

## Linear differential systems with singularities and an application to transitive Lie algebras

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### Introduction

In this note we will first define linear differential systems *with* singularities, and then give a sufficient condition for their integrability (Theorem 1 below), similar to Frobenius' theorem. As an application we will prove that a compact manifold  $M$  is completely determined by the Lie algebra  $L(M)$  of all analytic vector fields on  $M$  along with an isotropy subalgebra of  $L(M)$  (Theorem 2). On this regard we recall that Myers [4] proved that  $M$  is completely determined by the ring  $F(M)$  of all analytic functions on  $M$ .

We shall always be in the analytic category. The Lie algebra  $L(M)$  of the vector fields is a module over the function ring  $F(M)$ .

DEFINITION. A linear differential system on  $M$  is an  $F(M)$ -submodule of  $L(M)$ .

This notion is a generalization of the usual linear differential system (without singularities), i. e., a vector subbundle of the tangent bundle  $T(M)$  (indeed any real analytic vector bundle has sufficient ample sections, [5]), or a "distribution" as Chevalley calls [2]. A vector subspace  $L$  of  $L(M)$  over the reals  $\mathbf{R}$  generates a unique  $F(M)$ -module  $L^F$ . We shall understand  $L^F$  under the linear differential system of  $L$ . Given a subspace  $L$  of  $L(M)$  and a point  $x$  of  $M$ , we put  $L(x) = \{u(x) \mid u \in L\}$  and  $L_x = \{u \in L \mid u(x) = 0\}$  ( $0 \rightarrow L_x \rightarrow L \rightarrow L(x) \rightarrow 0$  is exact).  $L(x)$  is a subspace of the tangent space  $T_x(M)$  to  $M$  at  $x$ , which we call *the integral element* of  $L$  at  $x$ . An *integral manifold*  $N$  of  $L$  is by definition a connected submanifold of  $M$  such that each tangent space to  $N$  is an integral element of  $L$ ;  $T_x(N) = L(x)$ ,  $x \in N$ . The linear differential system  $L^F$  of  $L$  has the same integral elements and integral manifolds as  $L$ . If  $L$  is a Lie subalgebra of  $L(M)$ , then so is  $L^F$  and  $L_x$  is a subalgebra of  $L$ , which we call *the isotropy subalgebra* of  $L$  at  $x$ .

THEOREM 1. If  $L$  is a Lie subalgebra of  $L(M)$ , then  $M$  has a unique partition  $\mathfrak{N} = \{N\}$  by integral manifolds  $N$  of  $L$ . (That is,  $M$  is the disjoint union

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of the members  $N$  of  $\mathfrak{R}$ .)

This is exactly the Frobenius theorem as described in [2] (Theorem 2, p. 94), in the special case where  $\dim L(x)$  is independent of  $x$ . Theorem 1 is false in the  $C^\infty$ -category (see § 1).

An abstract transitive Lie algebra  $(L, L_0)$  is a pair of a Lie algebra  $L$  and its subalgebra  $L_0$  without containing nonzero ideals of  $L$  and having  $\dim L/L_0$  finite. We say that another abstract transitive Lie algebra  $(L', L'_0)$  is isomorphic with  $(L, L_0)$  when there exists an isomorphism of  $L'$  onto  $L$  which sends  $L'_0$  onto  $L_0$ . When  $M$  is connected,  $(L(M), L(M)_x)$  gives an example of an abstract transitive Lie algebra for any point  $x$  of  $M$ .

**THEOREM 2.** *Let  $M, N$  be two compact connected analytic manifolds. If there exists an isomorphism  $\alpha$  of  $(L(M), L(M)_a)$  onto  $(L(N), L(N)_b)$  for some  $a \in M$  and  $b \in N$ , then there exists an isomorphism  $f$  of an analytic manifold  $M$  onto  $N$  which induces  $\alpha$ .*

The proofs will be elementary except the usage for Theorem 2 of the fact that  $T(M)$  and its jet bundles have sufficiently ample (analytic) sections.

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**§ 1. Proof of Theorem 1**

Let  $\mathcal{N}_x$  be the set of all the integral manifolds of  $L$  which contain the point  $x$  of  $M$ .

(1.1)  $\mathcal{N}_x$  is not empty for any point  $x$ .

