J. Math. Soc. Japan
Vol. 19, No. 3, 1967

A singular flow with countable Lebesgue spectrum

By Gisiro MARUYAMA

(Received Jan. 31, 1967)

§ 1. Preliminaries.

There is given a real stationary process $\xi(t, \alpha)$, $t \in T \equiv (-\infty, \infty)$, $\alpha \in S$, on certain probability space $(S, \mathscr{F}, P)$, which is continuous in probability, i. e.

$$\lim_{t \to 0} P(|\xi(0, \alpha) - \xi(t, \alpha)| > \varepsilon) = 0, \tag{1.1}$$

for every $\varepsilon > 0$.

Consider a corresponding invariant measure μ and flow $\{S_t\}$ over R^T: (i) $x = (x_t, t \in T) \to S_\tau x = (x_{t+\tau}, t \in T)$ for any real τ, (ii) let $\lambda = (t_1, \cdots, t_n)$ be a subset of T, A a Borel set in R^λ, put

$$\tilde{A} = \{x : (x_{t_1}, \cdots, x_{t_n}) \in A\},$$

and define

$$\mu(\tilde{A}) = P((\xi_{t_1}, \cdots, \xi_{t_n}) \in A).$$

$\{S_t\}$ is a flow on $(R^T, \mathscr{B}, \mu)$, where $\mathscr{B}$ is the completion under μ of the σ-algebra generated by the cylinder sets $\tilde{A}$. When ξ satisfies (1.1) a Lebesgue subspace of R^T can be taken such that it is S_t-invariant. Define Ω to be the space of Lebesgue measurable real functions over T, then $\Omega \subset R^T$, the outer μ-measure of Ω is equal to 1, and Ω is a (strictly) S_t-invariant subspace of R^T. It is important to observe that Ω can be made into a complete metric separable space which endowed with μ becomes Lebesgue, and over which S_τ acts as a shift, $f(t) \to S_\tau f(t) = f(t+\tau)$, $f \in \Omega$ (c. f. [3], § 2—§ 4). S_t over (Ω, μ) is understood as a flow generated by ξ. One can also define, as its coordinate representation, a stationary process $(x_t(\omega), \omega \in \Omega, -\infty < t < \infty)$ over (Ω, μ), which is equivalent (in probability law) to the given $\xi(t)$ [3].

Suppose that $\xi(t)$ has the finite second moment, $E\xi(t) = 0$, and let its correlation function and spectral measure be $R(t)$, $\sigma(d\lambda)$,

$$R(t) = \int_{-\infty}^{\infty} e^{i\lambda t} \sigma(d\lambda). \tag{1.2}$$

Correspondingly to (1.2), x_t can be put in the form

$$x_t(\omega) = \int_{-\infty}^{\infty} e^{i\lambda t} \beta(d\lambda),\ E|\beta(d\lambda)|^2 = \sigma(d\lambda). \tag{1.3}$$

Let $L^2(\mu)$ be the set of square-integrable complex functions over (Ω, μ), then when A ranges over all Borel subsets of R, the system $\{\beta(A)\}$ generates $L^2(\mu)$, and so do the Baire functions of $\beta(A)$ the space of μ-measurable functions.

From now on assume further that ξ is Gaussian. Then $\beta(A)$ is complex Gaussian, and fundamental concepts in the following analysis are Itō's multiple Wiener integrals [2] with respect to $\beta(d\lambda)$. The integrals, roughly speaking, are polynomials constructed on the products

$$\beta(d\lambda_1)\beta(d\lambda_2)\cdots\beta(d\lambda_n), \qquad 1\leqq n<\infty. \tag{1.4}$$

When we make a summation of such products, two kinds of summation over "diagonals" should be distinguished, symbolically writing

(a) $$\sum_{\substack{\lambda_k+\lambda_l=0\\ -a\leqq\lambda_k,\ \lambda_l\leqq a}} \beta(d\lambda_k)\beta(d\lambda_l)=\sum_{-a\leqq\lambda_k\leqq a}|\beta(d\lambda_k)|^2,$$

(b) $$\sum_{\substack{\lambda_k-\lambda_l=0\\ a\leqq\lambda_k,\ \lambda_l\leqq b}} \beta(d\lambda_k)\beta(d\lambda_l)=\sum_{a\leqq\lambda_k\leqq b}(\beta(d\lambda_k))^2.$$

