence relation contradicts the assumption that $(j(1), j(2), \cdots, j(n))$ is reduced.

COROLLARY 1. *There actually exists $w \in W$ satisfying the assumption* $(*)$ *of Lemma* 1 *if and only if the group W_S is of finite order.*

PROOF. If there exists w satisfying the assumption $(*)$ of Lemma 1. $l(w') \leqq l(w)$ for any $w' \in W_S$. By the property ($\boldsymbol{C5}$), W_S is of finite order. Conversely if W_S is finite, there exists w with the maximal length $l(w)$ in W_S, this w obviously satisfies the assumption $(*)$ of Lemma 1.

COROLLARY 2. *Assume W itself is of finite order.* (i) *Then the following two conditions* (a), (b) *for $w \in W$ are equivalent,*

(a) $l(r_i w) < l(w)$ *for any* $r_i \in R$.

(b) *w has the maximal length.*

(ii) *An element w satisfying one of the above conditions* (a), (b) *is uniquely determined and consequently such a w is involutive.* (iii) *Let w be the element of the maximal length. Then for any $(j(1), j(2), \cdots, j(n)) \in \Sigma$, w has a reduced expression $w = r_{i(1)} r_{i(2)} \cdots r_{i(l)} = r_{k(1)} r_{k(2)} \cdots r_{k(l)}$ s. t. $i(1) = j(1)$, $i(2) = j(2), \cdots, i(n) = j(n)$ and $j(n) = k(l)$, $j(n-1) = k(l-1), \cdots, j(1) = k(l-n+1)$.*

PROOF. (b)$\Rightarrow$(a) is obvious. Let w satisfy (a), and $w' = r_{i(1)} r_{i(2)} \cdots r_{i(l)}$ have the maximal length l. By Lemma 1, w has an expression starting with $r_{i(1)} r_{i(2)} \cdots r_{i(l)}$, then by the maximality of $l(w')$, $w = r_{i(1)} r_{i(2)} \cdots r_{i(l)} = w'$. This proves (a)$\Rightarrow$(b) and at the same time our claim (ii). Once we know that w is involutive, (iii) is immediate from Lemma 1.

COROLLARY 3*. *Let S be a subset of R. Assume the group W_S generated*

* The referee has noticed that the statement of Cor. 3 of Lemma 1 is true without the assumption of the finiteness of W_S, and in that stronger form it is already proved in the lecture given by N. Iwahori at the University of Paris in 1966. The way of proof is similar to our proof of Lemma 1.

by S is finite. If $l(rw)>l(w)$ for any $r_j \in S$, then $l(sw)=l(s)+l(w)$ for any $s \in W_S$.

PROOF. Let s_0 be the element of the maximal length n of W_S. In view of Corollary 2 (iii), it suffices to see that $l(s_0w)=n+l(w)$. Let $w=r_{i(1)}r_{i(2)}\cdots r_{i(l)}$ be a reduced expression of w. Since W_S is finite, $\{l(sw): s \in W_S\}$ is bounded. Let $s=r_{j(1)}r_{j(2)}\cdots r_{j(t)}$ be an element s.t. $l(sw)$ attains the maximal length $t+l(w)$, then $l(r_jsw)<l(sw)$ for any $r_j \in S$. If $s \neq s_0$, by Corollary 3 (iii), there exists $r_{i(t+1)} \in S$ s.t. $(j(1), j(2), \cdots, j(t), j(t+1)) \in \Sigma$. By Lemma 1 sw has a reduced expression, $sw=r_{i(1)}r_{i(2)}\cdots r_{i(t+l(w))}$ s.t. $i(1)=j(1)$, $i(2)=j(2), \cdots, i(t+1)=j(t+1)$, i.e. w has a reduced expression starting with $r_{j(t+1)}$ which contradicts our assumption that $l(r_jw)>l(w)$ for any $r_j \in S$.

§2. Let Γ be a group of automorphisms of W. Throughout this section assume each $\gamma \in \Gamma$ keeps R invariant i.e. $\gamma \in \Gamma \Rightarrow {}^{\gamma}R=R$. Let R_j ($j \in \boldsymbol{J}'$) be Γ-orbits of R, i.e. each R_j is a minimal Γ-invariant subset of R, and $R_j \neq R_k$ if $j \neq k$. For any Γ-invariant subset V of W, let V^{Γ} denote the set of Γ-fixed elements of V, $V^{\Gamma}=\{v: v \in V, {}^{\gamma}v=v$ for any $\gamma \in \Gamma\}$.

