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Hypersurface with parallel Ricci tensor in a space of constant holomorphic sectional curvature

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Introduction. It is well known that the Ricci tensor of a symmetric space is parallel, that is, the covariant derivative of the Ricci tensor vanishes, which is also true in an Einstein space if the dimension of the space is greater than 2. But, in general, the converse is not always true.

The purpose of this paper is to prove the equivalence of the following statements for a complex hypersurface in a space of constant holomorphic sectional curvature: 1) *The hypersurface is a locally symmetric space.* 2) *The hypersurface is an Einstein space.* 3) *The Ricci tensor of the hypersurface is parallel.*

Recently B. Smyth has shown in his thesis that the statement (2) above implies (1) and also he has clasified such a hypersurface [**2**].

§1. Formulas for a complex hypersurface.

In this section we shall summarize the fundamental formulas for a complex hypersurface which will be used in §2. All of them are well known and easily proved as in the case of a real hypersurface [**1**]. The indices $A, B, C, \cdots$ take the values $1, 2, \cdots, n+1$ and the indices $i, j, k, \cdots$ take the values $1, 2, \cdots, n$.

Consider a Kähler manifold M' of complex dimension $n+1$ and a complex hypersurface M in M' which is a complex submanifold of M' of complex codimension 1. M is considered as a Kähler manifold by the induced metric from M'. In terms of local complex coordinates $(z^1, \cdots, z^n)$ of M and $(w^1, \cdots, w^{n+1})$ of M', w^A is a holomorphic function of $(z^1, \cdots, z^n)$. If we denote $\partial w^A/\partial z^i$ by B_i^A, the induced metric tensor $g_{\bar{j}i}$ is given by

$$g_{\bar{j}i} = B_{\bar{j}}{}^{\bar{B}} B_i{}^A g'_{\bar{B}A},$$

where $g'_{\bar{B}A}$ is the metric tensor of M' and $B_{\bar{i}}{}^{\bar{A}} = \overline{B_i{}^A}$.

Let N^A be a complex unit normal vector to M, that is, N^A is defined locally and satisfies

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$$g'_{\bar{B}A}B_{\bar{i}}^{\bar{B}}N^A=0 \qquad \text{and} \qquad g'_{\bar{B}A}N^{\bar{B}}N^A=1\,,$$

where $N^{\bar{A}}=\overline{N^A}$.

The covariant derivative of B_i^A and N^A by the so-called van der Waerden-Bortolotti covariant differntial operator ∇_j are

$$\nabla_j B_i^A=H_{ji}N^A\,,\qquad \nabla_{\bar{j}}B_i^A=0\,,$$

$$\nabla_j N^A=c_jN^A\,,\qquad \nabla_{\bar{j}}N^A=-H_{\bar{j}}{}^iB_i^A-c_{\bar{j}}N^A\,,$$

where H_{ji} is symmetric with respect to j and i, $H_{\bar{j}\bar{i}}=\overline{H_{ji}}$, $H_{\bar{j}}{}^i=H_{\bar{j}\bar{a}}g^{\bar{a}i}$ and $c_{\bar{j}}=\overline{c_j}$. H_{ji} and c_j are depend on the choice of N^A. Then denoting the curvature tensors of M and M' by $K_{\bar{k}ji\bar{h}}$ and $K'_{\bar{D}CB\bar{A}}$ respectively, we have the Gauss, Codazzi and Ricci formulas.

$$K_{\bar{k}ji\bar{h}}=B_{\bar{k}}^{\bar{D}}B_j^CB_i^BB_{\bar{h}}^{\bar{A}}K'_{\bar{D}CB\bar{A}}+H_{\bar{k}\bar{h}}H_{ji} \tag{1.1}$$

$$\begin{cases}\nabla_kH_{ji}+c_kH_{ji}=\nabla_jH_{ki}+c_jH_{ki}\\ \nabla_{\bar{k}}H_{ji}+c_{\bar{k}}H_{ji}=B_{\bar{k}}^{\bar{D}}B_j^CB_i^BN^{\bar{A}}K'_{\bar{D}CB\bar{A}}\end{cases} \tag{1.2}$$

$$\begin{cases}\nabla_jc_i-\nabla_ic_j=0\\ \nabla_{\bar{j}}c_i+\nabla_ic_{\bar{j}}=-H_{\bar{j}}{}^aH_{ia}+B_{\bar{j}}^{\bar{D}}B_i^CN^BN^{\bar{A}}K'_{\bar{D}CB\bar{A}}\,.\end{cases} \tag{1.3}$$