(a) is asymptotically equal to $\sum_{-a\leqq\lambda_k\leqq a}\sigma(d\lambda_k)$, whereas (b) to zero. So that when we speak of the products (1.4) we may impose the restriction that $\lambda_i\pm\lambda_j\neq 0$, $1\leqq i\neq j\leqq n$, and accordingly the polynomial

$$I_n(f)=\int f(\lambda_1,\cdots,\lambda_n)\beta(d\lambda_1)\cdots\beta(d\lambda_n) \tag{1.5}$$

means the multiple integral

$$\int\cdots\int_{\lambda_i\pm\lambda_j\neq 0,\ 1\leqq i\neq j\leqq n} f(\lambda_1,\cdots,\lambda_n)\beta(d\lambda_1)\cdots\beta(d\lambda_n).$$

Write now $\sigma^n(d\lambda)=\sigma(d\lambda_1)\times\cdots\times\sigma(d\lambda_n)$, $\lambda=(\lambda_1,\cdots,\lambda_n)$. The integral (1.5) is well defined for $f\in L^2(\sigma^n(d\lambda))$, and the followings are basic to computation with polynomials [2].

1° $(I_n(f), I_n(g))$

$$\begin{aligned} &=\int_{R^n} f(\lambda)\overline{g(\lambda)}\sigma^n(d\lambda) \quad \textit{if} \quad m=n,\\ &=0 \quad \textit{if} \quad m\neq n, \end{aligned} \tag{1.6}$$

where $(\xi,\eta)=E(\xi\bar{\eta})$, $\xi,\eta\in L^2(\mu)$.

Define a function

$$Z_n(f,\lambda)=\int_{\lambda_1+\cdots+\lambda_n\leqq\lambda} f(\lambda_1,\cdots,\lambda_n)\beta(d\lambda_1)\cdots\beta(d\lambda_n)$$

and corresponding orthogonal random measure of order n

$$Z_n(f,B)=\int f(\lambda)\chi_B(\lambda_1+\cdots+\lambda_n)\beta(d\lambda_1)\cdots\beta(d\lambda_n),$$

where χ_B is the characteristic function of a Borel subset B of R. The random measures of different orders are orthogonal each other, i. e.

$$(Z_n(f, B), \overline{Z_m}(g, C))=0$$

for $m \neq n$ and any Borel sets B, C.

2° *For a Borel set B*

$$\begin{aligned}\sigma_n(f, B) &\equiv \| Z_n(f, B)\|^2 \\ &= \int_B \sigma^{n*}(dx)\int_{R^n} |f(x)|^2 \sigma^n(d\lambda_1 \cdots d\lambda_n | \lambda_1 + \cdots + \lambda_n = x)\end{aligned} \tag{1.7}$$

where $\sigma^{n*}(d\lambda)$ is the n-th convolution of $\sigma(d\lambda)$, and $\sigma^n(d\lambda_1 \cdots d\lambda_n \mid \lambda_1 + \cdots + \lambda_n = x)$ is the measure induced by $\sigma^n(d\lambda)$ on the hyper-plane $\lambda_1 + \cdots + \lambda_n = x$.

One has

$$\sigma_n(f, d\lambda) \prec \sigma^{n*}(d\lambda), \tag{1.8}$$

and if $|f|$ is bounded away from zero, then

$$\sigma_n(f, d\lambda) \sim \sigma^{n*}(d\lambda), \tag{1.9}$$

where $\tau \prec \sigma$ means that the measure τ is absolutely continuous with respect to the measure σ, whereas $\tau \sim \sigma$ does $\tau \prec \sigma$ and $\sigma \prec \tau$.

Let $\tilde{S}_t$ be the one-parameter group of unitary operators on $L^2(\mu)$ generated by the flow S_t, $\tilde{S}_t h(\omega) = h(S_t\omega)$, $h \in L^2(\mu)$, and $H(h)$ be the closed linear manifold spanned by $(\hat{S}_t h, -\infty < t < \infty)$.

3° *If $h = I_n(f)$, $f \in L^2(\sigma^n)$, then*

$$H(h) = \left\{\int_{-\infty}^{\infty} \varphi(\lambda) Z_n(f, d\lambda) : \varphi \in L^2(\sigma_n(f, d\lambda))\right\}.$$

Let $k = I_n(g)$, $g \in L^2(\sigma^n)$, then a necessary and sufficient condition that $H(h) \perp H(k)$ ($H(h)$ is orthogonal with $H(k)$) is that $f \perp g$ as L^2-functions on $\lambda_1 + \cdots + \lambda_n = x$ under the measure $\sigma^n(d\lambda_1 \cdots d\lambda_n | \lambda_1 + \cdots + \lambda_n = x)$, for almost all x with respect to $\sigma^{n*}(dx)$.

