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Totally umbilical submanifolds of a Kaehlerian manifold

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Introduction. It is well known fact that an $(n-1)$-dimensional totally umbilical submanifold with non-zero mean curvature of an n-dimensional Euclidean space is isometric with a sphere.

In the previous paper [3], making use of Obata's theorem, the author proved that an $(n-1)$-dimensional complete, simply connected totally umbilical submanifold with non-zero constant mean curvature of an n-dimensional locally product Riemannian manifold is isometric with a sphere.

Now it is natural to try to solve the similar problem in another Riemannian manifold.

On the other hand there are many papers studying submanifolds of an almost complex manifold. However, most of them deals with an invariant submanifold with respect to the almost complex structure. So, this submanifold is necessarily a minimal submanifold. Thus it is expected to study another submanifold of an almost complex manifold.

In this paper, from the above two points of view, the author discusses a totally umbilical submanifold of a Kaehlerian manifold and prove that under some conditions, a $2n$-dimensional totally umbilical submanifold of a $(2n+2)$-dimensional Kaehlerian manifold is isometric with a sphere of $(2n+1)$-dimensional Euclidean space. To prove this we find a function satisfying a certain differential equation. The discovery of such a function enables us to use a famous theorem about an infinitesimal concircular transformation. Thus we can prove the above mentioned theorem completely.

In § 1 we state general properties of $2n$-dimensional submanifold of a $(2n+2)$-dimensional Kaehlerian manifold and in § 2 we give preliminaries of the theory of an infinitesimal concircular transformation.

Finally in § 3 we prove the above theorem under the preparation of some properties of the second fundamental tensor.

§ 1. Submanifolds of a Kaehlerian manifold.

An almost complex manifold $\tilde{M}$ is a differentiable manifold on which there

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exists a tensor field J of (1.1)-type satisfying the condition

$$J^2 = -I, \tag{1.1}$$

where I denotes the unit tensor field. Such a manifold M is necessarily orientable and even-dimensional.

In an almost complex manifold $\tilde{M}$, there always exists a positive definite Riemannian metric tensor $\tilde{g}$ which is Hermitian, that is

$$\tilde{g}(J\tilde{X}, J\tilde{Y}) = \tilde{g}(\tilde{X}, \tilde{Y}), \tag{1.2}$$

for all $\tilde{X}, \tilde{Y} \in T(\tilde{M})$, where $T(\tilde{M})$ denotes the tangent bundle of $\tilde{M}$.

Replacing $\tilde{Y}$ by $J\tilde{Y}$ in (1.2), we get easily

$$\tilde{g}(J\tilde{X}, \tilde{Y}) = -\tilde{g}(\tilde{X}, J\tilde{Y}). \tag{1.3}$$

An almost complex manifold with the Riemannian metric satisfying (1.2) is called an almost Hermitian manifold.

A Kaehlerian manifold is an almost Hermitian manifold in which

$$\tilde{\nabla}_{\tilde{X}} J = 0, \tag{1.4}$$

for any $\tilde{X} \in T(\tilde{M})$, where $\tilde{\nabla}_{\tilde{X}}$ denotes the covariant differentiation with respect to the Riemannian metric $\tilde{g}$.

Let M be a connected orientable real differentiable manifold of dimension $2n$ which is a submanifold of a $(2n+2)$-dimensional Kaehlerian manifold $\tilde{M}$, that is, there exists a differentiable mapping $\phi: M \to \tilde{M}$ whose differential $d\phi: T_p(M) \to T_{\phi(p)}(\tilde{M})$ is one-to-one at each point of M, where $T_p(M)$ and $T_{\phi(p)}(M)$ denote the tangent space of M at p and the tangent space of $\tilde{M}$ at $\phi(p)$ respectively.

A Riemannian metric g is naturally induced on M by the immersion ϕ in such a way that $g(X, Y) = \tilde{g}(d\phi(X), d\phi(Y))$. In order to simplify the presentation we identify, for each $p \in M$, the tangent space $T_p(M)$ with $d\phi(T_p(M)) \subset T_{\phi(p)}(\tilde{M})$ by means of $d\phi$.

