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Remarks on evolution inequalities

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Introduction

The Lax-Milgram Lemma was extended by G. Stampacchia [4] to the following: if $a(u, v)$ is a continuous bilinear form *coercive* on a (real) Hilbert space V and if K is a closed convex set in V, then, given a continuous linear form $v \to L(v)$ on V, there exists a unique element u in K such that $a(u, v-u) \geqq L(v-u)$ $\forall v \in K$ (the Lax-Milgram Lemma corresponds to the case when $K = V$).

In a joint work with Stampacchia [10] [11] we studied similar problems for bilinear forms which are (i) either $\geqq 0$ but *not coercive* (ii) either defined *on two different Hilbert spaces.*

In this paper we give some complements to the result of [11] on (ii). This will solve, as a particular case—see Section 3.3. below—the following *non linear boundary value problem*: find a function $u(x, t)$, $x \in \Omega \subset \boldsymbol{R}^n$, $t \in (0, T)$, $T < \infty$, such that:

$$\frac{\partial u}{\partial t} - \Delta u + u = f \qquad \left(\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}\right), \tag{1}$$

$$u \geqq 0 \quad \text{on} \quad \Gamma \times (0, T) \qquad (\Gamma = \text{boundary of } \Omega) \tag{2}$$

$$\frac{\partial u}{\partial \nu} \geqq 0 \quad \text{on} \quad \Gamma \times (0, T) \qquad \left(\frac{\partial}{\partial \nu} = \text{exterior normal derivative}\right)$$

$$u \cdot \frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \Gamma \times (0, T),$$

$$u(x, 0) = u(x, T). \tag{3}$$

(The case when instead of the "*periodic problem*" (3) we consider the "*initial value problem*" $u(x, 0) =$ given was solved in [11].)

Section 1 gives a general existence theorem; Section 2 gives applications to "ordinary" evolution equations and Section 3 gives a general existence

1) This paper develops technical details of part of a lecture given at the Annual Meeting of the Math. Soc. Japan, Kyoto, May 1966. Other parts of the lecture corresponded to a joint work with G. Stampacchia [10] [11].

and uniqueness theorem (Theorem 3.1) which solves, as a rather particular case, problem (1) (2) (3).

1. An existence theorem

1.1. Hypothese. Existence result.

Let $\mathcal{V}$ and $\mathcal{H}$ be two Hilbert spaces on $\boldsymbol{C}$[2], with $\mathcal{V}\subset\mathcal{H}$, the injection $\mathcal{V}\rightarrow\mathcal{H}$ being continuous and $\mathcal{V}$ being dense in $\mathcal{H}$. We denote by $(\,,)$ the scalar product in $\mathcal{H}$. If we identify $\mathcal{H}$ to its anti-dual, and if $\mathcal{V}'$ denotes the anti-dual of $\mathcal{V}$, we have

$$\mathcal{V}\subset\mathcal{H}\subset\mathcal{V}';$$

if $f\in\mathcal{V}'$, $v\in\mathcal{V}$, (f, v) denotes the scalar product of f and v; in case $f\in\mathcal{H}$, (f, v) coincides with the scalar product in $\mathcal{H}$, which justifies the notation. The three main data are the operators $\mathcal{A}$ and ∂ and the convex set $\mathcal{K}$.

The operator $\mathcal{A}$ is given in $\mathcal{L}(\mathcal{V};\mathcal{V}')$ (i.e. space of continuous linear mappings from $\mathcal{V}$ to $\mathcal{V}'$), such that

$$\mathrm{Re}\,(\mathcal{A}v, v)\geqq\alpha\|v\|_{\mathcal{V}}^{2},\ \alpha>0,\ \forall v\in\mathcal{V}. \tag{1.1}$$

The operator ∂ is an unbounded operator in $\mathcal{V}'$, with domain $D(\partial)$ dense in $\mathcal{V}'$; we assume that ∂ is closed and that

$$\mathrm{Re}\,(v, \partial v)\geqq 0 \qquad \forall v\in\mathcal{V}\cap D(\partial). \tag{1.2}$$

The set $\mathcal{K}$ is closed and convex in $\mathcal{V}$ such that

$$\mathcal{K}\cap D(\partial)\neq\phi. \tag{1.3}$$

We can now state:

THEOREM 1.1. *Let* $\mathcal{A}$, ∂, $\mathcal{K}$ *be given, satisfying* (1.1) (1.2) (1.3). *There exists* $u\in\mathcal{K}$ *such that*

$$\mathrm{Re}\,[(\mathcal{A}u, v-u)+(u, \partial v)-(f, v-u)]\geqq 0 \qquad \forall v\in\mathcal{K}\cap D(\partial). \tag{1.4}$$

1.2. PROOF OF THEOREM 1.1.

1) As a first step we use the *elliptic regularization* [7] analogously to [11], Section 7. Let $\varepsilon>0$. We consider on $\mathcal{W}=\mathcal{V}\cap D(\partial)$ the sesquilinear form

$$\pi_{\varepsilon}(u, v)=(\mathcal{A}u, v)+(u, \partial v)+\varepsilon(\partial u, \partial v)_{\mathcal{V}'}.$$

We provide $\mathcal{W}$ with the *Hilbertian* norm

$$(\|v\|_{\mathcal{V}}^{2}+\|\partial v\|_{\mathcal{V}'}^{2})^{1/2}=\|v\|_{\mathcal{W}}.$$

We have, thanks to (1.1) and (1.2):

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2) In the case of *real* Hilbert spaces, one has just to drop the "Real part" in the inequalities of the paper.

(1.5) $$\text{Re}\,\pi_\varepsilon(v, v) \geqq \alpha \| v \|^2_{\mathcal{V}} + \varepsilon \| \partial v \|^2_{\mathcal{V}'} \geqq \inf(\alpha, \varepsilon) \| v \|^2_{\mathcal{W}};$$

therefore, due to the Stampacchia's theorem [14], there exists a unique element $u_\varepsilon \in \mathcal{K} \cap D(\partial)$ such that

(1.6) $$\text{Re}\,\pi_\varepsilon(u_\varepsilon, v-u_\varepsilon) \geqq \text{Re}\,(f, v-u_\varepsilon) \qquad \forall v \in \mathcal{K} \cap D(\partial).$$

2) In the second step, we prove that we can extract from u_ε a subsequence which converges (weakly) to a solution of (1.4). Let v be choosen *fixed* in $\mathcal{K} \cap D(\partial)$. It follows from (1.6) and (1.5) that

$$\begin{aligned} \alpha \| u_\varepsilon \|^2_{\mathcal{V}} + \varepsilon \| \partial u_\varepsilon \|^2_{\mathcal{V}'} &\leqq \text{Re}\,\pi_\varepsilon(u_\varepsilon, v) - \text{Re}\,(f, v-u_\varepsilon) \\ &\leqq \| \mathcal{A} u_\varepsilon \|_{\mathcal{V}'} \| v \|_{\mathcal{V}} + \| u_\varepsilon \|_{\mathcal{V}} \| \partial v \|_{\mathcal{V}'} + \varepsilon \| \partial u_\varepsilon \|_{\mathcal{V}'} \| \partial v \|_{\mathcal{V}'} \\ &\quad + \| f \|_{\mathcal{V}'} \| u_\varepsilon \|_{\mathcal{V}} + \| f \|_{\mathcal{V}'} \| v \|_{\mathcal{V}} \text{ [5]} \\ &\leqq C \| u_\varepsilon \|_{\mathcal{V}} + C\varepsilon \| \partial u \|_{\mathcal{V}'} + C \\ &\leqq \frac{\alpha}{2} \| u_\varepsilon \|^2_{\mathcal{V}} + \frac{\varepsilon}{2} \| \partial u_\varepsilon \|^2_{\mathcal{V}'} + C \end{aligned}$$

hence

(1.7) $$\| u_\varepsilon \|^2_{\mathcal{V}} + \varepsilon \| \partial u \|^2_{\mathcal{V}'} \leqq C.$$

Therefore we can extract a subsequence, say u_η, $\eta \to 0$, such that

(1.8) $$u_\eta \to w \quad \text{weakly in } \mathcal{V}.$$

Since $u_\eta \in \mathcal{K}$ and since $\mathcal{K}$ is weakly closed in $\mathcal{V}$, it follows that

(1.9) $$w \in \mathcal{K}.$$

We deduce from (1.6)

$$\text{Re}\,(\mathcal{A} u_\varepsilon, u_\varepsilon) + (u_\varepsilon, \partial u_\varepsilon) + \varepsilon \| \partial u_\varepsilon \|^2_{\mathcal{V}'} \leqq \text{Re}\,\pi_\varepsilon(u_\varepsilon, v) - \text{Re}\,(f, v-u_\varepsilon).$$