**PROOF.** Take  $m$  elements  $u_1, \dots, u_m$  of  $L$  such that  $(u_\alpha(x))_{1 \leq \alpha \leq m}$  is a base of  $L(x)$ . Fix a coordinate system  $(x^i)$  around  $x$ . Let  $u_\alpha^i$  be the components of  $u_\alpha$ . We may assume that  $\det(u_\alpha^i)_{1 \leq \alpha, \beta \leq m}$  never vanishes on a neighborhood  $U$  of  $x$ . For simplicity, throughout the proof of (1.1), we denote by the same symbol  $L$  the linear differential system of  $L$  considered as a subspace of  $L(U)$ , so that  $L$  is an  $F(U)$ -module. Then  $L$  contains the vector fields  $v_\alpha, 1 \leq \alpha \leq m$ , of the form:  $v_\alpha = \partial/\partial x^\alpha + \sum_{m < \lambda \leq n} v_\alpha^\lambda \partial/\partial x^\lambda$  with all  $v_\alpha^\lambda \in F(U)$  vanishing at  $x$ . Let  $L'_{(-1)}$  be the space spanned by these  $v_\alpha$  over  $\mathbf{R}$ , and  $L'_0$  be the subspace of  $L$  defined by  $L'_0 = \{v \in L \mid v = \sum_{m < \lambda \leq n} v^\lambda \partial/\partial x^\lambda, v^\lambda \in F(U)\}$ . We put  $L' = L'_{(-1)} + L'_0$ . ( $L'_0$  might well be denoted by  $L'_x$ , though it shall not.)  $L'$  is a subalgebra of  $L$ . In fact we readily have

$$(1.2) \quad [L', L'] \subset L'_0.$$

Besides  $L$  is clearly the linear differential system of  $L'$ . Therefore (1.1) follows from

(1.3)  $L'$  has an integral manifold which contains  $x$ .

(1.3) will be proved after the demonstration of (1.4), in which the analyticity will be used. Let  $N$  be  $\exp L'_{(-1)}(x)$ , i. e., the image of the imbedding:  $v \rightarrow (\exp v)(x)$  of some neighborhood of the zero in  $L'_{(-1)}$  ( $\approx \mathbf{R}^m$ ) into  $M$ , where  $(\exp v)(x)$  is as usual the point  $x(1)$  on the curve  $x(t)$  given by the ordinary differential equation  $dx(t)/dt = v(x(t))$  with  $x(0) = x$ .

(1.4) *The vector fields in  $L'_0$  vanish everywhere on  $N$ .*

Take any vector field  $w$  in  $L'_0$ . We know that  $w$  vanishes at  $x$ . Let  $v$  be any member of  $L'_{(-1)}$ . It suffices to show that  $w$  vanishes on the integral curve  $(\exp tv)(x)$  of  $v$  with initial point  $x$ . The successive derivatives of  $w$  (or its components in terms of  $(x^i)$ ) with respect to the parameter  $t$  are obtained by applying  $ad(v)$  to  $w$  repeatedly;  $(d/dt)^k w = [(ad v)^k, w]$ , which is inductively defined by  $[v, [(ad v)^{k-1}, w]]$ ,  $k = 1, 2, 3, \dots$ . They vanish at  $x$  by (1.2). Since  $w$  is analytic,  $w$  vanishes on that curve. (1.4) is proved.

We will complete the proof of (1.3) by showing that  $(v_\alpha(y))_{1 \leq \alpha \leq m}$  span  $T_y(N)$  at each point  $y$  of  $N$  so that  $N$  is an integral manifold of  $L'$ , hence of  $L$ . Let  $u$  and  $v$  be any two members of  $L'_{(-1)}$ . Then the point  $f(s, t) = \exp(t(su + v))(x)$  lies on  $N$  whenever this makes sense for  $(s, t) \in \mathbf{R}^2$ . To show the above, we claim that