Transvecting (1.1) with $g^{i\bar{h}}$ we have

$$K_{\bar{k}j}=B_{\bar{k}}^{\bar{B}}B_j^AK'_{\bar{B}A}+B_{\bar{k}}^{\bar{D}}B_j^CN^BN^{\bar{A}}K'_{\bar{D}CB\bar{A}}-H_{\bar{k}}{}^aH_{ja}\,, \tag{1.4}$$

where $K_{\bar{k}j}$ and $K'_{\bar{B}A}$ are the Ricci tensors of M and M' respectively.

If M' has a constant holomorphic sectional curvature k, the curvature tensor and Ricci tensor of M' have the form

$$K'_{\bar{D}CB\bar{A}}=-\frac{k}{2}(g'_{\bar{D}C}g'_{B\bar{A}}+g'_{\bar{D}B}g'_{C\bar{A}})$$

$$K'_{\bar{B}A}=\frac{n+2}{2}kg'_{\bar{B}A}\,.$$

Also in this case we have

$$B_{\bar{k}}^{\bar{D}}B_j^CB_i^BB_{\bar{h}}^{\bar{A}}K'_{\bar{D}CB\bar{A}}=-\frac{k}{2}(g_{\bar{k}j}g_{i\bar{h}}+g_{\bar{k}i}g_{j\bar{h}})$$

$$B_{\bar{k}}^{\bar{D}}B_j^CB_i^BN^{\bar{A}}K'_{\bar{D}CB\bar{A}}=0$$

$$B_{\bar{k}}^{\bar{D}}B_j^CN^BN^{\bar{A}}K'_{\bar{D}CB\bar{A}}=-\frac{k}{2}g_{\bar{k}j}\,.$$

On account of these equations, for a complex hypersurface in a space of constant holomorphic sectional curvature k we have the following formulas:

$$K_{\bar{k}ji\bar{h}}=-\frac{k}{2}(g_{\bar{k}j}g_{i\bar{h}}+g_{\bar{k}i}g_{j\bar{h}})+H_{\bar{k}\bar{h}}H_{ji} \tag{1.5}$$

(1.6) $$\nabla_{\bar{k}} H_{ji} = c_{\bar{k}} H_{ji}$$

(1.7) $$\nabla_{\bar{j}} c_i + \nabla_i c_{\bar{j}} = -\frac{k}{2} g_{\bar{j}i} - H_{\bar{j}}{}^a H_{ia}$$

(1.8) $$K_{\bar{j}i} = \frac{n+1}{2} k g_{\bar{j}i} - H_{\bar{j}}{}^a H_{ia} .$$

§ 2. Hypersurfaces with parallel Ricci tensor.

In this section we shall assume that M is a connected complex hypersurface in a space M' of constant holomorphic sectional curvature k of complex dimension $n+1$.

Differentiating (1.8) and taking account of (1.6), we have

(2.1) $$\nabla_k K_{\bar{j}i} = -H_{\bar{j}}{}^a (\nabla_k H_{ia} + c_k H_{ia}) .$$

Now we define the scalars α and β on M by the following:

(2.2) $$\alpha = H_{kj} H_{\bar{i}\bar{h}} g^{k\bar{i}} g^{j\bar{h}} \quad \text{and} \quad \beta = H_{\bar{k}}{}^j H_j{}^{\bar{i}} H_{\bar{i}}{}^h H_h{}^{\bar{k}} .$$

PROPOSITION 1. *If the Ricci tensor of M is parallel, the scalars α and β are real constant.*

PROOF. By the definition it is easily verified that these scalars are real. Transvecting (1.8) with $g^{\bar{j}i}$ we have

$$\alpha = \frac{n(n+1)}{2} - K_{\bar{j}i} g^{\bar{j}i}$$

and also from (1.8) we have

$$\beta = K_{\bar{j}i} K^{\bar{j}i} - \frac{n(n+1)^2}{4} k^2 + (n+1) k\alpha .$$

Thus α and β are constant, since $K_{\bar{j}i} g^{\bar{j}i}$ and $K_{\bar{j}i} K^{\bar{j}i}$ are constant.