Proof. Since

$$\begin{aligned}T_t h &= \int f(\lambda) e^{i(\lambda_1 + \cdots + \lambda_n)t} \beta(d\lambda_1) \cdots \beta(d\lambda_n) \\ &= \int_{-\infty}^{\infty} e^{i\lambda t} Z_n(f, d\lambda),\end{aligned}$$

$H(h) \perp H(k)$ if and only if

$$\int_{-\infty}^{\infty} \varphi(\lambda) Z_n(f, d\lambda) \perp \int_{-\infty}^{\infty} \psi(\lambda) Z_n(g, d\lambda),$$

i. e.

$$\int_{-\infty}^{\infty} \varphi(\lambda)\overline{\psi(\lambda)} E(Z_n(f, d\lambda)\overline{Z_n(g, d\lambda)}) = 0 \tag{1.10}$$

for every $\varphi \in L^2(\sigma_n(f, d\lambda))$ and $\psi \in L^2(\sigma_n(g, d\lambda))$. (1.10) is equivalent to

$$E(Z_n(f, B)\overline{Z_n(g, B)}) = 0 \tag{1.11}$$

for every Borel set B. Since $Z_n(f, B)$ is linear in f, in view of (1.7), (1.11) is in turn equivalent to

$$\int_B \sigma^{n*}(dx) \int_{R^n} f(x)\overline{g(x)}\sigma^n(d\lambda_1 \cdots d\lambda_n \mid \lambda_1 + \cdots + \lambda_n = x) = 0$$

for every B, which proves the requested statement.

Incidentally we may notify that (1.11) is equivalent to the orthogonality between the random measures:

$$(Z_n(f, B), \overline{Z_n(g, C)}) = 0 \tag{1.12}$$

for any Borel sets B and C.

Let us define

L_0 = space of complex numbers,

L_n = closed linear manifold spanned by

$$\{I_n(f) : f \in L^2(\sigma^n)\}, \qquad n \geqq 1.$$

If

$$f \in L^2(\sigma), \qquad |f(\lambda)| > 0,$$

almost everywhere $(\sigma(d\lambda))$, then

$$L_1 = H(h), \qquad h = I_1(f).$$

Proposition 3° enables us to illustrate Girsanov's construction of a Gaussian process [**1**], for which $\tilde{S}_t$ has a continuous simple spectrum.

§ 2. The theorem.

Our main purpose is to prove the

THEOREM. *There exists a real Gaussian stationary process with zero entropy whose certain factor flow has a countable Lebesgue spectrum*)*.

PROOF. Given an arbitrary $\varepsilon > 0$, as in the time discrete case [**5**], there exists a symmetric continuous singular measure $\sigma(d\lambda)$ on R, $\sigma(R) < \infty$, such that

$$R(t) = \int_{-\infty}^{\infty} e^{i\lambda t}\sigma(d\lambda) = O(|t|^{-\frac{1}{2}+\varepsilon}), \qquad |t| \to \infty.$$

Suppose $0 < \varepsilon < 1/8$, then $R^2 \in L^2(-\infty, \infty)$, since

$$R^2(t) = \int_{-\infty}^{\infty} e^{i\lambda t}\sigma^{2*}(d\lambda) = O(|t|^{-1+2\varepsilon}),$$

* Shortly after sending the manuscript to the editor, the author was pointed out by H. Totoki that D. Newton and W. Parry obtained a similar result in Ann. Math. Statist., 4 (1966), 1528–1533, for the spectrum of an automorphism.

and its transform

$$p(\lambda)=\frac{1}{2\pi}\int_{-\infty}^{\infty}R^2(t)e^{-i\lambda t}dt$$

satisfies

$$\sigma^{2*}(\mu)-\sigma^{2*}(\lambda)=\int_{-\infty}^{\infty}R^2(t)\frac{e^{-i\mu t}-e^{-i\lambda t}}{-it}dt=\int_{\lambda}^{\mu}p(x)dx\,, \tag{2.1}$$

i.e. $\sigma^{2*}(d\lambda)$ is absolutely continuous. We shall prove that the Gaussian process ξ in § 1 with this $\sigma(d\lambda)$ as its spectral measure is the required one; the entropy of the corresponding flow is zero [**4**].