$$(1.5) \quad [\partial f(s, t)/\partial s]_{s=0} = tu(f(0, t)).$$

The left hand side of (1.5) is a vector field,  $u'$ , on the curve  $f(0, t)$ .  $u'$  vanishes at  $x = f(0, 0)$  and satisfies the ordinary differential equation  $du'/dt = u + \sum(\partial v/\partial x^i)u'^i$  in terms of the fixed coordinate system  $(x^i)$ , since we have  $\partial f/\partial t = (su + v)$  at  $f(s, t)$  by the definition of  $f$  so that  $\partial^2 f/\partial s \partial t = u + \sum(\partial(su + v)/\partial x^i) \cdot (\partial f^i/\partial s)$ . On the other hand  $tu$  satisfies the same equation with the same initial condition along the curve  $f(0, t)$ . In fact we have  $\partial(tu)/\partial t = u + t(\partial u/\partial t) = u + t \sum(\partial u/\partial x^i)v^i = u - t[u, v] + t \sum(\partial v/\partial x^i)u^i = u + \sum(\partial v/\partial x^i)(tu^i)$ , the last equality being due to (1.2) and (1.4). Thus we obtain  $u' = tu$  and (1.5) is proved. Hence (1.3) is proved, and so is (1.1).

In passing to the global part, we need the local uniqueness, or more precisely

(1.6) *For two members  $N_1, N_2$  of  $\mathcal{N}_x$ ,  $\mathcal{N}_x$  contains some  $N \subset N_1 \cap N_2$ .*

PROOF OF (1.6). Let  $L_{(-1)}$  be a subspace of  $L$  with dimension  $= L_{(-1)}(x) = L(x)$ . For any vector field  $v$  in  $L_{(-1)}$  the restriction of  $v$  to  $N_1, N_2$  are vector fields  $v_1, v_2$  on these integral manifolds  $N_1, N_2$  of  $L$ . We have three exponential maps of  $L_{(-1)}$  into  $M, N_1$  and  $N_2$  respectively, which are of course given by the equations  $dx(t)/dt = v(x(t))$ ,  $dx_1/dt = v_1$  and  $dx_2/dt = v_2$  with  $x(0) = x_1(0) = x_2(0) = x$  respectively. Each of them is an imbedding defined on a neighborhood of 0 in  $L_{(-1)}$ . We may assume the definition domains are the same,

$U$ . Restricted to  $U$ , these imbeddings are the same if they are considered as maps into  $M$ , since then they are all given by integral curves of  $\nu$  passing through  $x$ . The common image  $N$  is what is wanted.

Now we conclude the proof of Theorem 1. For any point  $x$  of  $M$ , we put  $N_x = \bigcup_{N \in \mathfrak{N}_x} N$ .  $N_x$  is non-empty by (1.1). The topology and the analytic structures are given by the charts of all  $N$  in  $\mathfrak{N}_x$  by virtue of (1.6). The topology of  $N_x$  is stronger than the one induced from  $M$ . Hence  $N_x$  is a Hausdorff space. Thus  $N_x$  is a submanifold of  $M$ . When  $M$  is paracompact,  $M$  has a Riemannian metric, which can be induced on  $N_x$ , so that  $N_x$  is metrizable and therefore  $N_x$  is paracompact. Finally  $\mathfrak{N} = \{N_x | x \in M\}$  is a partition of  $M$ . In fact the union of  $\mathfrak{N}$  is clearly  $M$ , and the distinct members of  $\mathfrak{N}$  are disjoint. We will prove this. Suppose  $N_x \cap N_y$  contains a point  $z$ . Then  $N_x$  and  $N_y$  belong to  $\mathfrak{N}_z$  so that  $N_x, N_y$  are contained in  $N_z$ . This argument also gives that  $N_z$  is contained in  $N_x$  and  $N_y$ . Thus we have  $N_x = N_z = N_y$ . The uniqueness of  $\mathfrak{N}$  is obvious by (1.6). The proof of Theorem 1 is complete.

Counter-example for the  $C^\infty$ -category. Let  $M$  be  $\mathbf{R}^2$  with the usual coordinate system  $(x, y)$ . Let  $L$  be the space of  $C^\infty$  vector fields on  $M$  spanned by  $\partial/\partial x$  and  $\{f(x)\partial/\partial y | f \text{ is a } C^\infty \text{ function on } M \text{ with all the derivatives } f^{(k)} \text{ vanishes at } (0, 0), k = 0, 1, 2, \dots\}$ . If a point  $p$  of  $M$  lies on the  $y$ -axis, then we have  $\dim L(p) = 1$  and  $L(p)$  is parallel to the  $x$ -axis, while we have  $\dim L(p) = 2$  otherwise. Therefore, there does not exist an integral manifold through a point on the  $y$ -axis. This example shows that Theorem 1 is false in the  $C^\infty$ -category. Theorem 1 holds good in a special case where  $M$  is compact and  $L$  is finite dimensional, as is well known.