THEOREM 1. *Let W_j be the group generated by R_j. When W_j is of infinite order, W_j^{Γ} is trivial $(=\{1\})$. When W_j is of finite order, W_j^{Γ} is a cyclic group of order two generated by the element s_j of the maximal length of W.*

PROOF. If $w \in W_j^{\Gamma}$ and $l(w) \geqq 1$, there exists some $r_i \in R_j$ s.t. $l(r_iw)<l(w)$. Since Γ is transitive on R_j, applying $\gamma \in \Gamma$ to the both sides of $l(r_iw)<l(w)$, we get $l(rw)<l(w)$ for any $r \in R_j$. Existence of such an element w implies W_j is finite (Corollary 1). When W_j is finite, the above conditions for w means that w is of the maximal length in W_j (Corollary 2). An element of the maximal length is unique and involutive (Corollary 2 (ii)) and certainly invariant by Γ.

THEOREM 2. *Let $\boldsymbol{J}$ denote the set of index j for which W_j is finite, $\boldsymbol{J}=\{j; j \in \boldsymbol{J}', |W_j|<\infty\}$. If $w \in W^{\Gamma}$ $w \neq 1$, then there exists $j \in \boldsymbol{J}$ s.t. $l(s_jw)=l(w)-l(s_j)$. Consequently W^{Γ} is generated by involutive elements $s_j (j \in \boldsymbol{J})$.*

PROOF. Since $l(w)>1$, there exists some $r_i \in R_j$ s.t. $l(r_iw)<l(w)$, and consequently $l(rw)<l(w)$ for any $r \in R_j$. By Corollary 1 W_j is finite, i.e. $j \in \boldsymbol{J}$. Now $l(s_jw)=l(w)-l(s_j)$ is a direct consequence of Lemma 1.

Let $R^*=\{s_j; j \in \boldsymbol{J}\}$. R^* is a system of involutive generators of W^{Γ}. Let Σ^* denote the set of reduced sequences of index set $\boldsymbol{J}$ with respect to W^{Γ} and R^*. Our next aim is to show that R^* is a good system of involutive generators of W^{Γ}. For any index $j \in \boldsymbol{J}$, let j^* denote a sequence $(i(1), i(2), \cdots, i(l))$, s.t. $s_j=r_{i(1)}r_{i(2)}\cdots r_{i(l)}$. For such sequences $j(1)^*=(i(1), i(2), \cdots, i(l))$, $j(2)^*=(i'(1), i'(2), \cdots, i'(l')), \cdots$, we use a convention that $(j(1)^*, j(2)^*, \cdots)$ denote the sequence $(i(1), i(2), \cdots, i(l), i'(1), i'(2), \cdots, i'(l'), \cdots)$

LEMMA 2. *If* $(j(1)^*, j(2)^*, \cdots, j(l)^*) \in \Sigma$ *and* $(j^*, j(1)^*, \cdots, j(l)^*) \notin \Sigma$, *then there exists an integer* m, $1 \leqq m \leqq l$ *s.t.* $(j^*, j(1)^*, \cdots, j(m-1)^*)$ *is reduced, and* $(j(1)^*, j(2)^*, \cdots, j(m)^*) \sim (j^*, j(1)^*, \cdots, j(m-1)^*)$. *Consequently* j^* *has the same length as* $j(m)^*$.

