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On contraction semi-groups and (*di*)-operators

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Lumer and Phillips [5] have studied semi-groups of linear contraction operators in a Banach space by virtue of the notation of semi-inner product introduced by Lumer.

The infinitesimal generator of such a semi-group is dissipative in their terminology. In a Banach lattice Phillips [8] have studied semi-groups of positive contraction operators by virtue of a special semi-inner product and the infinitesimal generator of such a semi-group is dispersive.

In this article we characterize the infinitesimal generators of such semi-groups of operators by virtue of tangent functionals.

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1. We begin this section with a study of some properties of tangent functionals in a Banach space X. For more general results, see Dunford and Schwarz [1].

PROPOSITION 1. *The functional*

$$u(x, y, a)=a^{-1}(\|x+ay\|-\|x\|)$$

is an increasing function of the positive real variables a for any x and y in X. The limit

$$\tau(x, y)=\lim_{a\to 0+} u(x, y, a)$$

exists for any x and y in X.

PROOF. Let $a'\geqq a>0$; then

$$u(x, y, a')-u(x, y, a)\geqq (aa')^{-1}(a\|x+a'y\|-a\|x\|$$
$$-\|ax+aa'y\|-(a'-a)\|x\|+a'\|x\|)=0\,.$$

Thus $u(x, y, a)$ decreases as a decreases. Since

$$u(x, y, a)\geqq -\|y\|,$$

the assertion is proved.

DEFINITION 1. To each pair $\{x, y\}$ of a Banach space X, we associate a real number $\tau'(x, y)$ as follows:

$$\tau'(x, y)=2^{-1}\{\tau(x, y)-\tau(x, -y)\}\,.$$

PROPOSITION 2. *For any x, y and z in X,*

(1) $$\tau(x, y) \leqq \| y \| ,$$

(2) $$\tau(x, y+z) \leqq \tau(x, y)+\tau(x, z) ,$$

(3) $$\tau(x, ay) = a\tau(x, y) \qquad (a \geqq 0) ,$$

(4) $$\tau(x, ax+y) = a\| x \| + \tau(x, y) \qquad (a \text{ real}) ,$$

(5) $$\tau'(x, y) \leqq \tau(x, y) \leqq \| y \| ,$$

(6) $$\tau'(x, ay) = a\tau'(x, y) \qquad (a \text{ real}) ,$$

(7) $$\tau'(x, ax+y) = a\| x \| + \tau'(x, y) \qquad (a \text{ real}) .$$

PROOF. Statements (1)-(3) are obvious. To prove (4) we note that

$$\begin{aligned} \tau(x, ax+y) &= \lim_{b \to 0+} b^{-1}(\| x+bax+by \| - \| x \|) \\ &= \lim_{b \to 0+} (1+ab)b^{-1}(\| x+(1+ab)^{-1}by \| - \| x \|) + a\| x \| \\ &= a\| x \| + \tau(x, y) . \end{aligned}$$

(5)-(7) are readily follows from (1)-(4).

DEFINITION 2. A linear operator A with domain $\mathfrak{D}(A)$ in a Banach space X is called a (dl)-operator if

(dl) $$\tau'(x, Ax) \leqq 0 \qquad (x \in \mathfrak{D}(A)) .$$

LEMMA 1. *If A is a* (dl)*-operator and $\lambda > 0$, then $(\lambda I - A)^{-1}$ exists and is bounded with norm $\leqq \lambda^{-1}$.*

PROOF. Suppose $y \in \mathfrak{D}(A)$ and $x = \lambda y - Ay$. Then

$$\begin{aligned} \lambda \| y \| &= \tau'(y, \lambda y) \\ &\leqq \tau'(y, \lambda y) - \tau'(y, Ay) \\ &= \tau'(y, x) \leqq \| x \| . \end{aligned}$$

DEFINITION 3. Let $\Sigma = \{T_t ; t \geqq 0\}$ be a family of bounded linear operators on X satisfying

(1) $$T_t T_s = T_{t+s}, \ T_0 = I \qquad (t, s \geqq 0) ,$$

(2) $$\lim_{t \to 0+} T_t x = x \qquad (x \in X) ,$$

(3) $$\| T_t \| \leqq 1 \qquad (t \geqq 0) .$$

We shall refer to Σ as a strongly continuous semi-group of contraction operators.

THEOREM 1. *A necessary and sufficient condition for a linear operator A*

with dense domain to generate a strongly continuous semi-group of contraction operators is that A be a (d1)*-operator and that* $\mathfrak{R}(I-A)=X$.