Since $(u_\varepsilon, \partial u_\varepsilon) \geqq 0$, it follows that (taking $\varepsilon = \eta$)

(1.10) $$\text{Re}\,(\mathcal{A} u_\eta, u_\eta) \leqq \text{Re}\,\pi_\eta(u_\eta, v) - \text{Re}\,(f, v-u_\eta).$$

But

$$\liminf_{\eta \to 0} \text{Re}\,(\mathcal{A} u_\eta, u_\eta) \geqq \text{Re}\,(\mathcal{A} w, w)$$

so that (1.10) implies

$$\text{Re}\,(\mathcal{A} w, w) \leqq \text{Re}\,\pi(w, v) - \text{Re}\,(f, v-w)$$

where $\pi(w, v) = (\mathcal{A} w, v) + (w, \partial v)$. In other words

$$\text{Re}\,[(\mathcal{A} w, v-w) + (w, \partial v) - (f, v-w)] \geqq 0$$

and one can take $u = w$ as a solution of (1.4).

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5) The C's denote various constants (which do not depend on ε).

1.3. The set of solutions.

It is not known whether there is *uniqueness* or not of the solution of the inequation (1.4). We will give below in Section 3 examples where the uniqueness holds. For the time being let us notice (this is a variant of Theorem 3.1 of [11]):

(1.11) The set X of all solutions $u\in\mathcal{K}$ of (1.4) is closed and convex.

Let us check that X is convex (it is obviously closed). Let u_1 and u_2 be two elements of X; if $0<\Theta<1$, we have (after an easy calculation)

$$\begin{aligned}&\operatorname{Re}(\mathcal{A}(\Theta u_1+(1-\Theta)u_2),\ v-(\Theta u_1+(1-\Theta)u_2))\\ &\qquad=\operatorname{Re}[\Theta(\mathcal{A}u_1,\ v-u_1)+(1-\Theta)(\mathcal{A}u_2,\ v-u_2)+\Theta(1-\Theta)(\mathcal{A}(u_2-u_1),\ u_2-u_1)]\end{aligned}$$

so that

$$\begin{aligned}&\operatorname{Re}[(\mathcal{A}(\Theta u_1+(1-\Theta)u_2),\ v-(\Theta u_1+(1-\Theta)u_2))+(\Theta u_1+(1-\Theta)u_2,\ \partial v)\\ &\qquad-(f,\ v-(\Theta u_1+(1-\Theta)u_2))]\geqq\Theta(1-\Theta)\alpha\|u_2-u_1\|^2_{\mathcal{V}}\geqq 0\end{aligned}$$

hence (1.11) follows.

1.4. Example of operator ∂.

Let $G(s)$ be a continuous semi-group [4] [16] in $\mathcal{V}'$ and in $\mathcal{V}$; in other words $G(s)\in\mathcal{L}(\mathcal{V}';\mathcal{V}')\cap\mathcal{L}(\mathcal{V};\mathcal{V})$, $G(s)G(t)=G(s+t)$ and $\forall v\in\mathcal{V}$ (resp. $\mathcal{V}'$) $s\to G(s)v$ is continuous from $s\geqq 0\to\mathcal{V}$ (resp. $\mathcal{V}'$). It follows from interpolation theory in Hilbert spaces [6] that $G(s)$ is a continuous semi-group in $\mathcal{H}$. We assume that

$$\|G(s)\|_{\mathcal{L}(\mathcal{H};\mathcal{H})}\leqq 1,\qquad s\geqq 0\,. \tag{1.12}$$

Let $G^*(s)$ be the adjoint semi-group of $G(s)$; it has analogous properties. We now define:

$$-\Lambda\ (\text{resp. } -\Lambda^*)=\text{infinitesimal generator of } G\ (\text{resp. } G^*)\,. \tag{1.13}$$

More precisely, $D(\Lambda;\mathcal{V})$ (resp. $D(\Lambda;\mathcal{V}')$, resp. $D(\Lambda;\mathcal{H})$) denotes the domain of Λ in $\mathcal{V}$ (resp. $\mathcal{V}'$, resp. $\mathcal{H}$). Same thing for Λ^*.