Define an infinite-dimensional stationary process with complex components

$$\eta_t=\left\{\eta_t^{(n)}=\iint_{B_n}e^{i(\lambda_1+\lambda_2)t}\beta(d\lambda_1)\beta(d\lambda_2),\ 1\leqq n<\infty\right\}, \tag{2.2}$$

where B_n runs over all closed intervals of the form $B_n=\{a\leqq x\leqq b,\ c\leqq y\leqq d\}$ with rationals a, b, c, d. As in § 1, the corresponding flow T_t is built on a Lebesgue space Ω, formed by Lebesgue measurable complex functions $f(t)$, $-\infty<t<\infty$, [**3**]. This flow is a factor of S_t. The unitary operators $\tilde{T}_t$ generated by T_t are equivalent to $\hat{S}_t$ generated by S_t over the Hilbert space

$$H_\eta=\text{closed linear manifold determined by } \eta_t^{(n)}\,,$$
$$1\leqq n<\infty\,,\qquad -\infty<t<\infty\,.$$

By the definition of η_t, H_η is nothing but the closure of the linear space of all even degree polynomials in $\beta(d\lambda)$, i.e.

$$H_\eta=\sum_{n=0}^{\infty}\oplus L_{2n}\,. \tag{2.3}$$

For every $n\geqq 1$, there exists a sequence of $L^2(\sigma^{2n})$— functions $f_{n0}, f_{n1}, f_{n2}, \cdots$ $(f_{n0}\equiv 1)$ such that

$$L_{2n}=\sum_{k=0}^{\infty}\oplus H(h_{nk}),\ h_{nk}=I_{2n}(f_{nk})\,, \tag{2.4}$$

and from § 1 one obtains an orthogonal system of random measures

$$Z_{2n}(f_{nk}, d\lambda)\,,\qquad 1\leqq n<\infty\,,\qquad 0\leqq k<\infty\,,$$

which satisfy

$$\begin{aligned}\sigma_{2n}(f_{n0}, d\lambda)&=\sigma^{2n*}(d\lambda)\prec d\lambda\,,\\ \sigma_{2n}(f_{nk}, d\lambda)&\prec\sigma^{2n*}(d\lambda)\prec d\lambda\,,\qquad 1\leqq k<\infty\,.\end{aligned} \tag{2.5}$$

Since $\sigma^{2*}(d\lambda)=p(\lambda)d\lambda$, if we take $p_0(\lambda)=\min(1, p(\lambda))$, we have

$$\sigma^{4*}(d\lambda)/d\lambda\geqq\int_{-\infty}^{\infty}p_0(\lambda-\mu)p_0(\mu)d\mu\,.$$

The right-hand member is continuous in λ, and for $\lambda=0$ it is equal to

$$\int_{-\infty}^{\infty} p_0(-\mu)p_0(\mu)d\mu=\int_{-\infty}^{\infty}(p_0(\mu))^2d\mu>0\,.$$

So that there exists a constant $c_0>0$ such that

$$\sigma^{4*}(d\lambda)/d\lambda\geqq c_0>0$$

almost everywhere around the origin, say for $|\lambda|\leqq\lambda_0$, $\lambda_0>0$, and similarly

$$\sigma^{4n*}(d\lambda)/d\lambda>0 \quad \text{for} \quad |\lambda|\leqq n\lambda_0\,. \tag{2.6}$$

By the carrier of $Z_{2n}(f_{nk}, d\lambda)$ is meant that of $\sigma_{2n}(f_{nk}, d\lambda)$, the complement of the maximal open set G such that $\sigma_{2n}(f_{nk}, G)=0$, and will be denoted as $\text{Car}\, Z_{2n}(f_{nk}, d\lambda)$. From $Z_{2n}(f_{n0}, d\lambda)$, $1\leqq n<\infty$, we first define an orthogonal set of random measures

$$\tilde{Z}_n(d\lambda)\,, \qquad 1\leqq n<\infty\,, \quad \text{with} \quad \text{Car}\,\tilde{Z}_n=(-\infty,\infty)\,.$$