PROOF. We see by Lemma 1 that $(\lambda I-A)^{-1}$ satisfies the norm condition

$$\|(\lambda I-A)^{-1}\|\leqq\lambda^{-1}\qquad(\lambda>0).$$

By assumption $\mathfrak{R}(I-A)=X$, $\lambda=1$ is in the resolvent set of A. Denoting the resolvent of A at λ by $R(\lambda;A)$, it readily follows that

$$R(\lambda;A)=R(1;A)\sum_{n=0}^{\infty}\{(1-\lambda)R(1;A)\}^n$$

for $|\lambda-1|<1$. (See [4] and [10].) And the method of analytic continuation shows that $R(\lambda;A)$ exists and satisfies the norm condition $\|R(\lambda;A)\|\leqq\lambda^{-1}$ for any $\lambda>0$.

Since $\mathfrak{D}(A)$ is dense in X by hypothesis, it follows from the Hille-Yosida theorem (see [4] and [10]) that A generates a strongly continuous semi-group of contraction operators.

Suppose $\{T_t;t\geqq0\}$ is a strongly continuous semi-group of contraction operators. Then

$$\tau'(x,T_tx-x)=\tau'(x,T_tx)-\|x\|\leqq\|T_tx\|-\|x\|\leqq0.$$

Thus, for any $x\in\mathfrak{D}(A)$, we have

$$\tau'(x,Ax)=\lim_{t\to0+}t^{-1}\tau'(x,T_tx-x)\leqq0.$$

Hence A is a (d1)-operator. Moreover it is known that $\mathfrak{D}(A)$ is dense and that $\mathfrak{R}(I-A)=X$.

REMARK 1. A necessary and sufficient condition for a linear operator A with dense domain to generate a strongly continuous group of contraction operators is that $\mathfrak{R}(I\pm A)=X$ and that A satisfies

(d0) $$\tau'(x,Ax)=0\qquad(x\in\mathfrak{D}(A)).$$

The following proposition is essentially due to Lumer and Phillips [5].

PROPOSITION 3. *Let $\{T_t;t\geqq0\}$ be a strongly continuous semi-group of operators with infinitesimal generator A of the local type $\omega(A)$. If we define*

$$\omega=\lim_{t\to0+}t^{-1}\log\|T_t\|$$

then $\omega=\omega(A)$ whenever $\omega<\infty$ and

$$\omega(A)=\theta(A)\equiv\sup\{\tau'(x,Ax);x\in\mathfrak{D}(A),\|x\|=1\}.$$

PROOF. If we define

$$T'_tx=\exp(-\omega t)T_tx\qquad(x\in X),$$

then $\{T'_t;t\geqq0\}$ defines a strongly continuous semi-group of contraction operators with infinitesimal generator $A'=A-\omega I$. It follows from Theorem 1 that

$$0\geqq\theta(A')=\theta(A)-\omega\,.$$

On the other hand if A is a infinitesimal generator then so is $A''=A-\theta(A)I$ and A'' is a (dl)-operator. In fact, for any $x\in\mathfrak{D}(A'')$,

$$\begin{aligned}\tau'(x, A''x)&=\tau'(x, Ax)-\theta(A)\|x\|\\&=\|x\|\{\tau'\Big(\frac{x}{\|x\|}, A\frac{x}{\|x\|}\Big)-\theta(A)\}\leqq 0\,.\end{aligned}$$

As a consequence $\{T_t''=\exp(-\theta(A)t)T_t\}$ is a strongly continuous semi-group of contraction operators and thus

$$\omega(A'')=-\theta(A)+\omega(A)\leqq 0\,.$$

Hence we obtain $\omega(A)=\theta(A)$.

2. In this section we are concerned with the problem of semi-groups of positive contraction operators in a Banach lattice.

Let X be a Banach lattice, that is, X be a complete normed real vector lattice for which the order relation and the norm are related by

$$|x|\leqq|y| \qquad \text{implies} \qquad \|x\|\leqq\|y\|\,;$$

here we have used the notation

$$|x|=x^{+}+x^{-}$$

where $x^{+}=x\vee 0$ and $x^{-}=(-x)\vee 0$.