QUESTION. Does there exist "a prolongation" so that  $L$  gets free of singularities? In other words, does there exist an analytic manifold  $\tilde{M}$  with an analytic surjection  $\pi: \tilde{M} \rightarrow M$  which has a linear differential system  $\tilde{L}$  without singularities such that  $\tilde{L}$  is isomorphic with a given Lie algebra  $L \subset L(M)$  in some natural way? When  $\dim L$  is finite, the answer is affirmative. Put  $\tilde{M} = G \times M$  to see this, where  $G$  is the simply connected Lie group with Lie algebra  $L$ .  $L$  is isomorphic with the right-invariant vector fields on  $G$ . From this, it is obvious how to define  $\tilde{L}$  on  $\tilde{M}$ .

ANOTHER QUESTION. The converse of Theorem 1 is false. The partition  $\mathfrak{N}$  should have some further property in order to be given by the maximal integral manifolds of a linear differential system.

## § 2. Proof of Theorem 2

Our method is basically the one used by E. Cartan frequently. Indeed we will construct in  $M \times N$  the graph of the mapping  $f: M \rightarrow N$  by means of a

suitable linear differential system. We lift up the vector fields  $L(M)$  and  $L(N)$  onto  $M \times N$  via the identification  $T(M \times N) = T(M) \times T(N)$ , obtaining the Lie algebra  $L^* = \{ \text{the vector fields } u^* | u^*(x, y) = (u(x), \alpha(u)(y)) \text{ for } (x, y) \in M \times N, u \in L(M) \}$  where  $\alpha$  is the given isomorphism:  $(L(M), L(M)_a) \rightarrow (L(N), L(N)_b)$ . By Theorem 1, there exists a unique maximal integral manifold  $\Gamma$  of the linear differential system  $L$  passing through  $(a, b)$ .  $\Gamma$  will be the graph of  $f$  whose existence we have to show. Given a positive integer  $p$ , we denote by  $J^p(T(M))$  the  $p$ -jet bundle  $\{j_x^p u | x \in M, u \in L(U), \text{ and } U \text{ is some neighborhood of } x \text{ in } M\}$  where  $j_x^p u$  denotes as usual the equivalence class of the local vector fields  $v$  around  $x$  which has the same derivatives as  $u$  of degrees up to  $p$  (including 0) at  $x$  with respect to some (and so every) coordinate system.

(2.1) *If two jets  $j_x, j_y \in J^p(T(M))$  have the distinct sources  $x, y$  then there exists some  $u$  in  $L(M)$  such that one has  $j_x^p u = j_x$  and  $j_y^p u = j_y$ .*

This is an immediate consequence of H. Cartan's theorem [1] combined with Grauert's [3] or of Shiga's [5] and which might be more or less obvious if one admits that an analytic manifold can be analytically and regularly imbedded into some  $\mathbf{R}^m$ .

We say that a subalgebra  $L$  of  $L(M)$  is *transitive* at a point  $x$  of  $M$  when  $L(x)$  equals  $T_x(M)$ . Hereafter we will write  $L$  for  $L(M)$ . The following (2.2) and (2.3) are corollaries to (2.1).

(2.2)  *$L$  is transitive at each point of  $M$ .*

(2.3) *If  $x$  and  $y$  are distinct points of  $M$ , then  $L_x$  is transitive at  $y$ .*

(2.4) *If  $L'$  is a subalgebra of  $L$  with  $\dim L/L' = n = \dim M$ , then, for any point  $x$  of  $M$ ,  $L'$  is either transitive at  $x$  or equal to  $L_x$ .*

PROOF. (The compactness of  $M$  is not necessary.) Suppose neither is the case. Then  $q = \dim L'/L'_x$  differs from 0 and  $n$ . Let  $L_1$  be the ideal  $\{u \in L_x | [u, L] \subset L_x\}$  of  $L_x$ . Then the Lie algebra  $L_{(0)} = L_x/L_1$  naturally acts on  $L_{(-1)} = L/L_x$  as the general linear Lie algebra on  $L_{(-1)}$ ,  $\dim L_{(0)} = n^2$ , by (2.1). Its subalgebra  $L'_x/L'_x \cap L_1$  leaves invariant the subspace  $L'/L'_x$  of  $L/L_x$ . It follows that  $\dim L'/L_1 \cap L' \leq (n+n^2) - ((n-q) + (n-q)q) = \dim L/L_1 - (n-q)(q+1)$ . Hence we have  $q = n-1$  by the assumption  $\dim L/L' = n$ . Thus  $L' \supset L_1$ . It is not hard to see that this leads to a contradiction by showing that  $[L', L_1]$  must necessarily contain  $L_x$  in view of (2.1). (2.4) is proved.