A vector in $T_{\phi(p)}(\tilde{M})$ which is orthogonal, with respect to $\tilde{g}$, to the subspace $d\phi(T_p(M))$ is said to be normal to M at p. Let U be a coordinate neighbourhood of M, in which there exist two fields of unit normal vectors to M which are mutually orthogonal at each point of U. We denote these two unit normal vector fields by C and D. Then we have

$$\begin{cases} \tilde{g}(X, C) = \tilde{g}(X, D) = \tilde{g}(C, D) = 0, \\ \tilde{g}(C, C) = \tilde{g}(D, D) = 1, \end{cases} \tag{1.5}$$

where and throughout the paper X, Y denote either tangent vector fields to M or vectors tangent to M at a point $p \in M$.

If X and Y are tangent to M, we can write

(1.6) $$\tilde{\nabla}_{d\phi(X)} d\phi(Y) = \nabla_X Y + h(X, Y)C + k(X, Y)D\,,$$

where $\nabla_X Y$ denotes the components of $\tilde{\nabla}_{d\phi(X)} d\phi(Y)$ tangent to M.

It is easily verified that $\nabla_X Y$ is identical with the covariant differentiation of Y with respect to the induced Riemannian metric g. Thus we write (1.6) as

(1.7) $$\nabla_X Y = \tilde{\nabla}_X Y - h(X, Y)C - k(X, Y)D\,,$$

by means of the above identification.

On the other hand the identity $\tilde{g}(C, C) = 1$ implies that $\tilde{g}(\tilde{\nabla}_X C, C) = 0$ for any tangent vector X and so we can write

(1.8) $$\tilde{\nabla}_X C = -A(X) + L(X)D\,,$$

where $A(X)$ is tangent component of $\tilde{\nabla}_X C$ to M and L is the connection form of the normal bundle to M. In the same way, from $\tilde{g}(D, D) = 1$ and $\tilde{g}(C, D) = 0$, we have

(1.9) $$\tilde{\nabla}_X D = -A'(X) - L(X)C\,.$$

The curvature form Ω of the normal bundle to M is given by

(1.10) $$\Omega(X, Y) = \tilde{\nabla}_Y L(X) - \tilde{\nabla}_X L(Y) - L([Y, X])\,,$$

and when $\Omega(X, Y) = 0$ for any $X, Y \in T(M)$, the connection of the normal bundle to M is said to be flat.

PROPOSITION 1.1. *Let $h(X, Y)$, $k(X, Y)$, $A(X)$ and $A'(X)$ be as above. Then we have*

(1.11) $$h(X, Y) = g(A(X), Y)$$

(1.12) $$k(X, Y) = g(A'(X), Y)\,.$$

PROOF. From $\tilde{g}(Y, C) = 0$, we get

(1.13) $$\tilde{g}(\tilde{\nabla}_X Y, C) + \tilde{g}(Y, \tilde{\nabla}_X C) = 0\,.$$

Substituting (1.6) and (1.8) into the above equation, we get

$$h(X, Y) = \tilde{g}(Y, A(X)) = g(A(X), Y)\,.$$

In the same way, we get by (1.6), (1.9) and (1.13),

$$k(X, Y) = g(A'(X), Y)\,.$$ Q. E. D.

Now consider the transform JX of X by J. Then we can put

(1.14) $$JX = (JX)^T + \xi(X)C + \eta(X)D\,.$$

The coefficients $\xi(X)$ and $\eta(X)$ are given by

(1.15) $$\xi(X)=\tilde{g}(JX, C),$$

(1.16) $$\eta(X)=\tilde{g}(JX, D).$$

The transform JC of C by J and JD of D by J are respectively perpendicular to C and D, and we get easily

(1.17) $$JC=(JC)^{T}+fD,$$

(1.18) $$JD=(JD)^{T}-fC,$$

where

(1.19) $$f=\tilde{g}(JC, D).$$

The function f seems to depend of the choice of mutually orthogonal unit vectors to the submanifold M. However we state the