PROPOSITION 4. *For any x in X, we have;*

(1) *If $x\geqq 0$ then $\tau'(x, y)\geqq 0$ for any $y\geqq 0$,*

(2) $$\tau'(x^{+}, x)=\|x^{+}\|\,,$$

(3) $$\tau'(x^{+}, x^{-})=0\,.$$

PROOF. To prove (1) we consider the relation

$$|x-ay|-|x+ay|=(-2ay)\vee(-2x)\leqq 0$$

and so that

$$\|x-ay\|\leqq\|x+ay\| \qquad (a\geqq 0)\,.$$

Thus we have

$$\begin{aligned}\tau'(x, y)&=2^{-1}\{\tau(x, y)-\tau(x, -y)\}\\&=\lim_{a\to 0+}(2a)^{-1}(\|x+ay\|-\|x-ay\|)\geqq 0\,.\end{aligned}$$

(2) is readily follows from the equality

$$\tau(x^{+}, -x^{-})=\tau(x^{+}, x^{-})\,.$$

The last assertion is follows from the fact that

$$x^+ \wedge x^- = 0,$$

that is, $x = x^+ - x^-$ is the Jordan decomposition of x.

PROPOSITION 5. *If T is a positive linear operator and satisfies*

$$\| Tx \| \leqq \| x \| \qquad (x \geqq 0),$$

then T is a contraction operator.

PROOF. We see that

$$|Tx| \leqq |Tx^+| + |Tx^-| = T|x|$$

and hence

$$\| Tx \| \leqq \| T|x| \| \leqq \| |x| \| = \| x \| .$$

DEFINITION 4. A linear operator A with domain $\mathfrak{D}(A)$ is called a (d2)-operator if A satisfies the following condition (d2):

$$\text{(d2)} \qquad \tau'(x^+, Ax) \leqq 0 \qquad (x \in \mathfrak{D}(A)).$$

THEOREM 2. *A necessary and sufficient condition for a linear operator A with dense domain $\mathfrak{D}(A)$ to generate a strongly continuous semi-group of positive contraction operators is that A be a* (d2)*-operator and that $\mathfrak{R}(I-A) = X$.*

PROOF. If A generates a strongly continuous semi-group of positive contraction operators $\Sigma = \{T_t\,;\, t \geqq 0\}$ then $\mathfrak{R}(I-A) = X$ by the Hille-Yosida theorem. Moreover we have

$$\begin{aligned} 2\tau'(x^+, T_t x - x) &= 2\tau'(x^+, T_t x + x^-) - 2\| x^+ \| \\ &= \tau(x^+, T_t x + x^-) - \tau(x^+, -T_t x - x^-) - 2\| x^+ \| \\ &\leqq 2\tau(x^+, T_t x^+) - 2\| x^+ \| + \tau(x^+, x^- - T_t x^-) - \tau(x^+, T_t x^- - x^-) \\ &\leqq \tau(x^+, x^- - T_t x^-) - \tau(x^+, T_t x^- - x^-) . \end{aligned}$$

If we set

$$w = |x^+ + a x^- - a T_t x^-| - |x^+ - a x^- + a T_t x^-|$$

then

$$w \leqq \{(2x^+) \wedge (2a x^-)\} \vee 0 = 0$$

and so that

$$\| x^+ + a x^- - a T_t x^- \| \leqq \| x^+ - a x^- + a T_t x^- \| \qquad (a \geqq 0) .$$

Thus we have

$$\tau'(x^+, T_t x - x) \leqq 0$$

and, for any $x \in \mathfrak{D}(A)$,

$$\tau'(x^+, Ax) \leqq 0$$

which proves that A is a (d2)-operator.

We next prove the inverse assertion. Let A be a (d2)-operator and suppose that $\mathfrak{R}(\lambda I - A) = X$ for some $\lambda > 0$. Then the inverse $(\lambda I - A)^{-1}$ exists and

$$\|(\lambda I-A)^{-1}y\| \leqq \lambda^{-1}\|y\| \qquad (y\in X).$$

In fact, for fixed $y\geqq 0$ in X there is an $x\in\mathfrak{D}(A)$ such that $\lambda x-Ax=y$ and

$$\begin{aligned}\lambda\|x^-\| &= \tau'(x^-, \lambda x^-)\\ &\leqq \tau'(x^-, \lambda x^-)-\tau'(x^-, -Ax)\\ &= \tau'(x^-, -y+\lambda x^+)\\ &\leqq 0,\end{aligned}$$