PROOF. Choose m so that $(j^*, j(1)^*, \cdots, j(m-1)^*) \in \Sigma$ and $(j^*, j(1)^*, \cdots, j(m)^*) \notin \Sigma$. Let $j(m)^* = (i(1), i(2), \cdots, i(p))$. If $l(r_i w) > l(w)$ for any $r_i \in R_j$ and $w = s_{j(1)} s_{j(2)} \cdots s_{j(m)}$, then $l(s_j w) = l(s_j) + l(w)$ by Corollary 3, i.e. $(j^*, j(1)^*, \cdots, j(m)^*) \in \Sigma$ contradicting our assumption. So $(i, j(1)^*, j(2)^*, \cdots, j(m)^*)$ is not reduced for some $r_i \in R_j$, and consequently for any $r \in R_j$. Since $(j^*, j(1)^*, \cdots, j(m-1)^*) \in \Sigma$, there must be some $r_i \in R_j$, s.t. $(i, j(1)^*, j(2)^*, \cdots, j(m-1)) \in \Sigma$. For this r_i, the condition $(\boldsymbol{C})$ implies that there exists an integer q, $1 \leqq q \leqq p$ s.t. $(i, j(1)^*, \cdots, j(m-1)^*, i(1), \cdots, i(q-1)) \sim (j(1)^*, j(2)^*, \cdots, j(m-1)^*, i(1), \cdots, i(q))$. That means $r_i = s_{j(1)} \cdots s_{j(m-1)} r_{i(1)} \cdots r_{i(q-1)} r_{i(q)} r_{i(q-1)} \cdots r_{i(1)} s_{j(m-1)} \cdots s_{j(1)}$. In particular,

$$r_i \in s_{j(1)} \cdots s_{j(m-1)} W_{j(m)} s_{j(m-1)} \cdots s_{j(1)} .$$

Since the right hand side is invariant by Γ, and Γ is transitive on R_j, the right hand side contains whole R_j and consequently contains W_j. Taking Γ-fixed points of both sides, we get, $s_j = s_{j(1)} \cdots s_{j(m-1)} s_{j(m)} s_{j(m-1)} \cdots s_{j(1)}$, i.e. $(j(1)^*, j(2)^*, \cdots, j(m)^*) \sim (j^*, j(1)^*, \cdots, j(m-1)^*)$. By our choice of m, $(j^*, j(1)^*, \cdots, j(m-1)^*)$ is reduced.

COROLLARY 1. *If* $(j(1)^*, j(2)^*, \cdots, j(l)^*) \notin \Sigma$, *then* $(j(1), j(2), \cdots, j(1)) \notin \Sigma^*$.

PROOF. Assume $(j(1)^*, j(2)^*, \cdots, j(l)^*) \notin \Sigma$, there exists n $2 \leqq n \leqq l$ s.t. $(j(n)^*, j(n+1)^*, \cdots, j(l)^*) \in \Sigma$ but $(j(n-1)^* j(n)^*, \cdots, j(l)^*) \notin \Sigma$. By Lemma 2, there exists m $n \leqq m \leqq l$ s.t. $(j(n)^*, j(n+1)^*, \cdots, j(m)^*) \sim (j(n-1)^*, j(n)^*, \cdots, j(m-1)^*)$ therefore $(j(1)^*, j(2)^*, \cdots, j(l)^*) \sim (j(1)^*, j(2)^*, \cdots, j(m-1)^*, j(m+1)^*, \cdots, j(l)^*)$. That means $s_{j(1)} s_{j(2)} \cdots s_{j(l)} = s_{j(1)} s_{j(2)} \cdots s_{j(m-1)} s_{j(m+1)} \cdots s_{j(l)}$, i.e. $(j(1), j(2), \cdots, j(l)) \notin \Sigma$.

COROLLARY 2. *If* $(j(1)^*, j(2)^*, \cdots, j(l)^*) \in \Sigma$, $(k(1)^*, k(2)^*, \cdots, k(n)^*) \in \Sigma$, *and* $(j(1)^*, j(2)^*, \cdots, j(l)^*) \sim (k(1)^*, k(2)^*, \cdots, k(n)^*)$, *then* $l = n$.

PROOF. Induction on n. Obvious for $n=0$. By Lemma 2, $(j(1)^*, j(2)^*, \cdots, j(l)^*) \sim (k(1)^*, j(1)^*, \cdots, j(m-1)^* j(m+1)^*, \cdots, j(l)^*) \sim (k(1)^*, k(2)^*, \cdots, k(n)^*)$. Drop out $k(1)^*$ and $(j(1)^*, \cdots, j(m-1)^*, j(m+1)^*, \cdots, j(l)^*) \sim (k(2)^*, k(3)^*, \cdots, k(n)^*)$. Since the right hand side of the above relation is reduced, and $j(m)^*$ and $k(l)^*$ has the same length, the left hand side is also reduced. By induction assumption $l-1 = n-1$, i.e. $l = n$.

COROLLARY 3. $(j(1)^*, j(2)^*, \cdots, j(l)^*) \in \Sigma$ *if and only if* $(j(1), j(2), \cdots, j(l)) \in \Sigma^*$.