LEMMA 1.2. *The function f is independent of the choice of mutually orthogonal unit normal vectors to the submanifold M.*

PROOF. Let C' and D' be mutually orthogonal unit normal vectors to the submanifold M. Then they can be expressed as linear combinations of C and D. So, we can write

(1.20) $$\begin{cases} C'=l_1C+l_2D, \\ D'=m_1C+m_2D. \end{cases}$$

Thus we get

$$\begin{aligned} f' &= \tilde{g}(JC', D')=\tilde{g}(l_1JC+l_2JD, m_1C+m_2D) \\ &= l_1m_2\tilde{g}(JC, D)+l_2m_1\tilde{g}(JD, C) \\ &= (l_1m_2-l_2m_1)\tilde{g}(JC, D)=(l_1m_2-l_2m_1)f, \end{aligned}$$

because of (1.3) and (1.19). On the other hand the transformation (1.20) being special orthogonal transformation, we get $l_1m_2-l_2m_1=1$. This shows that $f=f'$. This completes the proof.

In the discussions of the paragraph we use the vector fields defined on some neighbourhood U of $\phi(p)\in\tilde{M}$. Let U' be another neighbourhood of $\phi(p)\in\tilde{M}$. Then we have, at each point of $U\cap U'$, two orthonormal frames $\{X_1, \cdots, X_{2n}, C, D\}$ and $\{Y_1, \cdots, Y_{2n}, C', D'\}$, with respect to U and U' respectively. However, at $\phi(q)\in U\cap U'$, C' and D' being expressed as linear combinations of C and D, from Lemma 1.2, we know that the function f is identical in both U and U'. Thus f is globally defined function over M.

Now consider the covariant derivatives of $(JC)^{T}$ and $(JD)^{T}$ in the direction of $Y\in T(M)$. Differentiating $(JC)^{T}$ covariantly and making use of (1.17), we have

$$\begin{aligned}\nabla_Y(JC)^T &= \nabla_Y(JC-fD)\\ &= \tilde{\nabla}_Y(JC-fD)-h(Y, JC-fD)C-k(Y, JC-fD)D\\ &= J\tilde{\nabla}_Y C-(\tilde{\nabla}_Y f)D-f\tilde{\nabla}_Y D-h(Y, JC-fD)C-k(Y, JC-fD)D\,.\end{aligned}$$

Substituting (1.8) and (1.9) into the above, we get

$$\begin{aligned}\nabla_Y(JC)^T = &-JA(Y)+fA'(Y)+L(Y)JD+fL(Y)C\\ &-h(Y, JC-fD)C-(\tilde{\nabla}_Y f)D-k(Y, JC-fD)D\,.\end{aligned} \tag{1.21}$$

In entirely the same way, we have

$$\begin{aligned}\nabla_Y(JD)^T = &-JA'(Y)-fA(Y)+L(Y)JC+(\tilde{\nabla}_Y f)C-h(Y, JD+fC)C\\ &+fL(Y)D-k(Y, JD+fC)D\,.\end{aligned} \tag{1.22}$$

§ 2. Infinitesimal concircular transformations.

Let M be an m-dimensional Riemannian manifold with positive definite Riemannian metric g. An infinitesimal transformation X of M is called a gradient vector field and is denoted by $\operatorname{grad} f$, if there exists a differentiable function f satisfying

$$g(X, Y)=df(Y)\,, \tag{2.1}$$

for any vector field Y on M.

PROPOSITION 2.1. *For a gradient vector field Z, we have*

$$g(\nabla_X Z, Y)=g(\nabla_Y Z, X)\,, \tag{2.2}$$

where X and Y are any vector fields on M.