(the last inequality follows from the similar calculation as that of w), since $x\geqq 0$ and

$$\begin{aligned}\lambda\|x\| &= \tau'(x, \lambda x)\\ &\leqq \tau'(x, \lambda x)-\tau'(x, Ax)\\ &= \tau'(x, y)\leqq \|y\|.\end{aligned}$$

Thus, by Proposition 5, this inequality implies that $(\lambda I-A)$ is one-to-one and

$$\lambda R(\lambda; A)\equiv \lambda(\lambda I-A)^{-1}$$

is a positive contraction operator. By hypothesis, $\mathfrak{R}(I-A)=X$ so that the above discussions are true for $\lambda=1$. If $|\lambda-1|<1$, then the resolvent $R(\lambda; A)$ exists and is given by

$$R(\lambda; A)=R(1; A)\sum_{n=0}^{\infty}\{(1-\lambda)R(1; A)\}^n.$$

Since (d2) implies $\|R(\lambda; A)\|\leqq\lambda^{-1}$, the method of analytic continuation shows that $R(\lambda; A)$ exists and satisfies the norm condition $\|R(\lambda; A)\|\leqq\lambda^{-1}$ for $\lambda>0$. It follows from the Hille-Yosida theorem that A generates a strongly continuous semi-group of contraction operators $\{T_t; t\geqq 0\}$. It also follows from the proof of the Hille-Yosida theorem that

$$T_t x=\lim_{\lambda\to\infty}\exp(-\lambda t)\sum_{n=0}^{\infty}\frac{(\lambda t)^n}{n!}\{\lambda R(\lambda; A)\}^n x$$

and this expression implies that T_t is a positive operator if $R(\lambda; A)$ is positive.

REMARK 2. To each pair $\{x, y\}$ in a Banach lattice X, we can associate a real number $\tau''(x, y)$ such that

$$\tau''(x, y)=\tau'(x^+, y)+\tau'(x^-, y),$$

then this functional also characterizes the infinitesimal generator of a semi-group of positive contraction operators. In particular, if X is an abstract (L)-space, then the functional

$$\varphi(x)=\tau''(x, x)=\|x^+\|-\|x^-\|$$

completely characterizes the infinitesimal generators of such semi-groups on X. See Reuter [**9**] and Miyadera [**6**].

3. In this section we concern with a generation of positive contraction semi-groups which dominate a given semi-group in a Banach lattice.

DEFINITION 5. Given a semi-group $\Sigma=\{T_t;\, t\geqq 0\}$ of positive contraction operators and $\Sigma'=\{T'_t;\, t\geqq 0\}$ is another one, we say that Σ' dominates Σ, if

$$T'_t x\geqq T_t x \qquad (x\geqq 0,\, t\geqq 0).$$

The following lemma in a Banach space will be required in the sequel.

PROPOSITION 6. *Suppose that a linear operator A generates a strongly continuous semi-group of contraction operators on a Banach space X and that B is a linear operator with domain $\mathfrak{D}(B)\supset\mathfrak{D}(A)$.*

If $A'=A+B$ has a closed extension $\bar{A}'$ then

$$\| BR(\lambda;\, A)\| \leqq K<\infty\,,$$

where K is independent of $\lambda>1$ and

$$\lim_{\lambda\to\infty} \| BR(\lambda;\, A)x\| =0 \qquad (x\in X).$$

PROOF. Using the closed graph theorem, the formula

$$\begin{aligned} BR(\lambda;\, A) &= A'R(\lambda;\, A)-AR(\lambda;\, A) \\ &= \bar{A}'R(\lambda;\, A)-AR(\lambda;\, A) \end{aligned}$$

implies that $BR(\lambda;\, A)$ is a bounded linear operator for any $\lambda>0$. By the resolvent equation, we have

$$BR(\lambda;\, A)=BR(1;\, A)-(\lambda-1)BR(1;\, A)R(\lambda;\, A)$$

and

$$\| BR(\lambda;\, A)\| \leqq \| BR(1;\, A)\| (1+\lambda^{-1}|\lambda-1|)\leqq K<\infty\,.$$

Since, for any $x\in\mathfrak{D}(A)$, there is $y\in X$ such that $x=R(1;\, A)y$, we have

$$\begin{aligned} \| BR(\lambda;\, A)x\| &= \| BR(\lambda;\, A)R(1;\, A)y\| \\ &\leqq K\| R(\lambda;\, A)y\| \leqq \lambda^{-1}K\| y\|\,. \end{aligned}$$

The assertion is proved by this inequality and $\overline{\mathfrak{D}(A)}=X$.