We now define:

$$\partial=\Lambda^*\qquad D(\partial)=D(\Lambda^*;\mathcal{V}')\,. \tag{1.14}$$

PROPOSITION 1.1. *If* (1.12) *holds true, then one has* (1.2) *for the choice* (1.14) *of* ∂.

PROOF. Let v be given in $\mathcal{V}\cap D(\Lambda^*;\mathcal{V}')$. Let ρ_n be a regularizing sequence of C^∞ functions of t, with compact support in $t>0$, $\rho_n(t)\geqq 0$, $\int_0^\infty\rho_n(t)dt=1$, support of $\rho_n\in[0,\varepsilon_n]$, $\varepsilon_n\to 0$. We define

(1.15) $$G^*(\rho_n)\cdot v=\int_0^\infty G^*(s)v\cdot\rho_n(s)ds.$$

Then $G^*(\rho_n)v\in D(\Lambda^*;\mathcal{V})$, (hence in $D(\Lambda^*;\mathcal{H})$) and $\Lambda^*G^*(\rho_n)v\to\Lambda^*v$ in $\mathcal{V}'$ as $n\to\infty$. We have

(1.16) $$\operatorname{Re}(v,\Lambda^*v)=\lim_{n\to\infty}\operatorname{Re}(G^*(\rho_n)v,\Lambda^*G^*(\rho_n)v).$$

But $G^*(s)$ being a contraction semi group in $\mathcal{H}$, one has ($-\Lambda^*$ is the infinitesimal generator of $G^*(s)$) $\operatorname{Re}(G^*(\rho_n)v,\Lambda^*G^*(\rho_n)v)\geqq 0$ and the desired result follows from (1.16).

2. The case of equations

2.1. A general result.

THEOREM 2.1. *Let* $\mathcal{A}$ *be given satisfying* (1.1); *let* ∂ *be given by* (1.14) *with* (1.12) (1.13) *and let* f *be given in* $\mathcal{V}'$. *There exists one element* u *and only one such that*

(2.1) $$u\in\mathcal{V}\cap D(\Lambda;\mathcal{V}')$$

(2.2) $$\mathcal{A}u+\Lambda u=f.$$

PROOF.

1) Existence.

We apply Theorem 1.1 with $\mathcal{K}=\mathcal{V}$; there exists $u\in\mathcal{V}$ such that

$$\operatorname{Re}[(\mathcal{A}u,v-u)+(u,\Lambda^*v)-(f,v-u)]\geqq 0\qquad\forall v\in\mathcal{V}\cap D(\Lambda^*;\mathcal{V}').$$

Then

$$\operatorname{Re}[(\mathcal{A}u,v)+(u,\Lambda^*v)-(f,v)]\geqq\operatorname{Re}[(\mathcal{A}u,u)-(f,u)]\quad\forall v\in\mathcal{V}\cap D(\Lambda^*;\mathcal{V}')$$

which implies

(2.3) $$(\mathcal{A}u,v)+(u,\Lambda^*v)=(f,v)\qquad\forall v\in\mathcal{V}\cap D(\Lambda^*;\mathcal{V}')$$

and

(2.4) $$(\mathcal{A}u,u)=(f,u).$$

It follows from (2.3) that the form $v\to(u,\Lambda^*v)=(f-\mathcal{A}u,v)$ is continuous on $D(\Lambda^*;\mathcal{V})$ provided with the topology of $\mathcal{V}$, so that $u\in D(\Lambda;\mathcal{V}')$ (hence u satisfies (2.1)) and $(u,\Lambda^*v)=(\Lambda u,v)$. Therefore

$$(\mathcal{A}u+\Lambda u-f,v)=0\qquad\forall v\in\mathcal{V}\cap D(\Lambda^*;\mathcal{V}').$$

But $\mathcal{V}\cap D(\Lambda^*;\mathcal{V}')$ is dense in $\mathcal{V}$ and (2.2) is satisfied.

2) Uniqueness.

Let u be in $\mathcal{V}\cap D(\Lambda;\mathcal{V}')$ satisfying

$$\mathcal{A}u+\Lambda u=0.$$

We have to prove that $u=0$. But

$$\mathrm{Re}(\mathcal{A}u, u)+\mathrm{Re}(\Lambda u, u)=0,$$

and it will be enough to prove that

$$\mathrm{Re}(\Lambda u, u)\geqq 0 \qquad \forall u\in \mathcal{V}\cap D(\Lambda;\mathcal{V}').$$

This is Proposition 1.1 (with Λ^* replaced by Λ).

