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Groups with a certain type of Sylow 2-subgroups[1)]

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1. The purpose of this paper is to prove the following theorem:

THEOREM. *Let G be a finite group. If a Sylow 2-group S of G has the following form*

$$S=A\times B$$

where A is a non trivial cyclic 2-group and B is a 2-group with a cyclic subgroup of index 2. Then one of the following possibilities holds:

(a) *S is an elementary abelian 2-group of order 4 or 8.*

(b) *The index $[G:G']$ is even, where G' is the commutator subgroup of G.*

(c) *The group $G/0_{2'}(G)$ has a normal 2-subgroup, where $0_{2'}(G)$ is the maximal normal subgroup of G of odd order.*

In particular, a 2-group S satisfying the assumption of the theorem can be a Sylow 2-subgroup of a simple group only when S is an elementary abelian group of order 4 or 8. Using the argument in proving our theorem, we shall get next proposition.

PROPOSITION. *Let G be a finite group and τ a central involution of a Sylow 2-subgroup of G. If the centralizer of τ $C_G(\tau)$ is isomorphic to the group $\langle\tau\rangle\times PSL(2, q)$ where $q\geqq 5$, then one of the following possibilities holds:*

(a) *S is an elementary abelian 2-group.*

(b) *The factor group $G/0_{2'}(G)$ is isomorphic to the group $\langle\tau\rangle\times PSL(2, q)$. In particular the index $[G:G']=2$.*

This proposition generalizes more or less the following theorem of Z. Janko and J. G. Thompson [5]:

THEOREM. *Let G be a finite group with the following properties:*

(i) *2-Sylow subgroups are abelian,*

(ii) *the index $[G:G']$ is odd,*

(iii) *G has an element τ of order 2 such that*

$$C_G(\tau)=\langle\tau\rangle\times PSL(2, q),\ where\ q>5\,.$$

Then G is a non-abelian simple group with $q=3^{2n+1}$ $(n\geqq 1)$.

NOTATION. All the groups considered are finite.

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1) The author thanks to Dr. T. Kondo for pointing out a gap in an original proof of the Theorem.

$Z(X)$ ········the center of a group X.
$D(X)$ ········the Frattini subgroup of a group X.
$\Omega_1(X)$ ······the group generated by all the elements of order p of a p-group X.
$o(X)$ ········the order of an element X.
$\langle a, b, \cdots\rangle$ ···the group generated by the elements $a, b, \cdots$.
$X < Y$ ······a set X is properly contained in a set Y.

Next theorem due to G. Glauberman [3] is very useful in our proof.

THEOREM. *Let G be a finite group of even order. Assume that a Sylow 2-subgroup S of G contains an involution τ which is not conjugate in G to any involution σ $(\neq \tau)$ of S. Then τ is contained in the center of $G/0_{2'}(G)$.*

2. Let B be a 2-group with a cyclic subgroup of index 2. Then B has one of the following forms (see M. Hall [4]).

(I) Cyclic 2-group.
(II) Abelian group of type $(2, 2^n)$, $n \geqq 1$.
(III) Generalized quaternion group.
(IV) $n \geqq 4$, $B = \langle a, b \mid a^{2^{n-1}} = b^2 = 1,\ bab = a^{1+2^{n-2}}\rangle$.
(V) $n \geqq 4$, $B = \langle a, b \mid a^{2^{n-1}} = b^2 = 1,\ bab = a^{-1+2^{n-2}}\rangle$.
(VI) Dihedral group of order $\geqq 8$.

We call a group G a group of type (N) $(N = \mathrm{I} \sim \mathrm{VI})$, if a Sylow 2-subgroup of G has the form $S = A \times B$ where A is a non trivial cyclic 2-group and B is a 2-group of type (N) in the above list. We shall prove our theorem in each case of type (N) $(N = \mathrm{I} \sim \mathrm{VI})$. In the rest of this note we assume that S is not an elementary abelian 2-group.