Now we begin with the proof of Theorem 2. (It might be worth noting that  $M$  and  $N$  are not assumed to be simply connected. By this reason,  $L(M)$  or  $L(N)$  cannot be replaced by an arbitrary transitive subalgebra.) We have  $\dim M = \dim N$  by (2.2). Let  $\pi^M, \pi^N$  be the projections of  $M \times N$  onto  $M, N$  respectively. The differential  $d\pi^M$  sends  $L^*(x, y) \subset T_{(x,y)}(M \times N)$  onto  $T_x(M)$

for any point  $(x, y)$  of  $M \times N$  by (2.2), and the analogous for  $d\pi^N$ . The kernel of the restriction  $d\pi^M|L^*(x, y)$  equals  $\{u^* \in L^* | u \in L(M)_x\}$ . The analogous for  $d\pi^N$  is  $\{u^* \in L^* | \alpha(u) \in L(N)_y\}$ . Hence, in case  $\alpha(L(M)_x) = L(N)_y$ ,  $L^*(x, y)$  is isomorphic with  $T_x(M)$  through  $d\pi^M$ . And, in the contrary case,  $\alpha(L(M)_x)$  is transitive at  $y$  by (2.4) (applied to  $N$ ), which shows that  $L^*$  is transitive at  $(x, y)$ . Thus we have proved

(2.5) *The following four conditions are equivalent for a point  $(x, y)$  of  $M \times N$ :*

- (1)  $\alpha(L(M)_x) = L(N)_y$ ,
- (2)  $L^*$  is not transitive at  $(x, y)$ ,
- (3)  $d\pi^M$  induces an isomorphism of  $L^*(x, y)$  onto  $T_x(M)$ , and
- (4)  $d\pi^N$  induces an isomorphism of  $L^*(x, y)$  onto  $T_y(N)$ .

To conclude the proof of Theorem 2, we denote by  $\Gamma$  the maximal integral manifold passing through  $(a, b)$  of the linear differential system  $L^*$ . By (2.5) and the assumption that  $\alpha(L(M)_a) = L(N)_b$ ,  $\Gamma$  is immersed as an open submanifold into  $M$  (resp.  $N$ ) by  $\pi^M$  (resp.  $\pi^N$ ). Hence the image  $\pi^M(\Gamma)$  (resp.  $\pi^N(\Gamma)$ ) will coincide with  $M$  (resp.  $N$ ) if  $\Gamma$  is proved to be compact. And  $\Gamma$  will be compact if it is closed. Suppose  $(x, y)$  adheres to  $\Gamma$ . Clearly we have  $\dim L^*(x, y) = n$  by (2.5). By (2.5) and the argument above, there exists a point  $x'$  of  $M$  such that  $(x', y)$  belongs to  $\Gamma$ . Again by (2.5) we have  $\alpha(L(M)_{x'}) = L(N)_y = \alpha(L(M)_x)$ , hence  $L(M)_{x'} = L(M)_x$ . We thus have  $x' = x$ , by (2.3) and (2.5). Thus,  $(x, y)$  belongs to  $\Gamma$ . Finally it remains to show that  $\pi^M$  (resp.  $\pi^N$ ) restricted to  $\Gamma$  is injective. (Then  $\Gamma$  will be the graph of the required map  $f: M \rightarrow N$ . That  $f$  induces  $\alpha$  will be obvious.) But the proof of the injectiveness is implicit in that of the closedness of  $\Gamma$ . Theorem 2 is proved.

QUESTION 3. Can  $L(M)$ ,  $L(N)$  in Theorem 2 be replaced by transitive Lie algebras when  $M$ ,  $N$  are simply connected?

QUESTION 4. Is Theorem 2 true in the  $C^\infty$ -category? Moreover can such a question be discussed in terms of some relation between Lie algebras in the two categories?

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