The following theorem is previously obtained by the author when A' is dispersive with respect to some semi-inner product [3]. Some modifications are necessary to apply his proof for the present case.

THEOREM 3. *In a weakly complete Banach lattice X let A be a generator of a positive contraction semi-group Σ and let B be a linear operator with domain $\mathfrak{D}(B)\supset\mathfrak{D}(A)$. Then $A'=A+B$ or its closed extension generates a positive contraction semi-group Σ' which dominates Σ if and only if*

(1) $$Bx \geqq 0 \qquad (x \geqq 0,\ x \in \mathfrak{D}(A)),$$

(2) $$\tau'(x, A'x) \leqq 0 \qquad (x \geqq 0,\ x \in \mathfrak{D}(A)),$$

(3) *A' has a closed extension.*

PROOF. We define a sequence of linear operators $\{A_{n,\lambda}\}$ by

$$A_{n,\lambda} = A + (n-\lambda)BR(n\,;A) \qquad (n \geqq \lambda)$$

and $\{B_{n,\lambda}\}$ by

$$\begin{aligned} B_{n,\lambda} &= A_{n+1,\lambda} - A_{n,\lambda} \\ &= BR(n+1\,;A)(\lambda - A)R(n\,;A) \qquad (n \geqq \lambda). \end{aligned}$$

Then it follows from Proposition 6 that

$$\begin{aligned} \|B_{n,\lambda}\| &\leqq \|BR(n+1\,;A)\|\{1+n^{-1}(n-\lambda)\} \\ &\leqq L < +\infty \end{aligned}$$

where L is independent of n and λ.

If we assume that the resolvent $R(\lambda\,;A_{n,\lambda})$ exists which acts on X and is positive for some λ and n $(n \geqq \lambda)$, then we have, for any $x \geqq 0$,

$$\begin{aligned} \lambda\|R(\lambda\,;A_{n,\lambda})x\| &= \tau'(R(\lambda\,;A_{n,\lambda})x,\ \lambda R(\lambda\,;A_{n,\lambda})x) \\ &\leqq \tau'(R(\lambda\,;A_{n,\lambda})x,\ \lambda R(\lambda\,;A_{n,\lambda})x) \\ &\quad - \tau'(R(\lambda\,;A_{n,\lambda})x,\ A'R(\lambda\,;A_{n,\lambda})x) \\ &= \tau'(R(\lambda\,;A_{n,\lambda})x,\ (\lambda - A')R(\lambda\,;A_{n,\lambda})x) \\ &= \tau'(R(\lambda\,;A_{n,\lambda})x,\ x - BR(n\,;A)(\lambda - A)R(\lambda\,;A_{n,\lambda})x) \\ &\leqq \|x\|, \end{aligned}$$

where the last inequality holds by virtue of the formula

$$(\lambda - A)R(\lambda\,;A_{n,\lambda})x = x + (n-\lambda)BR(n\,;A)R(\lambda\,;A_{n,\lambda})x.$$

In fact, if we set

$$y = BR(n\,;A)(\lambda - A)R(\lambda\,;A_{n,\lambda})x$$

then we see that

$$\begin{aligned} &2\tau'(R(\lambda\,;A_{n,\lambda})x,\ x-y) - 2\|x\| \\ &\quad = \tau(R(\lambda\,;A_{n,\lambda})x,\ x-y) - \tau(R(\lambda\,;A_{n,\lambda})x,\ y-x) - 2\|x\| \\ &\quad = \lim_{a\to 0+} a^{-1}(\|R(\lambda\,;A_{n,\lambda})x + ax - ay\| \\ &\qquad\qquad - \|R(\lambda\,;A_{n,\lambda})x - ax + ay\| - 2a\|x\|) \\ &\quad \leqq \overline{\lim_{a\to 0+}}\, a^{-1}(\|R(\lambda\,;A_{n,\lambda})x + ax - ay\| \\ &\qquad\qquad - \|R(\lambda\,;A_{n,\lambda})x + ax + ay\|) \\ &\quad \leqq 0. \end{aligned}$$

Thus we obtain

$$\lambda\| R(\lambda ; A_{n,\lambda})x\| \leqq \| x\| \qquad (x \geqq 0)$$

and so that, for any $x \in X$,

$$\lambda\| R(\lambda ; A_{n,\lambda})x\| \leqq \lambda\| R(\lambda ; A_{n,\lambda})|x|\| \leqq \| |x| \| = \| x\| .$$