2.2. Examples.

EXAMPLE 2.1. Let V and H be two Hilbert spaces, with $V\subset H$, $V\to H$ continuous, V dense in H. Let V' be the anti-dual of V; if we identify H to its anti-dual, we have

$$V\subset H\subset V'.$$

We define, for T given $<\infty$,

$\mathcal{V}=L_2(0, T; V)$ (square integrable functions in $(0, T)$ with values in V),

$\mathcal{H}=L_2(0, T; H)$; then $\mathcal{V}'=L_2(0, T; V')$.

Let $A(t)$, $t\in(0, T)$, be a family of operators which satisfy:

$$A(t)\in\mathcal{L}(V; V') \tag{2.5}$$

$$t\to\langle A(t)u, v\rangle \text{ is measurable } \forall u, v\in V^{4)} \tag{2.6}$$

$$\mathrm{Re}\langle A(t)v, v\rangle\geqq\alpha\|v\|_V^2 \quad \text{for a.e. } t. \quad \forall v\in V,\ \alpha>0. \tag{2.7}$$

We define $\mathcal{A}$ by

$$\mathcal{A}v(t)=A(t)v(t) \quad \text{for a.e. } t., \quad v\in\mathcal{V} \tag{2.8}$$

and $\mathcal{A}$ satisfies (1.1).

We define next the semi-group $G(s)$ by

$$G(s)f(t)=\begin{cases} 0 & \text{if } t<s \\ f(t-s) & \text{if } s<t<T \end{cases}, \quad f\in\mathcal{V} \text{ (or } \mathcal{H} \text{ or } \mathcal{V}'). \tag{2.9}$$

This is a continuous contraction semi group in $\mathcal{H}$ (and $\mathcal{V}$, and $\mathcal{V}'$). The adjoint semi-group is defined by

$$G^*(s)f(t)=\begin{cases} f(t+s) & \text{if } 0<t<T-s \\ 0 & \text{if } T-s<t<T. \end{cases} \tag{2.10}$$

We have:

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4) $\langle , \rangle$ denotes the anti-linear scalar product between V' and V and the scalar product in H.

(2.11) $$D(\Lambda\,;\,\mathcal{V}')=\left\{f\,|\,f\in\mathcal{V}',\ \frac{df}{dt}\in\mathcal{V}',\ f(0)=0\right\}$$

and

(2.12) $$\Lambda f=\frac{df}{dt}.$$

THEOREM 2.1. gives: *For f given in* $\mathcal{V}'$, *there exists an element u in* $\mathcal{V}$ *and only one such that*

(2.13) $$A(t)u(t)+\frac{du}{dt}(t)=f(t) \qquad \text{a.e.}$$

(2.14) $$u(0)=0,$$

if the family $A(t)$ *satisfies* (2.5) (2.6) (2.7). This result was proved, by a different method in [5] and by essentially the same method in [7].

EXAMPLE 2.2. We choose $\mathcal{V}$, $\mathcal{H}$, $\mathcal{V}'$, $\mathcal{A}$ as in Example 2.1. We define now $G(s)$ by

(2.15) $$G(s)f(t)=\begin{cases} f(t-s+T) & \text{if } \ 0<t<s \\ f(t-s) & \text{if } \ s<t<T. \end{cases}$$

In this case it is a *group*; if $s<0$,

$$G(s)f(t)=\begin{cases} f(t-s) & \text{if } \ 0<t<T+s \\ f(t+s-T) & \text{if } \ T+s<t<T \end{cases}$$

and $G^*(s)=G(-s)$.

We have

(2.16) $$D(\Lambda\,;\,\mathcal{V}')=\left\{f\,|\,f\in\mathcal{V}',\ \frac{df}{dt}\in\mathcal{V}',\ f(0)=f(T)\right\}$$

and Theorem 2.1 gives: *for f given in* $\mathcal{V}'$, *there exists an element u in* $\mathcal{V}$ *and only one such that*

(2.17) $$A(t)u(t)+\frac{du(t)}{dt}=f(t) \qquad \text{a.e.}$$

(2.18) $$u(0)=u(T).$$

We obtain the existence and uniqueness of a *periodic* solution of (2.17), cf. another proof in [5].