PROOF. Since Z is a gradient vector field there exists a differentiable function f which satisfies (2.1). So we have

$$\begin{aligned}g(\nabla_X Z, Y)-g(\nabla_Y Z, X) &= \nabla_X(g(Z, Y))-g(Z, \nabla_X Y)-\nabla_Y(g(Z, X))+g(Z, \nabla_Y X)\\ &= \nabla_X(df(Y))-\nabla_Y(df(X))-g(Z, \nabla_X Y-\nabla_Y X)\\ &= \nabla_X(Yf)-\nabla_Y(Xf)-df(\nabla_X Y-\nabla_Y X)\\ &= X(Yf)-Y(Xf)-(\nabla_X Y-\nabla_Y X)f\,.\end{aligned}$$

On the other hand we have denoted by ∇ the Riemannian connection and so it is torsionless. This shows that

$$\nabla_X Y-\nabla_Y X=[X, Y]\,. \tag{2.3}$$

Substituting (2.3) into the above equation, we have (2.2). Q. E. D.

An infinitesimal transformation Z of M is called an infinitesimal conformal transformation, if it satisfies the equation

$$(2.4)\qquad (\mathcal{L}(Z)g)(X, Y)=2\phi g(X, Y),$$

for any vector fields X and Y on M, where $\mathcal{L}(Z)$ denotes the operator of Lie derivatives with respect to Z and ϕ is a differentiable function defined over M. If an infinitesimal conformal transformation is a gradient vector field the transformation is called an infinitesimal concircular transformation.

It is well known facts[1] that an infinitesimal concircular transformation is an infinitesimal conformal transformation preserving geodesic circles invariant and that any infinitesimal conformal transformation of an Einstein space is an infinitesimal concircular transformation.

PROPOSITION 2.2. *In order that a gradient vector field* grad f *be an infinitesimal concircular transformation, it is necessary and sufficient that the following condition be valid.*

$$(2.5)\qquad g(\nabla_X \operatorname{grad} f, Y)=\phi g(X, Y),$$

where X and Y are any vector fields on M.

PROOF. By means of the definition of Lie derivatives, for any vector field Z, we have

$$\begin{aligned}(\mathcal{L}(Z)g)(X, Y)&=\nabla_Z(g(X, Y))-g([Z, X], Y)-g(X, [Z, Y])\\ &=g(\nabla_Z X, Y)+g(X, \nabla_Z Y)-g([Z, X], Y)-g(X, [Z, Y]).\end{aligned}$$

This can be rewritten as

$$(2.6)\qquad (\mathcal{L}(Z)g)(X, Y)=g(\nabla_X Z, Y)+g(X, \nabla_Y Z),$$

because of (2.3). So, if grad f is an infinitesimal concircular transformation we have (2.5) because of (2.4) and Proposition 2.1.

Conversely if grad f satisfies (2.5) we have

$$(\mathcal{L}(\operatorname{grad} f)g)(X, Y)=2(\nabla_X \operatorname{grad} f, Y)=2\phi g(X, Y),$$

by virtue of (2.4), (2.6) and Proposition 2.1. This completes the proof.

In (2.5), if the function ϕ is of the form $\phi=-cf$ with positive constant coefficient c, an infinitesimal concircular transformation is called an infinitesimal special concircular transformation[2]. As to a Riemannian manifold admitting an infinitesimal special concircular transformation we know the following

THEOREM 2.3.[3] *Let M^m be a complete, connected Riemannian manifold of dimension $m(\geqq 2)$. In order for M^m to admit a non-constant function f with*

$$(2.6)\qquad g(\nabla_X \operatorname{grad} f, Y)=-cfg(X, Y),$$

1) K. Yano, [**6**].
2) Y. Tashiro, [**5**].
3) M. Obata, [**2**], Y. Tashiro, [**5**].

for any X and Y, it is necessary and sufficient that M^m be isometric with a sphere S^m of radius $1/\sqrt{c}$ in the Euclidean $(m+1)$-space.

§ 3. Totally umbilical submanifolds.