PROPOSITION 2. *If the Ricci tensor of M is parallel, there exists a constant λ such that*

(2.3) $$H_j{}^{\bar{a}} H_{\bar{a}}{}^b H_{bi} = \lambda H_{ji} .$$

PROOF. Since $\nabla_j K_{\bar{i}h} = 0$, using the Ricci's identity we have

$$K_{\bar{k}j\bar{i}}{}^{\bar{a}} K_{\bar{a}h} + K_{\bar{k}jh}{}^a K_{\bar{i}a} = 0 .$$

Substituing (1.5) and (1.8) in this equation, we have

(2.4) $$\frac{k}{2}(g_{\bar{k}h} H_j{}^{\bar{a}} H_{\bar{a}\bar{i}} - g_{j\bar{i}} H_{\bar{k}}{}^a H_{ah}) + H_{\bar{k}\bar{i}} H_j{}^{\bar{a}} H_{\bar{a}}{}^b H_{bh} - H_{jh} H_{\bar{k}}{}^a H_a{}^{\bar{b}} H_{\bar{b}\bar{i}} = 0 .$$

Transvecting (2.5) with $H^{\bar{k}\bar{i}}$, we obtain

$$\alpha H_j{}^{\bar a} H_{\bar a}{}^b H_{bh} - \beta H_{jh} = 0\,.$$

Thus we have

$$H_j{}^{\bar a} H_{\bar a}{}^b H_{bh} = \lambda H_{jh}$$

where $\lambda = \beta/\alpha$, if $\alpha \neq 0$. If $\alpha = 0$, then $H_{ji} = 0$ and the proposition is trivial.

PROPOSITION 3. *The Ricci tensor of M is parallel if and only if*

$$\nabla_k H_{ji} + c_k H_{ji} = 0\,. \tag{2.5}$$

PROOF. If (2.5) is satisfied, we can see directly from (2.1) that $\nabla_k K_{\bar j i} = 0$.

Conversely we assume that the Ricci tensor of M is parallel, then we have from (2.1)

$$(\nabla_k H_{ja}) H_{\bar i}{}^a = -c_k H_{ja} H_{\bar i}{}^a\,. \tag{2.6}$$

If M is totally geodesic, (2.5) is trivial. So we may assume that M is not totally geodesic. Then the formula (2.3) holds and $\lambda \neq 0$. Differentiating (2.3) and taking account of (1.6) and (2.6) we have

$$\begin{aligned}
\lambda \nabla_k H_{ji} &= \nabla_k H_j{}^{\bar a} H_{\bar a}{}^b H_{bi} + H_j{}^{\bar a} \nabla_k H_{\bar a}{}^b H_{bi} + H_j{}^{\bar a} H_{\bar a}{}^b \nabla_k H_{bi} \\
&= -c_k H_j{}^{\bar a} H_{\bar a}{}^b H_{bi} + c_k H_j{}^{\bar a} H_{\bar a}{}^b H_{bi} - c_k H_j{}^{\bar a} H_{\bar a}{}^b H_{bi} \\
&= -c_k H_j{}^{\bar a} H_{\bar a}{}^b H_{bi} \\
&= -\lambda c_k H_{ji}\,.
\end{aligned}$$

Since $\lambda \neq 0$, we get (2.5).

PROPOSITION 4. *M is locally symmetric if and only if the Ricci tensor of M is parallel.*

PROOF. It is well known that if M is locally symmetric, the Ricci tensor of M is parallel. We may assume that the Ricci tensor of M is parallel. Taking account of (1.6) and (2.5) we have

$$\begin{aligned}
\nabla_l K_{\bar k j i \bar h} &= \nabla_l H_{\bar k \bar h} H_{ji} + H_{\bar k \bar h} \nabla_l H_{ji} \\
&= c_l H_{\bar k \bar h} H_{ji} - c_l H_{\bar k \bar h} H_{ji} \\
&= 0\,.
\end{aligned}$$

This shows that M is locally symmetric.