By induction on n we next show that the resolvent $R(\lambda ; A_{n,\lambda})$ exists which acts on X and is positive for any $\lambda > L$ and $n \geqq \lambda$. It is obvious that

$$R(\lambda ; A_{\lambda,\lambda}) = R(\lambda ; A)$$

is a positive operator for any $\lambda > L$. Suppose that $R(\lambda ; A_{n,\lambda})$ is positive for any $\lambda > L$ and some n, then we have

$$\| B_{n,\lambda} R(\lambda ; A_{n,\lambda})\| \leqq \| B_{n,\lambda}\| \ \| R(\lambda ; A_{n,\lambda})\| < 1 .$$

It follows from this norm condition that $R(\lambda ; A_{n+1,\lambda})$ exists which acts on X and is given by

$$R(\lambda ; A_{n+1,\lambda}) = \sum_{k=0}^{\infty} R(\lambda ; A_{n,\lambda})\{B_{n,\lambda} R(\lambda ; A_{n,\lambda})\}^k$$

for any $\lambda > L$. See [4] and [**10**]. Moreover we have, for any $x \geqq 0$,

$$\begin{aligned} & B_{n,\lambda} R(\lambda ; A_{n,\lambda})x \\ & \quad = BR(n+1 ; A)R(n ; A)\{x + (n-\lambda)BR(n ; A)R(\lambda ; A_{n,\lambda})x\} \\ & \quad \geqq 0 . \end{aligned}$$

It follows that

$$R(\lambda ; A_{n+1,\lambda})x \geqq R(\lambda ; A_{n,\lambda})x \geqq 0 \qquad (x \geqq 0) .$$

Since X is a weakly complete Banach lattice, we have, for any $x \geqq 0$ and then for any $x \in X$,

$$\lim_{n,n' \to \infty} \| R(\lambda ; A_{n,\lambda})x - R(\lambda ; A_{n',\lambda})x\| = 0 .$$

To show that $\{R(\lambda' ; A_{n,\lambda})x\}$ $(0 < \lambda' < 2\lambda)$ is a Cauchy sequence for any $x \in X$, we make use of the relation

$$R(\lambda - \mu ; A_{n,\lambda}) = \sum_{k=1}^{\infty} \mu^{k-1} R(\lambda ; A_{n,\lambda})^k$$

where, provided that $|\mu| < \lambda$, the right hand side converges uniformly in n. See [4] and [**10**].

It also follows from this formula that $R(\lambda' ; A_{n,\lambda})$ $(0 < \lambda' < \lambda)$ is positive and that

$$\lambda'\| R(\lambda' ; A_{n,\lambda})x\| \leqq \| x\| \qquad (x \in X) .$$

We have already proved that a family of resolvents

$$\{R(\lambda\,;\,A_{n,k})\,;\,\lambda\leqq k\}_n \qquad (k=[L]+1,\,[L]+2,\,\cdots)$$

has the following properties:

(1) $$\lim_{n,n'\to\infty}\|R(\lambda\,;\,A_{n,k})x-R(\lambda\,;\,A_{n',k})x\|=0 \qquad (x\in X),$$

(2) $$R(\lambda\,;\,A_{n,k})-R(\lambda'\,;\,A_{n,k})=(\lambda'-\lambda)R(\lambda\,;\,A_{n,k})R(\lambda'\,;\,A_{n,k}),$$

(3) $$\lambda\|R(\lambda\,;\,A_{n,k})\|\leqq 1.$$

Setting

$$\tilde{R}(\lambda\,;\,A_k)x=\lim_{n\to\infty}R(\lambda\,;\,A_{n,k})x \qquad (x\in X),$$

we see that $\{\tilde{R}(\lambda\,;\,A_k)\,;\,\lambda\leqq k\}$ satisfies the above properties (2) and (3) and is a consistent family of resolvents in the following sense:

$$\tilde{R}(\lambda\,;\,A_{k'})x=\tilde{R}(\lambda\,;\,A_k)x \qquad (x\in X,\,\lambda<k<k').$$

In fact, we have

$$\begin{aligned}
&\|\tilde{R}(\lambda\,;\,A_{k'})x-\tilde{R}(\lambda\,;\,A_k)x\|\\
&\quad\leqq\|\tilde{R}(\lambda\,;\,A_{k'})x-R(\lambda\,;\,A_{n,k'})x\|\\
&\qquad+\|R(\lambda\,;\,A_{n,k})x-\tilde{R}(\lambda\,;\,A_k)x\|\\
&\qquad+\|R(\lambda\,;\,A_{n,k'})x-R(\lambda\,;\,A_{n,k})x\|.
\end{aligned}$$