Let M be a $2n$-dimensional submanifold of a $(2n+2)$-dimensional Riemannian manifold $\tilde{M}$. The mean curvature vector field H of M in $\tilde{M}$ is defined by

$$H=\alpha C+\beta D\,, \tag{3.1}$$

where $\alpha=\frac{1}{2n}\sum_{i=1}^{2n}h(E_i, E_i)$, $\beta=\frac{1}{2n}\sum_{i=1}^{2n}k(E_i, E_i)$ and $\{E_1, \cdots, E_{2n}\}$ is an orthonormal frame tangent to M. If the mean curvature vector field vanishes identically the submanifold M is called a minimal submanifold. The mean curvature of M in $\tilde{M}$ is the magnitude of the mean curvature vector field, that is,

$$\mu=g(H, H)\,. \tag{3.2}$$

When at each point of the submanifold M there exist differentiable functions α and β such that

$$h(X, Y)=\alpha g(X, Y), \qquad k(X, Y)=\beta g(X, Y)\,, \tag{3.3}$$

for any $X, Y\in T(M)$, we call the submanifold a totally umbilical submanifold.

Differentiating (3.1) covariantly, we have

$$\tilde{\nabla}_X H=(\tilde{\nabla}_X\alpha)C+\alpha(\tilde{\nabla}_X C)+(\tilde{\nabla}_X\beta)D+\beta(\tilde{\nabla}_X D)\,.$$

Substituting (1.8) and (1.9) into the above equation, we have

$$\tilde{\nabla}_X H=(\tilde{\nabla}_X\alpha-\beta L(X))C+(\tilde{\nabla}_X\beta+\alpha L(X))D-\alpha A(X)-\beta A'(X)\,. \tag{3.4}$$

Since $A(X)$ and $A'(X)$ are both tangent to M, we have the

LEMMA 3.1. *Let M be a submanifold of a Riemannian manifold $\tilde{M}$ such that* $\dim\tilde{M}-\dim M=2$. *In order that the covariant derivative $\tilde{\nabla}_X H$ of the mean curvature vector field be tangent to M, it is necessary and sufficient that*

$$\tilde{\nabla}_X\alpha=\beta L(X)\,, \tag{3.5}$$

and

$$\tilde{\nabla}_X\beta=-\alpha L(X) \tag{3.6}$$

are both valid, where $\alpha=\frac{1}{2n}\sum_{i=1}^{2n}h(E_i, E_i)$ and $\beta=\frac{1}{2n}\sum_{i=1}^{2}k(E_i, E_i)$.

THEOREM 3.2. *Under the same assumptions of Lemma 3.1, M has a constant mean curvature.*

PROOF. Substituting (3.1) into (3.2), we have

$$\mu=g(H, H)=g(\alpha C+\beta D, \alpha C+\beta D)=\alpha^2+\beta^2 .$$

Differentiating the above equation covariantly in the direction of X and making use of Lemma 3.1, we get

$$\tilde{\nabla}_X \mu=2(\alpha \nabla_X \alpha+\beta \nabla_X \beta)=2(\alpha \beta L(X)-\beta \alpha L(X))=0 .$$

This completes the proof.

THEOREM 3.3. *Under the same assumptions of Lemma* 3.1, *the connection of the normal bundle to a non-minimal submanifold M is flat.*

PROOF. Differentiating (3.5) covariantly in the direction of Y, we have

$$\tilde{\nabla}_Y \tilde{\nabla}_X \alpha=\tilde{\nabla}_Y \beta L(X)+\beta \tilde{\nabla}_Y L(X) ,$$

from which, together with (3.6), we get

$$\tilde{\nabla}_Y \tilde{\nabla}_X \alpha-\tilde{\nabla}_X \tilde{\nabla}_Y \alpha=\beta(\tilde{\nabla}_Y L(X)-\tilde{\nabla}_X L(Y)) .$$

Since the left hand members of the above equation is $[Y, X]\alpha$, using (3.5) again, we have

$$\Omega(X, Y)=\tilde{\nabla}_X L(X)-\tilde{\nabla}_X L(Y)-L([Y, X])=0 .$$

This shows that the connection of the normal bundle to M is flat.