2.3. Remarks.

REMARK 2.1. The solutions u of (2.2) depends continuously on f; one can state:

(2.19) $\mathcal{A}+\Lambda$ is an isomorphism form $\mathcal{V}\cap D(\Lambda\,;\,\mathcal{V}')$ onto $\mathcal{V}'$.

If $\mathcal{A}^*$ denotes the adjoint of $\mathcal{A}$, we have in the same way

(2.19 bis) $\mathcal{A}^*+\Lambda^*$ is an isomorphism form $\mathcal{V} \cap D(\Lambda^* ; \mathcal{V}')$ onto $\mathcal{V}'$.

REMARK 2.2. We can now transpose (2.19 bis) to obtain (we do not give explicit technical details).

(2.20) $\mathcal{A}+\Lambda$ is an isomorphism from $\mathcal{V}$ onto $(\mathcal{V} \cap D(\Lambda^* ; \mathcal{V}'))'$.

By interpolation in Hilbert spaces [**6**] (and with the notations of [**1**] [**8**]) we get

(2.21) $\mathcal{A}+\Lambda$ is an isomorphism from $[\mathcal{V} \cap D(\Lambda ; \mathcal{V}'), \mathcal{V}]_{1/2}=\Phi$ onto

$$[\mathcal{V}', (\mathcal{V} \cap D(\Lambda^* ; \mathcal{V}'))']_{1/2}=\Psi .$$

It is interesting to know if the two spaces Φ and Ψ are in duality; this is related to questions considered in [**3**] [**9**]; for parabolic partial differential operators, this question was considered in [**2**].

In case of Example 2.1 the spaces Φ and Ψ are *not* in duality. In case of Example 2.2, $D(\Lambda, \mathcal{V}')=D(\Lambda^* ; \mathcal{V}')$ and then $\Phi=\Psi'$. [In all these examples one can give a constructive definition of Φ and Ψ. See [**17**].]

3. A uniqueness theorem for inequalities

3.1. General result.

THEOREM 3.1. *Hypotheses of Theorem* 1.1. *We assume moreover that*

(3.1) $G(s)$ *is a group of unitary operators in* $\mathcal{H}$;

(3.2) $$G(s)\mathcal{K} \subset \mathcal{K} \quad \forall s .$$

Then (1.4) *admits a unique solution.*

PROOF (of uniqueness).

1) Let ρ be a C^∞ function on $\boldsymbol{R}^s$, with compact support and *even*. We define $G(\varphi)v=\int_{-\infty}^{+\infty} G(s)v\varphi(s)ds \quad \forall\varphi$ continuous with compact support, (and the analogous definition for $G^*(\varphi)$). We note that

(3.3) $$G(\rho')+G^*(\rho')=0 \qquad \left(\rho'=\frac{d\rho}{ds}\right)$$

(since $G^*(s)=G(-s)$ and ρ' is odd, ρ being even).

We remark now that

(3.4) $$\text{if} \quad \rho \geqq 0, \ \int_{-\infty}^{+\infty}\rho(s)ds=1, \quad \text{then} \quad G^*(\rho)v \in \mathcal{K} \cap D(\Lambda^* ; \mathcal{V}') .$$

The only thing to check is that $G^*(\rho)v \in \mathcal{K}$; but by hypothesis $G^*(s)v \in \mathcal{K}$ and then $\int_{-\infty}^{+\infty} G^*(s)v \cdot \rho(s)ds \in \mathcal{K}$ since $\mathcal{K}$ is convex (convexity theorem).

2) Let now u_1 and u_2 be two solutions of (1.4). Therefore:

$$(3.5)_j \quad \begin{cases} [\operatorname{Re}(\mathcal{A}u_j, v-u_j)+(u_j, \Lambda^*v)-(f, v-u_j)] \geqq 0 \quad \forall v \in \mathcal{V} \cap D(\Lambda^*; \mathcal{V}') \\ j=1, 2. \end{cases}$$

We take a regularizing sequence of C^∞ functions, say ρ_n, which are *even*, and we choose:

$$v = G^*(\rho_n)u_2 \quad \text{in } (3.5)_1, \qquad G^*(\rho_n)u_1 \quad \text{in } (3.5)_2$$

(which is allowed, thanks to (3.4)).