Here

$$\begin{aligned}
&\|R(\lambda\,;\,A_{n,k'})x-R(\lambda\,;\,A_{n,k})x\|\\
&\quad\leqq(k'-k)\|R(\lambda\,;\,A_{n,k'})BR(n\,;\,A)R(\lambda\,;\,A_{n,k})x\|\\
&\quad\leqq\lambda^{-1}(k'-k)\|BR(n\,;\,A)\tilde{R}(\lambda\,;\,A_k)x\|\\
&\qquad+\lambda^{-1}(k'-k)L\|R(\lambda\,;\,A_{n,k})x-\tilde{R}(\lambda\,;\,A_k)x\|.
\end{aligned}$$

Hence, we obtain the desired inequality

$$\begin{aligned}
&\|\tilde{R}(\lambda\,;\,A_{k'})x-\tilde{R}(\lambda\,;\,A_k)x\|\\
&\quad\leqq\|\tilde{R}(\lambda\,;\,A_{k'})x-R(\lambda\,;\,A_{n,k'})x\|\\
&\qquad+\{1+\lambda^{-1}(k'-k)L\}\|R(\lambda\,;\,A_{n,k})x-\tilde{R}(\lambda\,;\,A_k)x\|\\
&\qquad+\lambda^{-1}(k'-k)\|BR(n\,;\,A)\tilde{R}(\lambda\,;\,A_k)x\|.
\end{aligned}$$

Letting $n\to\infty$, we have, for any $\lambda<k<k'$,

$$\tilde{R}(\lambda\,;\,A_{k'})x=\tilde{R}(\lambda\,;\,A_k)x \qquad (x\in X).$$

Since $\{\tilde{R}(\lambda\,;\,A_k)\,;\,\lambda\leqq k\}$ is consistent, we obtain a family of resolvents $\{\tilde{R}(\lambda\,;\,A')\}$ which satisfies the following conditions:

(1) $$\tilde{R}(\lambda\,;\,A')=\tilde{R}(\lambda\,;\,A_k) \qquad (\lambda\leqq k),$$

(2) $$\tilde{R}(\lambda; A') - \tilde{R}(\lambda'; A') = (\lambda' - \lambda)\tilde{R}(\lambda; A')\tilde{R}(\lambda'; A'),$$

(3) $$\lambda \| \tilde{R}(\lambda; A') \| \leqq 1.$$

It follows from (2) that $\tilde{R}(\lambda; A')$ is a one-to-one transformation from X to $\mathfrak{R}(\tilde{R}(\lambda; A'))$ and

$$\tilde{A}_\lambda = \lambda - \tilde{R}(\lambda; A')^{-1}$$

is independent of λ, that is,

$$\tilde{A}x = \tilde{A}_\lambda x = \tilde{A}_{\lambda'} x \qquad (x \in \mathfrak{R})$$

where $\mathfrak{R} = \mathfrak{R}(\tilde{R}(\lambda; A')) = \mathfrak{R}(\tilde{R}(\lambda'; A'))$. Then, by the Hille-Yosida theorem, we find that $\tilde{A}$ generates a strongly continuous semi-group of contraction operators. It is readily verified that $\tilde{A}$ is a closed extension of A' and that Σ' dominates Σ.

The inverse part of this theorem is obvious.

THEOREM 4. *In a weakly complete Banach lattice X let A be a generator of a positive contraction semi-group Σ and let B be a linear operator with domain $\mathfrak{D}(B) \supset \mathfrak{D}(A)$.*

Then $A' = A + B$ or its closed extension generates a positive contraction semi-group Σ' which dominates Σ if and only if

(1) $$Bx \geqq 0 \qquad (x \geqq 0,\ x \in \mathfrak{D}(A)),$$

(2) *A' is a* (d3)*-operator, that is,*
A' satisfies the following condition

(d3) $$\tau(x, A'x) \leqq 0 \qquad (x \in \mathfrak{D}(A)).$$

PROOF. The proof of this theorem follows from the following lemma due to Lumer and Phillips when A is a dissipative operator in a semi-inner product space. Some modifications are necessary in their proof of Lemma 3.3 in [5] to apply for the case when A is a (d3)-operator.