Next we consider a totally umbilical submanifold of a Kaehlerian manifold. At first we prove the

LEMMA 3.4. *Let M be a $2n$-dimensional totally umbilical submanifold of a $(2n+2)$-dimensional Kaehlerian manifold $\tilde{M}$. If the mean curvature vector field H of M in $\tilde{M}$ does not vanish at any point of M, the function f defined by* (1.19) *is not constant over M.*

PROOF. For any tangent vector X to M, we have by (1.19)

$$\begin{aligned} g(\operatorname{grad} f, X)=\tilde{g}(\operatorname{grad} f, X)=\tilde{\nabla}_X f=\tilde{\nabla}_X(\tilde{g}(JC, D)) \\ =\tilde{g}(J\tilde{\nabla}_X C, D)+\tilde{g}(JC, \tilde{\nabla}_X D) . \end{aligned}$$

Making use of (1.3), (1.8) and (1.9), we get

$$g(\operatorname{grad} f, X)=\tilde{g}(A(X), JD)-\tilde{g}(A'(X), JC) .$$

$A(X)$ and $A'(X)$ being tangent to M, we have

$$g(\operatorname{grad} f, X)=g(A(X), (JD)^T)-g(A'(X), (JC)^T) .$$

from which

$$g(\operatorname{grad} f, X)=h(X, (JD)^T)-k(X, (JC)^T) , \tag{3.7}$$

because of (1.11) and (1.12).

Since M is a totally umbilical submanifold (3.3) and (3.7) imply that

$$g(\operatorname{grad} f, X) = \alpha g(X, (JD)^T) - \beta g(X, (JC)^T) \tag{3.8}$$
$$= g(X, \alpha(JD)^T - \beta(JC)^T).$$

Suppose that f is a constant over M. As X is any tangent vector to M, it follows that

$$\alpha(JD)^T - \beta(JC)^T = 0. \tag{3.9}$$

The Riemannian metric $\tilde{g}$ on $\tilde{M}$ being Hermitian, JC and JD are mutually orthogonal and consequently $(JC)^T$ and $(JD)^T$ are also mutually orthogonal.

This fact and (3.9) mean that $(JC)^T = (JD)^T = 0$, because of our assumptions. So, from (1.17) and (1.18) we have

$$JC = fD, \quad JD = -fC. \tag{3.10}$$

Substituting (3.10) into (1.15) and (1.16), we have

$$\xi(X) = \tilde{g}(JX, C) = -\tilde{g}(X, JC) = -\tilde{g}(X, fD) = 0, \tag{3.11}$$

$$\eta(X) = \tilde{g}(JX, D) = -\tilde{g}(X, JD) = \tilde{g}(X, fC) = 0. \tag{3.12}$$

Substituting (3.11) and (3.12) into (1.14), we get

$$JX = (JX)^T. \tag{3.13}$$

This shows that the submanifold M is an invariant submanifold of a Kaehlerian manifold. However we have known[4] that an invariant submanifold of a Kaehlerian manifold is necessarily a minimal submanifold. This contradicts to our assumptions. So, f cannot be constant over M. This completes the proof.

THEOREM 3.5. *Let M be a $2n$-dimensional totally umbilical submanifold of a $(2n+2)$-dimensional Kaehlerian manifold with non zero mean curvature. Suppose that for any tangent vector X to M the covariant derivatives of the mean curvature vector $\tilde{\nabla}_X H$ be tangent to M. Then the vector field* grad f *is an infinitesimal concircular transformation over M.*

PROOF. Differentiating (3.8) covariantly in the direction of $Y \in T(M)$, we have

$$g(\nabla_Y \operatorname{grad} f, X) = -g(\operatorname{grad} f, \nabla_Y X) + \nabla_Y \alpha g(X, (JD)^T) + \alpha g(\nabla_Y X, (JD)^T) \tag{3.14}$$
$$+ \alpha g(X, \nabla_Y (JD)^T) - \nabla_Y \beta g(X, (JC)^T)$$
$$- \beta g(\nabla_Y X, (JC)^T) - \beta g(X, \nabla_Y (JC)^T).$$

On the other hand, the submanifold M being totally umbilical, we have from (1.21)

4) J. A. Schouten and K. Yano, [4].