Adding up, we obtain, after setting

$$X_n = (\mathcal{A}u_1, G^*(\rho_n)u_2-u_1)+(\mathcal{A}u_2, G^*(\rho_n)u_1-u_2),$$

$$Y_n = (u_1, G^*(\rho_n')u_2)+(u_2, G^*(\rho_n')u_1),$$

$$Z_n = (f, G^*(\rho_n)u_2-u_1+G^*(\rho_n)u_1-u_2),$$

that

$$\operatorname{Re}(X_n+Y_n-Z_n) \geqq 0.$$

But

$$2\operatorname{Re} Y_n = ((G(\rho_n')+G^*(\rho_n'))u_1, u_2)+((G(\rho_n')+G^*(\rho_n'))u_2, u_1)=0$$

by using (3.3). As $n \to \infty$,

$$Z_n \to 0 \quad \text{and} \quad X_n \to -(\mathcal{A}(u_1-u_2), u_1-u_2)$$

so that we obtain at the limit

$$-\operatorname{Re}(\mathcal{A}(u_1-u_2), u_1-u_2) \geqq 0$$

i.e.

$$\alpha \| u_1-u_2 \|_{\mathcal{V}}^2 \leqq \operatorname{Re}(\mathcal{A}(u_1-u_2), u_1-u_2) \leqq 0, \qquad \text{hence} \quad u_1 = u_2.$$

REMARK 3.1. One can give a complement to Theorem 3.1 when $\mathcal{K}$ is a *cone*:

THEOREM 3.2. *Hypotheses of Theorem* 3.1. *We assume moreover that* (3.6) $\mathcal{K}$ *is a cone.*

Then (1.4) *is equivalent to the system*

$$(3.7) \qquad \operatorname{Re}[(\mathcal{A}u, v)+(u, \Lambda^*v)-(f, v)] \geqq 0 \qquad \forall v \in \mathcal{K} \cap D(\Lambda^*, \mathcal{V}'),$$

$$(3.8) \qquad \operatorname{Re}[(\mathcal{A}u, u)-(f, u)] = 0.$$

PROOF. We have only to prove that the unique solution u of (1.4) satisfies (3.7) and (3.8).

If w is arbitrarily chosen in $\mathcal{K} \cap D(\Lambda^*; \mathcal{V}')$, we take in (1.4)

$$v = v_n = G^*(\rho_n)u+w \qquad \text{(notations of the proof of Theorem 3.1).}$$

We get:

$$\operatorname{Re}[(\mathcal{A}u, C^*(\rho_n)u-u+w)+(u, G^*(\rho_n')u)+(u, \Lambda^*w)-(f, G^*(\rho_n)u-u+w)] \geqq 0$$

But (cf. (3.3)) $\operatorname{Re}(u, G^*(\rho_n')u)=0$ and if we let $n \to \infty$, we obtain

$$\operatorname{Re}[(\mathcal{A}u, w)+(u, \Lambda^* w)-(f, w)] \geqq 0$$

which proves (3.7).

If we take $w=G^*(\rho_n)u$ in (3.7) and let $n \to \infty$, we obtain

$$\operatorname{Re}[(\mathcal{A}u, u)-(f, u)] \geqq 0 .$$

On another hand, taking $v=0$ in (1.4) leads to the opposite inequality, hence (3.8) follows, which completes the proof.

3.2. Example.

We choose $\mathcal{V}$, $\mathcal{H}$, $\mathcal{V}'$, $\mathcal{A}$, $G(s)$ as in Example 2.2, Section 2.2. We now take:

(3.9) $$K = \text{closed convex set in } V,$$

and

(3.10) $$\mathcal{K} = \{v \mid v \in L_2(0, T, V),\ v(t) \in K \quad \text{for a.e. } t\} .$$

We define in this way a *closed* convex set in $\mathcal{V}$ and *condition* (3.2) *is satisfied.* Theorem 3.1 gives:

There exists an element u and only one in $\mathcal{K}$ such that

(3.11) $$\operatorname{Re}\int_0^T [\langle A(t)u(t), v(t)-u(t)\rangle - \left\langle u(t), \frac{dv(t)}{dt}\right\rangle - \langle f(t), v(t)-u(t)\rangle] dt \geqq 0$$

$$\forall v \in \mathcal{K} \text{ such that } \frac{dv}{dt} \in \mathcal{V}' \quad \text{and} \quad v(0)=v(T).$$

3.3. The case of partial differential operators.

Let Ω be an open set in $\boldsymbol{R}^n$. With notations of Example 2.1, Section 2.2, we take:

(3.12) $$V = H^1(\Omega) = \{v \mid v \in L_2(\Omega),\ \frac{\partial v}{\partial x_i} \in L_2(\Omega),\ i=1, \cdots, n\}^{5)}$$

(Sobolev space [13]), with the norm:

$$\|v\|_V = \left(\int_\Omega (|v|^2 + \sum_{i=1}^n \left|\frac{\partial v}{\partial x_i}\right|^2) dx\right)^{\frac{1}{2}};$$

we take next

(3.13) $$H = L_2(\Omega);$$

the anti dual V' is *not* a space of distributions on Ω [12].