To do this make up the defect of $Z_2(f_{10}, d\lambda)$ by $Z_4(f_{20}, d\lambda)$, i.e. define a measure

$$Z^{(1)}(A)=Z_2(f_{10}, A)+Z_4(f_{20}, A\cap G)\,, \qquad A \text{ Borel}\,, \tag{2.7}$$

where G is the complement of $\text{Car}\, Z_2(f_{10}, d\lambda)$. Then by (2.6)

$$\text{Car}\, Z^{(1)}\supset[-\lambda_0, \lambda_0]\,.$$

There remains also the residual measure $Z_4'(f_{20}, d\lambda)$ of $Z_4(f_{20}, d\lambda)$, where $Z_4'(f_{20}, A)=Z_4(f_{20}, A)-Z_4(f_{20}, G\cap A)$. Next, in the same way, make up $Z^{(1)}$ by means of $Z_6(f_{30}, d\lambda)$ to have a measure $Z^{(2)}$, then by (2.6)

$$\text{Car}\, Z^{(2)}\supset[-\lambda_0, \lambda_0]\,.$$

Continuing this way one has

$$\tilde{Z}_1(A)=\lim_{n\to\infty} Z^{(n)}(A)\,.$$

$\tilde{Z}_1$ is an orthogonal random measure with $\text{Car}\,\tilde{Z}_1=(-\infty,\infty)$. After these procedures, there remain residual measures $Z_{2n}'(f_{n0}, d\lambda)$ of $Z_{2n}(f_{n0}, d\lambda)$, $n\geqq 2$, with

$$\text{Car}\, Z_{4n}'\supset[(n-1)\lambda_0, (n-1)\lambda_0]\,, \qquad n\geqq 2\,.$$

So that, we can apply the same procedure as above to $Z_{2n}'(f_{n0}, d\lambda)$, $n\geqq 2$. One obtains an orthogonal random measure

$$\tilde{Z}_2(d\lambda) \quad \text{with} \quad \text{Car}\,\tilde{Z}_2=(-\infty,\infty)\,.$$

Continuing this way we get a requested orthogonal set of random measures $\tilde{Z}_n$, $1\leqq n<\infty$.

Rearrange $\tilde{Z}_n$, $1\leqq n<\infty$, into a double sequence $\tilde{Z}_{mn}$, $1\leqq m$, $n<\infty$, and $Z_{2n}(f_{nk}, d\lambda)$, $1\leqq n$, $k<\infty$, into a simple sequence $\tilde{Z}_{m0}(d\lambda)$, $1\leqq m<\infty$, and then put them together into the array (orthogonal set of ramdom measures)

$$\tilde{Z}_{m,n}\,, \qquad 0\leqq n<\infty\,, \qquad m=1, 2, \cdots\,. \tag{2.8}$$

Now apply the making up procedures to each row in (2.8), getting a sequence of random measures with common carrier $(-\infty, \infty)$. Then collecting these we finally obtain an orthogonal set of random measures $Z_n(d\lambda)$, $1 \leqq n < \infty$, with $\mathrm{Car}\, Z_n = (-\infty, \infty)$, $\sigma_n(d\lambda) = \| Z_n(d\lambda) \|^2 \prec d\lambda$; therefore $\sigma_n(d\lambda) \sim d\lambda$. By the above construction $h \in H_\eta \ominus L_0$ is represented as

$$h = \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} \varphi_n(\lambda) Z_n(d\lambda), \qquad \varphi_n \in L^2(\sigma_n),$$

$$\| h \|^2 = \sum_{n=1}^{\infty} \| \varphi_n \|_n^2, \qquad \| \varphi_n \|_n^2 = \int_{-\infty}^{\infty} | \varphi_n(\lambda) |^2 \sigma_n(d\lambda),$$

and

$$\hat{S}_t h = \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} \varphi_n(\lambda) e^{i\lambda t} Z_n(d\lambda).$$

Therefore $\hat{S}_t$ is isomorphic with unitary operators

$$\{\varphi_n(\lambda),\ 1 \leqq n < \infty\} \to \{\varphi_n(\lambda) e^{i\lambda t},\ 1 \leqq n < \infty\}$$

over

$$\sum_{n=1}^{\infty} \oplus L^2(\sigma_n(d\lambda)), \sigma_n(d\lambda) \sim d\lambda.$$

This proves the theorem.

Tokyo University of Education

References

[1] I. V. Girsanov, On spectra of dynamical systems generated by stationary Gaussian processes, Dokl. Akad. Nauk SSSR, **119** (1958), 851-853.

[2] K. Itō, Complex multiple Wiener integral, Japan. J. Math., **22** (1952), 63-86.

[3] G. Maruyama, Transformations of flows, J. Math. Soc. Japan, **18** (1966), 303-330.

[4] M. S. Pinsker, Dynamical systems with completely positive or zero entropy, Dokl. Akad. Nauk SSSR, **133** (1960), 1025-1026.

[5] A. C. Schaeffer, Fourier-Stieltjes coefficients of a function of bounded variation, Amer. J. Math., **61** (1939), 934-940.