Now assume that the Ricci tensor of M is parallel and M is not totally geodesic. On account of (1.6) and (2.5) we have

$$\nabla_j H_{ih} = -c_j H_{ih} \qquad \nabla_{\bar k} H_{ih} = c_{\bar k} H_{ih}$$

$$\nabla_{\bar k} \nabla_j H_{ih} = -\nabla_{\bar k} c_j H_{ih} - c_{\bar k} c_j H_{ih} \tag{2.7}$$

$$\nabla_j \nabla_{\bar k} H_{ih} = -\nabla_j c_{\bar k} H_{ih} - c_{\bar k} c_j H_{ih} \tag{2.8}$$

Subtracting (2.8) from (2.7) and taking account of Ricci's identity we have

$$-K_{\bar{k}ji}{}^{a}H_{ah}-K_{\bar{k}jh}{}^{a}H_{ia}=-(\nabla_{\bar{k}}c_{j}+\nabla_{j}c_{\bar{k}})H_{ih}\,.$$

Then taking account of (1.7) in the right hand side and (1.5) in the left hand side we have

$$(2.9)\qquad \Big(\frac{k}{2}g_{\bar{k}j}-H_{\bar{k}}{}^{a}H_{ja}\Big)H_{ih}+\Big(\frac{k}{2}g_{\bar{k}i}-H_{\bar{k}}{}^{a}H_{ia}\Big)H_{jh}+\Big(\frac{k}{2}g_{\bar{k}h}-H_{\bar{k}}{}^{a}H_{ha}\Big)H_{ji}=0\,.$$

Transvecting (2.9) with $g^{\bar{k}j}$ and taking account of (2.3) we obtain

$$\Big(\frac{n+2}{2}k-\alpha-2\lambda\Big)H_{ih}=0\,.$$

Since $H_{ji}\neq 0$ in our assumption, we have

$$(2.10)\qquad \frac{n+2}{2}k=\alpha+2\lambda\,.$$

As β and α are positive and then $\lambda=\beta/\alpha$ is also positive, we see that k must be positive. Transvecting (2.4) with $g^{\bar{k}h}$ we have

$$\frac{k}{2}(nH_{j}{}^{\bar{a}}H_{\bar{i}\bar{a}}-\alpha g_{j\bar{i}})=0$$

and this implies

$$(2.11)\qquad H_{j}{}^{\bar{a}}H_{\bar{i}\bar{a}}=\frac{\alpha}{n}g_{j\bar{i}}\,.$$

Transvecting (2.11) with $H_{k}{}^{\bar{i}}$ and taking account of (2.5) we have

$$\lambda H_{kj}=\frac{\alpha}{n}H_{kj}\,.$$

Thus we have $\lambda=\alpha/n$, and from (2.10) we have

$$(2.12)\qquad \frac{\alpha}{n}=\frac{k}{2}\,.$$

Therefore we have

$$(2.13)\qquad H_{\bar{j}}{}^{a}H_{ia}=\frac{k}{2}g_{\bar{j}i}\,.$$

Substituting this in (1.8) we obtain

$$K_{\bar{j}i}=\frac{n}{2}kg_{\bar{j}i}\,.$$

This means that M is an Einstein space.

If M is totally geodesic, it is easily seen that M is a space of constant holomorphic sectional curvature k and therefore M is an Einstein space.

Summarizing the results of this section we have the theorems.

THEOREM A. *If M is a connected complex hypersurface in a space of constant holomorphic sectional curvature of complex dimension $n+1$ and $n\geqq 2$, the* *following statements are equivalent:* 1) *M is a Locally symmetric space.* 2) *The Ricci tensor of M is parallel.* 3) *M is an Einstein space.*

The non totally geodesic case of M occurs only in a space of positive constant holomorphic sectional curvature, we have

THEOREM B. *If M is a connected complex hypersurface in a space M′ of non positive constant holomorphic sectional curvature k of complex dimension* $n+1$ *and* $n \geqq 2$, *the following statements are equivalent:* 1) *M is totally geodesic in M′.* 2) *M is a space of constant holomorphic sectional curvature k.* 3) *The Ricci tensor of M is parallel.* 4) *M is an Einstein space.* 5) *M is a locally symmetric.*

THEOREM C. *Let M be a connected complex hypersurface in a space M′ of positive constant holomorphic sectional curvature k. If the Ricci tensor of M is parallel, then either M is totally geodesic in M′ and has a constant holomorphic sectional curvature k, or M is a locally symmetric Einstein space with the scalar curvature* n^2k.

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Bibliography

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