$$\nabla_Y (JC)^T = JA(Y) + fA'(Y) + L(Y)JD + fL(Y)C$$
$$-\alpha g(Y, JC-fD)C - (\tilde{\nabla}_Y f)D - \beta g(Y, JC-fD)D\,.$$

As Y and $JC-fD$ are both tangent to M, it follows that

$$\begin{aligned}\nabla_Y (JC)^T &= -JA(Y) + fA'(Y) + L(Y)JD + fL(Y)C \\ &\quad -\alpha \tilde{g}(Y, JC)C - (\tilde{\nabla}_Y f)D - \beta \tilde{g}(Y, JC)D \\ &= -JA(Y) + fA'(Y) + L(Y)JD + fL(Y)C \\ &\quad -\alpha g(Y, (JC)^T)C - (\tilde{\nabla}_Y f)D - \beta g(Y, (JC)^T)D\,.\end{aligned}$$

Hence we have

$$\begin{aligned}g(X, \nabla_Y (JC)^T) &= \tilde{g}(X, -JA(Y)) + f\tilde{g}(X, A'(Y)) + L(Y)\tilde{g}(X, JD) \\ &= \tilde{g}(JX, A(Y)) + f\tilde{g}(X, A'(Y)) + L(Y)\tilde{g}(X, JD)\,.\end{aligned}$$

Substituting (1.11) and (1.12) into the above, we get

$$g(X, \nabla_Y (JC)^T) = h((JX)^T, Y) + fk(X, Y) + L(Y)\tilde{g}(X, JD)\,.$$

M being totally umbilical, we get

$$g(X, \nabla_Y (JC)^T) = \alpha g((JX)^T, Y) + f\beta g(X, Y) + L(Y)g(X, (JD)^T)\,, \tag{3.15}$$

because of the fact that X is tangent to M.

In entirely the same method we can easily see that

$$g(X, \nabla_Y (JD)^T) = \beta g((JX)^T, Y) - f\alpha g(X, Y) - L(Y)g(X, (JC)^T)\,. \tag{3.16}$$

Substituting (3.15) and (3.16) into (3.14) and using (3.8), we have

$$\begin{aligned}g(\nabla_Y \operatorname{grad} f, X) &= -\alpha g(\nabla_Y X, (JD)^T) + \beta g(\nabla_Y X, (JC)^T) \\ &\quad + \nabla_Y \alpha g(X, (JD)^T) + \alpha g(\nabla_Y X, (JD)^T) \\ &\quad + \alpha[\beta g((JX)^T, Y) - f\alpha g(X, Y) - L(Y)g(X, (JC)^T)] \\ &\quad - \nabla_Y \beta g(X, (JC)^T) - \beta g(\nabla_Y X, (JC)^T) \\ &\quad - \beta[\alpha g((JX)^T, Y) + f\beta g(X, Y) + L(Y)g(X, (JD)^T)] \\ &= -(\alpha^2+\beta^2) f g(X, Y) + (\nabla_Y \alpha - \beta L(Y))g(X, (JD)^T) \\ &\quad - (\nabla_Y \beta + \alpha L(Y))g(X, (JC)^T)\,.\end{aligned}$$

By means of Lemma 3.1 we have under our assumptions

$$g(\nabla_Y \operatorname{grad} f, X) = -(\alpha^2+\beta^2) f g(X, Y)\,. \tag{3.18}$$

Thus we have, by Proposition 2.2, the vector field $\operatorname{grad} f$ is an infinitesimal concircular transformation. This completes the proof.

Furthermore, by means of Theorem 3.2, under our conditions $\alpha^2+\beta^2=\text{const.}$

And so, combinig Theorem 2.3, 3.4 and 3.5, we have the

THEOREM 3.5. *Let M be a complete, connected $2n$-dimensional totally umbilical submanifold with non-zero mean curvature μ of a $(2n+2)$-dimensional Kaehlerian manifold. Suppose that for any tangent vector X to M the covariant derivatives of the mean curvature vector $\tilde{\nabla}_X H$ be tangent to M. Then M is isometric with a sphere S^{2n} of radius $1/\sqrt{\mu}$.*

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