PROOF. If part is Corollary 2. Let $w = s_{j(1)} s_{j(2)} \cdots s_{j(l)}$ and let $w = s_{k(1)} s_{k(2)} \cdots s_{k(m)}$ be a reduced expression of w w.r.t. R^*. $(k(1), k(2), \cdots, k(m)) \in \Sigma^*$ implies $(k(1)^*, k(2)^*, \cdots, k(m)^*) \in \Sigma$. By Corollary 2 $m = l$, i.e. $(j(1), j(2), \cdots, j(l)) \in \Sigma^*$.

THEOREM 3. $R^* = \{s_j ; j \in \boldsymbol{J}\}$ *is a good system of involutive generators of*

W^{Γ}.

PROOF. Assume $(j(1), j(2), \cdots, j(l)) \in \Sigma^*$ and $(j, j(1), \cdots, (j(l)) \notin \Sigma^*$. By Corollary 3, $(j(1)^*, j(2)^*, \cdots, j(l)^*) \in \Sigma$ and $(j^*, j(1)^*, \cdots, j(l)^*) \notin \Sigma$. Now our Theorem is immediate from Lemma 2.

Appendix

In this appendix, we intend to give more explicit structure of the group W^{Γ}, i.e. for any pair of j and $k \in \boldsymbol{J}$, we will determine the structure of $R_j \cup R_k$ and the action of Γ on it. For any pair of elements w, w' of W, let $m(w, w')$ denote the order of the element ww' in W. Since we already knew that W^{Γ} is generated by a good system of involutive generators s_j $(j \in \boldsymbol{J})$, the numbers $m(s_j, s_k)$ $j, k \in \boldsymbol{J}$ completely determine the structure of W^{Γ} as an abstract group.

LEMMA. *The group generated by* $\bigcup_{j \in J} R_j$ *is a finite group if and only if the group* W^{Γ} *is a finite group.*

PROOF. Only if part is obvious. Assume W^{Γ} is finite and let w be the element of the maximal length in W^{Γ} and $w = s_{j(1)} \cdots s_{j(n)}$ be a reduced expression of w by $s_{j(1)}, \cdots, s_{j(n)} \in R^*$. Since $(j(l)^*, \cdots, j(n)^*) \in \Sigma$ by Corollary 3 in § 2, we have $l(w) = l(s_{j(1)}) + \cdots + l(s_{j(n)})$. Since $s_{j(1)}$ is the element of the maximal length in $W_{j(1)}$, we have $l(rw) < l(w)$ for any $r \in R_{j(1)}$. But for any $j \in \boldsymbol{J}$, w has a reduced expression starting with s_j (Corollary 2. (iii) in § 1), and consequently we have $l(rw) < l(w)$ for any $r \in \bigcup_{j \in J} R_j$. Now the existence of such a w implies that the group generated by $\bigcup_{j \in J} R_j$ is finite (Corollary 1 in § 1).

COROLLARY. *For a pair of indices $j, k \in \boldsymbol{J}$,* let *$W_{j,k}$ denote the group generated by $R_j \cup R_k$. Then $m(s_j, s_k)$ is finite if and only if $W_{j,k}$ is a finite group.*

Assume $W_{j,k}$ is finite. Let $s_{j,k}$ denote the element of the maximal length in $W_{j,k}$, then $m(s_j, s_k) = 2\, l(s_{j,k})/(l(s_j) + l(s_k))$.

PROOF. The former part of our claim is immediate from Lemma. Assume $W_{j,k}$ is finite, then $(W_{j,k})^{\Gamma}$ is a dihedral group generated by s_j and s_k. Let $m = m(s_j, s_k)$, then $s_{j,k} = \overbrace{s_j s_k \cdots}^{m} = \overbrace{s_k s_j \cdots}^{m}$ is a reduced expression of $s_{j,k}$ by s_j and s_k. By Corollary 3 Lemma 2, $l(s_{j,k}) = \overbrace{l(s_j) + l(s_k) + \cdots}^{m} = \overbrace{l(s_k) + l(s_j) + \cdots}^{m}$. Hence m is odd only if $l(s_j) = l(s_k)$, and we have $l(s_{j,k}) = \frac{1}{2} m(l(s_j) + l(s_k))$. Q. E. D.