To simplify a little bit, let us take only *real valued functions* (we shall therefore suppress the Re in (3.11)).

5) Derivatives are taken in the sense of distributions in Ω.

We take

$$A(t)=A\left(x, t, \frac{\partial}{\partial x}\right)=-\sum \frac{\partial}{\partial x_i}\left(a_{ij}(x, t)\frac{\partial}{\partial x_j}\right)+a_0(x, t) \tag{3.14}$$

where

$$a_0,\ a_{ij}\in L_\infty(\Omega\times(0, T)),$$

$$a_0(x, t)\geqq\alpha>0 \qquad \text{a.e.,}$$

$$\sum_{i,j=1}^{n} a_{ij}(x, t)\xi_i\xi_j\geqq\alpha(\xi_1^2+\cdots+\xi_n^2),\ \alpha>0, \qquad \text{a.e..}$$

Then (*in the duality* between V and V'),

$$\langle A(t)u, v\rangle=\sum_{i,j=1}^{n}\int_\Omega a_{ij}(x, t)\frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_j}dx+\int_\Omega a_0(x, t)uvdx,$$

and one has (2.5) (2.6) (2.7).

We choose now (see [**15**])

$$K=\{v\,|\,v\in H^1(\Omega),\ v\geqq 0 \text{ a.e. on } \Gamma=\text{boundary of } \Omega\}. \tag{3.15}$$

(This makes sense for Ω with an arbitrary boundary; in any case it is simple to check that we define in this way *a closed* convex set of $H^1(\Omega)$ if Γ is "smooth".)

In this case $\mathcal{K}$ (defined by (3.10)) is a *cone* and Theorem 3.2 applies. There exists one element u and only one in $\mathcal{K}$ such that

$$\begin{cases} \displaystyle\int_0^T[\langle A(t)u(t), v(t)\rangle-\left\langle u(t), \frac{dv(t)}{dt}\right\rangle-\langle f(t), v(t)\rangle]dt\geqq 0 \\ \forall v\in\mathcal{K},\ \dfrac{dv}{dt}\in\mathcal{V}',\ v(0)=v(T) \end{cases} \tag{3.16}$$

and

$$\int_0^T[\langle A(t)u(t), u(t)\rangle-\langle f(t), u(t)\rangle]dt=0. \tag{3.17}$$

We can interpret (3.16) (3.17) in the following way: taking for v a C^∞ function with compact support in $\Omega\times[0, T]$, (3,16) implies

$$A\left(x, t, \frac{\partial}{\partial x}\right)u+\frac{\partial u}{\partial t}=f. \tag{3.18}$$

Next (see [**11**]) if we set

$$\frac{\partial u}{\partial\nu_A}=\sum_{i,j=1}^{n}a_{ij}(x, t)\frac{\partial u}{\partial x_j}\cos(\nu, x_i),\quad \nu=\text{exterior normal to } \Gamma,$$

we see that u must satisfy (formally)

$$u\geqq 0 \quad\text{on}\quad \Gamma\times(0, T),\quad \frac{\partial u}{\partial\nu_A}\geqq 0 \quad\text{on}\quad \Gamma\times(0, T) \tag{3.19}$$

(3.21) $$u \cdot \frac{\partial u}{\partial \nu_A} = 0 \quad \text{on} \quad \Gamma \times (0, T) \quad \text{(this follows from (3.17))}$$

and

(3.21) $$u(x, 0) = u(x, T).$$

REMARK 3.2. We can of course extend the preceeding example to higher order operators, and to other convex sets K. For the case of *initial data* (instead of (3.21)) see [**11**].

REMARK 3.3. Similar problems for hyperbolic operators were considered in a lecture at the Leray's Seminar, December 1965.

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