3. Let G be one of the groups of type (I). Then S is an abelian group of type $(2^m, 2^n)$ $(m \geqq n)$. If $m > n$, then $N_G(S) = C_G(S)$. The theorem of Burnside shows that G has a normal 2-complement. If $m = n \geqq 2$, by a theorem [1] of R. Brauer [1] we have

$$G/O_{2'}(G) \triangleright S \cdot O_{2'}(G)/O_{2'}(G)\,.$$

Hence we have proved the theorem in this case.

4. Let G be one of the groups of type (II). Then S is an abelian 2-group of type $(2^m, 2^n, 2)$ $(m \geqq n \geqq 1)$. If $m > n > 1$, then $N_G(S) = C_G(S)$. The theorem of Burnside shows that G has a normal 2-complement. Assume $m = n > 1$ or $m > n = 1$. We shall show that the group $Z(N_G(S)) \cap S$ is non-trivial. If so, the transfer theorem shows that the index $[G : G']$ is even. Form the group $N_G(S)/C_G(S)$. Then the group $N_G(S)/C_G(S)$ is a subgroup of the automorphism group of S of odd order. Since S is an abelian group of type $(2^m, 2^m, 2)$ or

$(2^m, 2, 2)$ where $m>1$, we can conclude the order of $N_G(S)/C_G(S)$ is 1 or 3. If $N_G(S)=C_G(S)$, we have $Z(N_G(S))\cap S=S$. Assume $|N_G(S)/C_G(S)|=3$. Let x be an element of $N_G(S)$ which is not contained in $C_G(S)$. Then x induces an automorphism $\bar{x}$ of S of order 3. Since $|\Omega_1(S)|=8$, $\bar{x}$ has a fixed point τ in $\Omega_1(S)$. Hence $\tau\in Z(N_G(S))\cap S$. Thus we have proved our theorem in this case.

5. Let G be one of the groups of type (III), of type (IV) or of type (V). Then $\Omega_1(Z(S))$ is an abelian group of type $(2, 2)$, and $\Omega_1(S')$ is a group of order 2. Therefore if $x\in N_G(S)$, x centralizes $\Omega_1(S')$. Since $\Omega_1(S')\subset\Omega_1(Z(S))$, x centralizes $\Omega_1(Z(S))$. Hence by the argument of Burnside, any two central involutions of S are not conjugate to each other in G. Comparing the structure of B, we conclude that S has at most 2 classes of non-central involutions. Therefore a certain involution of $\Omega_1(Z(S))$ is not conjugate to any involution of S. By the theorem of Glauberman [3], we have our theorem in this case.

6. Let G be one of the groups of type (VI). In this case the proof of the theorem is a little complicated. We need several lemmas.

LEMMA 1. *Any two central involutions of a Sylow 2-subgroup of G are not conjugate to one another in G.*

PROOF. As in the previous section, this lemma is easy to prove. We omit the proof.

Put $A=\langle\eta\rangle$, $B=\langle\rho,\ \sigma\,|\,\rho^{2^{n-1}}=\sigma^2=1,\ \sigma\rho\sigma=\rho^{-1},\ (n>2)\rangle$, $\rho^{2^{n-2}}=\pi$ and $\Omega_1(Z(S))=\langle\tau,\pi\rangle$. S has 7 conjugate classes (in S) of involutions. The representatives of 7 classes are $\pi, \sigma, \sigma\rho, \tau, \tau\pi, \tau\sigma, \tau\sigma\rho$. We first consider the fusion of involutions of S. Assume $Z^*=Z(G/O_{2'}(G))=\langle 1\rangle$. We write $a\overset{S}{\sim}b$ or $a\sim b$ if two elements a, b are conjugate to each other in S or in G respectively.

