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On some ideals of differentiable functions

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Many properties proved for analytic functions, or analytic sets, are in fact far more generally valid. Here we shall deal with the following property of analytical sets: the existence of "smooth" points. We recall the following definitions:

ANALYTIC SET. A closed set E in $\boldsymbol{R}^n$ is said to be analytic in an open set U of $\boldsymbol{R}^n$ if, at any point $x \in U \cap E$, there are local analytic functions $g_1, \cdots, g_s$ such that E is defined in a neighborhood of x by $g_1 = g_2 = \cdots = g_s = 0$.

SMOOTH POINT. A point x of a closed set $E \subset \boldsymbol{R}^n$ is called "smooth" if in a neighborhood of x the set E is locally a k-dimensional C^∞ imbedded submanifold (where k is the dimension of E at x).

We then have the following well known theorem:

THEOREM. *Any analytic set A in U has an open everywhere dense set A' of smooth points.*

The dimension of the set A at smooth points may vary; but, for a *complex* analytic set in $\boldsymbol{C}^n$ which is *irreducible* the set of smooth points is connected, and its dimension is the dimension of A. In that case $A - A'$ is itself an analytic set with lower dimension than that of A [1].

DEFINITION. Given an open set $U \subset \boldsymbol{R}^n$, an ideal $\mathcal{J}$ of the algebra $C^\infty(U)$ of C^∞ functions in U is said to be a Łojasiewicz ideal if it has a finite number of generators $f_1, f_2, \cdots, f_s$ and if for any compact $K \subset U$, there exist positive constants A, c such that for any $x \in K - E$,

$$G(x) > A(d(x))^c, \tag{1}$$

where $G(x)$ is the sum of squares of a system of generators $G(x) = \sum_{j=1}^{s} f_j^2$, $d(x)$ being the euclidean distance of x to the set of zeros of $\mathcal{J}$ in U.

This definition does not depend on the chosen system of generators $(f_1, f_2, \cdots, f_s)$. For, if $(h_1, h_2, \cdots, h_k)$ is another system of generators, $H = \sum_j h_j^2$ also satisfies a Łojasiewicz inequality in any compact $K \subset U$ with the same exponent c. Suppose this is not true. Then there exists a point x of the set E of zeros and a sequence of points $x_i \to x$ such that the quotient $H(x_i)/d(x_i)^c$ tends to zero as i tends to infinity; hence the $|h_j(x_i)|/d(x_i)^{c/2}$ also tends to zero. But we have $f_r(x) = \sum a_i^r(x) h_i(x)$, and any inequality of the form $|h_j(x_i)| < Md(x_i)^\beta$ implies an inequality of the form $|f_j(x_i)| < M'd(x_i)^\beta$, which contradicts the inequality (1). It should be observed that the property of an ideal to be Łojasiewicz is a local property: if the ideal $\mathcal{I}$ is Łojasiewicz in an open set U, then the induced ideal in any open set $U_1 \subset U$ is also Łojasiewicz.

Any analytic ideal is Łojasiewicz (this is the fundamental Lojasiewicz inequality for analytic functions); a finitely generated ideal in $C^\infty(U)$ which is closed (i.e., satisfies Whitney's spectral theorem: any function whose Taylor expansion belongs at any point to the induced ideal in the corresponding local algebra of formal power series belongs to the ideal) is also a Łojasiewicz ideal [**2**].

We now prove the

THEOREM 1. *Let E be the set of zeros of a Łojasiewicz ideal. Then E has an open everywhere dense set of smooth points.*

We have to suppose the ideal $\mathcal{I}$ to be non-zero, the inequality (1) being strict. We need the following lemma:

LEMMA 1. *Let $\mathcal{I}$ be a Łojasiewicz ideal in a neighborhood U and $j : \boldsymbol{R}^k \to U$ a differentiable imbedding of a k-dimensional manifold in U such that the set E of zeros of $\mathcal{I}$ is contained in the image $j(\boldsymbol{R}^k)$; then the induced ideal $j^{-1}(\mathcal{I})$ is also a Lojasiewicz ideal for $j^{-1}(U)$ in $\boldsymbol{R}^k$.*

If p_1, p_2 are two points of a compact K in $j^{-1}(U)$, it is clear that the ratio $|j(p_2)-j(p_1)|/|p_2-p_1|$ has on K upper and lower bounds of positive value depending only on the "curvature" of the imbedding j; hence if the function $G(j(p))$ satisfies on $j(K)$ a Łojasiewicz inequality with respect to the distance $|j(p_2)-j(p_1)|$, it also satisfies a Łojasiewicz inequality (with the same exponent c and a possibly bigger constant A) with respect to the distance $|p_2-p_1|$. For if x' is in $j(K) \cap E \cap U$ the nearest point of $E \cap U$ to x, $x' = j(p')$, $x = j(p)$, then $G(j(p)) > A|j(p)-j(p')|^c$. Now the nearest point of p on $j^{-1}(E)$ is a point p'' such that $|p-p''| \leqq |p-p'|$; hence the inequality $G(j(p)) > B|p-p'|^c \geqq B|p-p''|^c$.