A subset S of R will be called connected if S is not the union of two mutually elementwise commutative subsets. A maximal connected subset of S will be called a connected component of S. Let S_α $(\alpha \in A)$ be all the connected components of $R_j \cup R_k$. Now the following four conditions are obviously mutually equivalent.

(I) R_j and R_k are not mutually elementwise commutative.

(II) There exists $\alpha \in A$ s. t. neither $R_j \cap S_\alpha$ nor $R_k \cap S_\alpha$ is vacant.

(III) For any $\alpha \in A$, neither $R_j \cap S_\alpha$ nor $R_k \cap S_\alpha$ is vacant.

(IV) Γ permutes $\{S_\alpha ; \alpha \in A\}$ transitively.

Let Γ_α be the isotropy subgroup of S_α in Γ, $\Gamma_\alpha = \{\gamma ; \gamma \in \Gamma, {}^\gamma S_\alpha \in S_\alpha\}$ and let $s_{j,\alpha}$ (resp. $s_{k,\alpha}$) be the element of the maximal length in the group generated by $R_j \cap S_\alpha$ (resp. $R_k \cap S_\alpha$).

THEOREM. (i) $s_j = \prod_\alpha s_{j,\alpha}$, $s_k = \prod_\alpha s_{k,\alpha}$ *and* $s_j s_k = \prod_\alpha (s_{j,\alpha} s_{k,\alpha})$, *where each product* $\prod_\alpha$ *is commutative.* (ii) *If* R_j *and* R_k *are mutually elementwise commutative, then* $m(s_j, s_k) = 2$. (iii) *If* R_j *and* R_k *are not mutually elementwise commutative, then* $m(s_j, s_k) = m(s_{j,\alpha}, s_{k,\alpha})$ *for any* $\alpha \in A$. (iv) $m(s_j, s_k)$ *is infinite if and only if the group* $W_{j,k}$ *generated by* $R_j \cup R_k$ *is infinite. In this case, the group generated by* $S_\alpha \cap (R_j \cup R_k)$ *is infinite for any* $\alpha \in A$. (v) *Assume* R_j *and* R_k *are not mutually elementwise commutative and* $W_{j,k}$ *is finite then the only possibilities are following;*

($\boldsymbol{D}_m$) Γ_α *is trivial,* $S_\alpha \cap R_j = \{r\}$, $S_\alpha \cap R_k = \{s\}$ *and* $m(s_j, s_k) = m(r, s) = m$.

(A_3) Γ_α *is of order two,* S_α *consists of three elements and* $m(s_j, s_k) = 4$

(A_4) Γ_α *is of order two,* S_α *consists of four elements and* $m(s_j, s_k) = 4$

(D_4) Γ_α *is either a cyclic group of order three or the full permutation group of three letters,* S_α *consists of four elements and* $m(s_j, s_k) = 6$.

In the first case S_α *generates a dihedral group. In the other three cases,* S_α *can be identified as the set of fundamental reflexions of the Weyl group of type* A_3 (A_4 *or* D_4 *respectively*) *and* Γ_α *is the group of symmetries of its Dynkin diagram.*

PROOF. (i) and (ii) are obvious from their definitions. (iii) By our remark (IV), for any $\beta \in A$, there exists $\gamma \in \Gamma$ s. t. $\gamma(S_\beta) = S_\alpha$. Since $s_{j,\beta}$ (resp. $s_{k,\beta}$) is the element of the maximal length, γ necessarily transforms $s_{j,\beta}$ (resp. $s_{k,\beta}$) to $s_{j,\alpha}$ (resp. $s_{k,\alpha}$), therefore $m(s_{j,\beta}, s_{k,\beta}) = m(s_{j,\alpha}, s_{k,\alpha}) = m(s_j, s_k)$. (iv) follows immediately from Corollary in the appendix and the above (iii). (v) We know that S_α is connected and that S_α generates a finite group. Such a system was completely classified (cf. Witt [**5**]). Γ_α is obviously transitive on $S_\alpha \cap R_j$ or $S_\alpha \cap R_k$. If Γ_α is trivial, then S_α consists of two elements and we have the case ($\boldsymbol{D}_m$). If Γ_α is not trivial, the existence of non-trivial symmetries restricts S_α to the type of A_n, D_n, or E_6. But except D_4, Γ_α is of order two, and we know that S_α has two orbits under Γ_α, thus n (= the cardinality of S_α) must be at most four.

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