LEMMA 2. *If we choose the suitable elements τ, ρ, σ, we can set*

(a) $\pi\sim\sigma,\ \tau\sim\sigma\rho,\ \tau\pi\sim\tau\sigma$, *or*

(b) $\pi\sim\sigma,\ \tau\sim\sigma\rho,\ \tau\pi\sim\tau\sigma\rho\,.$

PROOF. By the theorem of Glauberman and Lemma 1, π is conjugate to a non central involution of S. Any non central involution of S has the form $\alpha\sigma\rho^j$, $\alpha\in\Omega_1(A)$, $j\geqq 0$. Therefore, if we choose a direct factor B, we can assume $\pi\sim\sigma$. Since π, τ, $\tau\pi$ are not conjugate to one another and by the assumption $Z^*=\langle 1\rangle$, there are the following six possibilities for the fusion of τ, $\tau\pi$:

(1) $\tau\sim\sigma\rho,\quad \tau\pi\sim\tau\sigma,$

(2) $\tau\sim\sigma\rho,\quad \tau\pi\sim\tau\sigma\rho,$

(3) $\tau\sim\tau\sigma,\quad \tau\pi\sim\sigma\rho,$

(4) $$\tau\sim\tau\sigma\,,\qquad \tau\pi\sim\tau\sigma\rho\,,$$

(5) $$\tau\sim\tau\sigma\rho\,,\qquad \tau\pi\sim\sigma\rho\,,$$

(6) $$\tau\sim\tau\sigma\rho\,,\qquad \tau\pi\sim\tau\sigma\,.$$

If we replace τ by $\tau\pi$, (3) goes to (1); (6) to (4); (5) to (2). If we replace ρ by $\rho\tau$, (4) goes to (1). Thus we have proved the lemma.

LEMMA 3. *Choosing the suitable elements τ, ρ, σ, the fusion of involutions of* (a), (b) *in Lemma* 2 *do not occur except the following one*[2].

$$\pi\sim\sigma\sim\tau\sigma\rho,\quad \tau\sim\sigma\rho,\quad \tau\pi\sim\tau\sigma\,.$$

PROOF. Assume that $\pi\sim\sigma$, $\tau\sim\sigma\rho$, $\tau\pi\sim\tau\sigma$. A group $C_G(\sigma\rho)$ contains an elementary 2-group $\langle\pi, \tau, \sigma\rho\rangle$. Let S_1 be a Sylow 2-subgroup of $C_G(\sigma\rho)$ which contains $\langle\pi, \rho, \sigma\rho\rangle$. Since $\tau\sim\sigma\rho$, S_1 is a Sylow 2-subgroup of G. By the structure of S_1, $\Omega_1(Z(S_1))$ is contained in $\langle\pi, \tau, \sigma\rangle$. Put $\langle\pi_1\rangle=\Omega_1(S_1')$. Then $\pi\sim\pi_1$. Since $\pi_1\in\langle\pi, \tau, \sigma\rangle$, we have, by Lemma 2.(a),

$$\pi_1=\pi,\ \tau\sigma\rho \text{ or } \tau\pi\sigma\rho\,.$$

If $\pi_1=\pi$ then $\Omega_1(Z(S_1))=\langle\pi, \sigma\rho\rangle$. Since $\sigma\rho\overset{S}{\sim}\pi\sigma\rho$, two central involutions of S_1 are conjugate. This is impossible by Lemma 1. Similarly $\pi_1\neq\tau\sigma\rho$. Hence we get

$$\pi_1=\tau\pi\sigma\rho \quad\text{and}\quad \pi\sim\sigma\sim\tau\pi\sigma\rho\overset{S}{\sim}\tau\sigma\rho\,.$$

Next assume $\pi\sim\sigma$, $\tau\sim\sigma\rho$, $\tau\pi\sim\tau\sigma\rho$. Let S_2 be a Sylow 2-group of $C_G(\sigma\rho)$ and $\langle\pi_2\rangle$ the group $\Omega_1(S_2')$. Then $\pi\sim\pi_2$ and $\pi_2\in\langle\pi, \tau, \sigma\rho\rangle$. By the assumption $\pi\sim\sigma$, $\tau\sim\sigma\rho$, $\tau\pi\sim\tau\sigma\rho$ we conclude $\pi_2=\pi$. By the same argument as in the case (a) this is impossible.