DEFINITION. An ideal $\mathcal{I}$ is called *flat* at a point x if all derivatives of any function f of the ideal $\mathcal{I}$ vanish at x.

LEMMA 2. *If $\mathcal{I}$ is a Łojasiewicz ideal in $V \subset \boldsymbol{R}^n$, U any open set relatively compact in V, then in $U \cap E$ there are points where $\mathcal{I}$ is not flat.*

Suppose $\mathcal{I}$ is flat at all points of $U \cap E$; then there exists a neighborhood U' of $U \cap E$ such that for all $x \in U'$ there exists a nearest point x' in $E \cap U$, with $G(x) < M|x-x'|^a$ where the exponent a is greater than c. Such an inequality contradicts the inequality (1) if the distance $|x-x'|$ is sufficiently small.

DEFINITION. Jacobian extension of an ideal.

Given an ideal $\mathcal{I}$ in $C^\infty(U)$, we define the jacobian extension of order k of $\mathcal{I}$ (notation $J^k(\mathcal{I})$) to be the ideal generated by $\mathcal{I}$ and all jacobians of order k of k functions $f_1, \cdots, f_k$ of $\mathcal{I}$ with respect to any set of k coordinates

$D(f_1, f_2, \cdots, f_k)/D(x_{i_1}, \cdots, x_{i_k})$. If $\mathcal{J}$ is finitely generated, so is also $J^k(\mathcal{J})$, and we have the sequence of inclusions:

$$\mathcal{J} \subset J^n(\mathcal{J}) \subset J^{n-1}(\mathcal{J}) \subset \cdots \subset J^1(\mathcal{J}).$$

Such an extension is independent of the coordinates and is invariant under diffeomorphisms of the ambient space.

We now prove Theorem 1 by induction on the dimension n of the ambient space. For $n=1$, we observe that because of Lemma 2, for any open interval U the set $U \cap E$, if not empty, contains points where the ideal $\mathcal{J}$ is not flat. Such a point x is obviously an isolated point of E: if x were not isolated, then any function f of $\mathcal{J}$ would vanish in an arbitrary small interval containing x and so would all derivatives of f. Hence the theorem is true for $n=1$.

Suppose now the theorem true up to dimension $n-1$, and let $\mathcal{J}$ be a Łojasiewicz ideal on an open set U of $\boldsymbol{R}^n$. Form the successive jacobian extensions of $\mathcal{J}$. If the set of zeros E does not become smaller by the extension $\mathcal{J} \to J^k(\mathcal{J})$ then *a fortiori* $J^k(\mathcal{J})$ is a Łojasiewicz ideal. If, in the extension $\mathcal{J} \to J^k(\mathcal{J})$ the set E becomes a smaller set E', then in a neighborhood of any point y of $E-E'$ there exists a system of k functions $f_1, \cdots, f_k$ of $\mathcal{J}$ with non-vanishing jacobian with respect to k coordinates; hence the set E is, in a neighborhood of y, contained in a submanifold of codimension k. As the induced ideal in this submanifold is a Łojasiewicz ideal (Lemma 1), induction can be used in this case.

It remains to prove that this procedure ends. If $J^1(\mathcal{J})$ has the same set of zeros as $\mathcal{J}$, we iterate the jacobian extensions of $J^1(\mathcal{J})$; namely we claim that there exists a power s such that $J^1 \cdots J^1(\mathcal{J}) = (J^1)^s(\mathcal{J})$ has a smaller set of zeros in U. Let x be a point of E where $\mathcal{J}$ is not flat. Then the Taylor expansion of a function g of $\mathcal{J}$ starts with a monomial of some degree t; let s be the smallest degree occurring in these expansions. Then $(J^1)^s(\mathcal{J})$ contains an invertible function at x, and the set of zeros of $(J^1)^s(\mathcal{J})$ does not contain x any longer. Hence the set of zeros does become smaller through jacobian extensions, and the proof is complete.

We shall now prove another theorem which shows the interest of the notion of jacobian extension. In some sense the notion plays for the real field the role of the "Nullstellensatz" in the complex case.

THEOREM 2. *Let $\mathcal{J}$ be a Łojasiewicz ideal in $C^\infty(U)$. For any point of an open everywhere dense set in the set E of zeros of $\mathcal{J}$, there exists an iterated jacobian extension of $\mathcal{J}$ which locally induces the ideal of definition of E.*

LEMMA 3. *Suppose the Łojasiewicz ideal $\mathcal{J}$ contains locally k coordinate functions $u_1, u_2, \cdots, u_k$. Let W be the local submanifold defined by $u_1 = u_2 = \cdots = u_k = 0$, and let $w(K)$ denote the restriction of an ideal K to W. Then locally* *the diagram*

$$\begin{array}{ccc} \mathcal{J} & \longrightarrow & J^{k+s}(\mathcal{J}) \\ \big\downarrow w & & \big\downarrow w \\ w(\mathcal{J}) & \longrightarrow & J^{s}(w(\mathcal{J})) \end{array}$$

commutes. More precisely, we prove that $w(J^{k+s}(\mathcal{J}))=J^s(w(\mathcal{J}))$.