LEMMA 4. *If an element $\alpha\in S$ is conjugate to ρ^i with $o(\rho^i)\geqq 4$, then $\alpha=\rho^i$ or ρ^{-i}.*

PROOF. Assume $\alpha\sim\rho^i$ and $o(\rho^i)\geqq 4$. Then we can conclude $\alpha=\beta\cdot\rho^j$ where $\beta\in A$ and $o(\beta)<o(\rho^j)$ because any two central involutions are not conjugate to each other. $S_1=\langle\eta\rangle\times\langle\rho\rangle=S\cap C_G(\alpha)=S\cap C_G(\rho^i)$. Since an element ρ^i does not conjugate to a central element of S, S_1 is a 2-Sylow group of $C_G(\rho^i)$. Let $\alpha^x=\rho^i$. Then there exist an element $y\in C_G(\rho^i)$ and $S_1^{xy}=S_1$, $\alpha^{xy}=\rho^i$ hold. Since any two involutions of $\Omega_1(S_1)$ are not conjugate to each other, we conclude $|N_G(S_1)/C_G(S_1)|=2$. Therefore the element xy is contained in $\langle\sigma\rangle\cdot C_G(S_1)$. Hence we have $\alpha=\rho^i$ or ρ^{-i}.

Now we shall prove our theorem in this case. Denote the focal subgroup of S by S^*. If $|A|=2$, then $S^*=\langle\tau\rho, \sigma\rangle$ (by Lemma 3 and Lemma 4). Hence

2) The group $G=S_6$ or S_7, symmetric group of degree 6 or 7 has this fusion of involutions.

$S > S^*$. By the transfer theorem, we have $[G : G'] = \text{even}$. Thus we have proved the theorem. Next, assume $|A| \geqq 4$. We consider the fusion of elements of order $\geqq 4$. Let an element α of S has the order larger than 4, then α is written in the from

$$\alpha = \eta^i,\ \eta^j \sigma \rho^k,\ \eta^l \rho^s,\ \rho^t, \qquad \text{where } i, j, k, l, s, t \text{ are suitable integers.}$$

In order to prove $S > S^*$, we can assume $\alpha \neq \rho^t$ by Lemma 4. If $\alpha = \eta^i \sim \eta^j \sigma \rho^k$ then $o(\eta^i) = o(\eta^j)$. Therefore $o(\eta^{-i}\eta^j \sigma \rho^k) = \max \{o(\eta^{-i}\eta^j),\ o(\sigma\rho^k)\} < o(\eta^i)$. If $\alpha = \eta^i \sim \eta^l \rho^s$, we have $o(\eta^i) = o(\eta^l) > o(\rho^s)$ because any two central involutions are not conjugate to each other. Therefore $o(\eta^{-i}\eta^l \rho^s) = \max \{o(\eta^{-i}\eta^l),\ o(\rho^s)\} < o(\eta^i)$. If $\alpha = \eta^j \sigma \rho^k \sim \eta^l \rho^s = \beta$ we have $o(\eta^j \sigma \rho^k) = o(\eta^l \rho^s) > o(\rho^s)$. Therefore $o(\alpha^{-1}\beta) < o(\eta^j)$. In any cases $\eta \notin S^*$. Hence the transfer theorem show that the index $[G : G']$ is even.

7. Next we shall prove the proposition stated in Section 1. Let G be a finite group satisfying the assumption of our proposition. Furthermore assume that Sylow 2-subgroups of G are not abelian. Then G is one of the groups of type (VI). Since the group $PSL(2, q)$, where $q \geqq 5$, has one class of involutions, the involution τ is not conjugate to any involution σ ($\neq \tau$) of $\langle \tau \rangle \times PSL(2, q)$. Hence τ is contained in $Z(G/O_{2'}(G))$. Hence $G = C_G(\tau) \cdot O_{2'}(G)$. Clearly $C_G(\tau) \cap O_{2'}(G) = \langle 1 \rangle$. Thus we have proved our proposition.

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