First, $w(J^{k+s}(\mathcal{J}))$ is contained in $J^s(w(\mathcal{J}))$. For, the restriction to W of a $(k+s)$ minor is a minor of the same order on W if it is taken with respect to coordinates (x_i) other than $u_1, \cdots, u_k$ (recall that $J^{k+s}\subset J^s$). If the minor contains some variables $u_{i_1}, \cdots, u_{i_m}$ like $D(f_1, f_2, \cdots, f_{k+s})/D(u_{i_1}, \cdots, u_{i_m}, x_{m+1}, \cdots, x_{k+s})$ we substitute for the f_i expressions of the form: $f(x, u)=f_w(x, 0)+ug(x, u)$, so that $f_u(x, u)=g(x, u)+ug_u(x, u)$; hence $w(D(f_1, \cdots, f_{k+s})/D(u, x))$ is a sum of the form:

$$\sum g_1(x, 0)g_2(x, 0)\cdots g_m(x, 0)D(f_{m+1}, \cdots, f_{k+s})/D(x_{m+1}, \cdots, x_{k+s}),$$

and $w(J^{k+s}(\mathcal{J}))$ is then contained in $J^s(w(\mathcal{J}))$. It remains to show that $w: J^{k+s}(\mathcal{J})\to J^s(w(\mathcal{J}))$ is surjective; however this results from the fact that any s-jacobian $D(g_1, \cdots, g_s)/D(x_1, \cdots, x_s)$ on W is the w-image of $D(u_1, \cdots, u_k, g_1, \cdots, g_s)/D(u_1, \cdots, u_k, x_1, \cdots, x_s)$.

Now the proof of Theorem 2 follows the proof of Theorem 1. If $n=1$, on any smooth point of E where $\mathcal{J}$ is not flat, let $s+1$ be the smallest order of the non-zero terms of the Taylor expansions of functions of $\mathcal{J}$. Then $(J^1)^s(\mathcal{J})$ obviously defines the maximal ideal at this point. For an arbitrary n, let y be a point of E where some jacobian extension $J^k(\mathcal{J})$ of $\mathcal{J}$ contains a non-zero function. Then $\mathcal{J}$ contains k local coordinate functions $u_1, u_2, \cdots, u_k$. Let W be the local submanifold of codimension k defined by $u_1=u_2=\cdots=u_k=0$, and let w denote the local restriction to W. By induction there exists an iterated jacobian extension $J^{\rho_1}\cdots J^{\rho_s}(w(\mathcal{J}))$ which gives on W the ideal of definition of E near y. Applying Lemma 3, we see that the image under w of $J^{\rho_1+k}\cdots J^{\rho_s+k}(\mathcal{J})$ gives locally the same ideal of definition. But, as $(u_1, \cdots, u_k)$ are in this extension $J^{\rho_1+k}\cdots J^{\rho_s+k}(\mathcal{J})$, it is also the ideal of definition of E near y on $\boldsymbol{R}^n$. Now this reasoning applies perhaps not to $\mathcal{J}$ itself, but at least to an iterated jacobian extension $\mathcal{J}'=(J^1)^{\nu_1}(\mathcal{J})$, which is also Łojasiewicz.

SOME OPEN QUESTIONS.

Let $\mathcal{J}$ be an analytic ideal; let us form the family of ideals containing $\mathcal{J}$, closed with respect to jacobian extension and sum. Because of the noetherian character of germs of analytic sets, this infinite set of ideals defines only a finite number of analytic sets A_i at any point. It is easy to check that given such a set A, if B is the union of all proper subsets of the family contained in A, then $A-B$ is an imbedded manifold; hence this family of sets defines what we may call a "primary stratification" of the analytic set defined by $\mathcal{J}$.

This type of stratification is not fine enough to present the tangential properties (properties A, B of Whitney [3]) required of a "stratification" in the strongest sense (see [4]). Nevertheless one may hope that there exist canonical extension operations X generalizing the jacobian extensions (and defined by polynomial ideals in the space $J^r(n, p)$ of local jets) and possessing the following property: all canonical extensions of an analytic ideal $\mathcal{J}$ define a "true" stratification.

The following seems to be an interesting class of differentiable ideals: $\mathcal{J}$ and all its canonical extensions $X\mathcal{J}$ are Łojasiewicz ideals. Such "almost analytic" ideals probably have the following property: given an "almost analytic" ideal $\mathcal{J}$ on $\boldsymbol{R}^k$, then for "almost any" differentiable map $F:\boldsymbol{R}^n \to \boldsymbol{R}^k$, the induced ideal $F^*(\mathcal{J})$ is "almost analytic".

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