PROPOSITION 7. *If A is a* (d3)*-operator with dense domain in a Banach space X, then A has a smallest closed linear extension $\bar{A}$.*

PROOF. If A does not have a closed extension, then there is a sequence $\{x_n\} \subset \mathfrak{D}(A)$ such that

$$x_n \to 0, \quad Ax_n \to y \qquad \text{and} \qquad \|y\| = 1.$$

Choose $u \in \mathfrak{D}(A)$ such that $\|u - y\| < 2^{-1}$, $\|u\| = 1$.

Since $\tau(u + cx_n, \cdot)$ is a real continuous functional on X such that

$$\tau(u + cx_n, y + z) \leqq \tau(u + cx_n, y) + \tau(u + cx_n, z)$$

and

$$\tau(u + cx_n, \alpha y) = \alpha \tau(u + cx_n, y) \qquad \text{for} \qquad \alpha \geqq 0,$$

it follows that there exists a continuous real linear functional $k_{c,n}^{*}$ on X satisfying

$$k_{c,n}^{*}(u+cx_{n})=\tau(u+cx_{n},\ u+cx_{n})=\|u+cx_{n}\|$$

and

$$-\tau(u+cx_{n},\ -y)\leqq k_{c,n}^{*}(y)\leqq\tau(u+cx_{n},\ y)\qquad(y\in X)\,.$$

It also follows that the bounded set $\{k_{c,n}^{*}\}$ in X^{*} has a limit point k_{c}^{*} in the X-topology. Moreover we have

$$\|k_{c}^{*}\|\leqq\varliminf_{n}\|k_{c,n}^{*}\|\leqq 1\,,$$

$$\begin{aligned}k_{c}^{*}(u)&=\lim_{n}k_{c,n}^{*}(u)\\&=\lim_{n}k_{c,n}^{*}(u+cx_{n})\\&=1\end{aligned}$$

and

$$\begin{aligned}k_{c}^{*}(Au+cy)&\leqq\varlimsup_{n}\tau'(u+cx_{n},\ Au+cy)\\&=\varlimsup_{n}\tau(u+cx_{n},\ Au+cAx_{n})\\&\leqq 0\,.\end{aligned}$$

On the other hand

$$\begin{aligned}k_{c}^{*}(y)&=k_{c}^{*}(u)-k_{c}^{*}(u-y)\\&\geqq 1-\|u-y\|>2^{-1}\end{aligned}$$

so that $k_{c}^{*}(Au+cy)\leqq 0$ is impossible if c is chosen larger than $2\|Au\|$.

REMARK 3. In Theorem 1, the condition (d1) can be replaced by (d3).

The following theorem is due to Miyadera [6] when X is an abstract (L)-space. Analogous result is obtained when X is a Banach lattice by Olubummo [7].

THEOREM 5. *Let A generate a strongly continuous semi-group Σ of positive contraction operators on a weakly complete Banach lattice X and let B be a linear operator with domain $\mathfrak{D}(B)\supset\mathfrak{D}(A)$.*

Then $A'=A+B$ will generate a strongly continuous semi-group Σ' of positive contraction operators dominating Σ if and only if

(1) $$Bx\geqq 0\qquad(x\geqq 0,\ x\in\mathfrak{D}(A))\,,$$

(2) $$\tau'(x,\ A'x)\leqq 0\qquad(x\geqq 0,\ x\in\mathfrak{D}(A))\,,$$

(3) $$\mathfrak{R}(I-BR(\lambda\,;\,A))=X\qquad(\lambda>0)\,.$$

PROOF. We note that

$$(I-BR(\lambda\,;\,A))X=(\lambda-A')\mathfrak{D}(A)=X\,,$$

thus the assertion is readily follows from the proof of Theorem 3.

4. As an application of Theorem 4, we remark a convergence theorem of a family of semi-groups of operators. See [3].

PROPOSITION 8. *Suppose that a family of linear operators* $\{A_n\}$ $(n=1, 2, \cdots)$ *which generate strongly continuous semi-groups of positive contraction operators on a weakly complete Banach lattice* X *satisfies the following conditions:*

(1) $$\mathfrak{D}(A_{n+1}) \subset \mathfrak{D}(A_n),$$

(2) $$A_{n+1}x = A_n x + B_n x,$$

$$B_n x \geqq 0 \qquad (x \geqq 0,\ x \in \mathfrak{D}(A_{n+1})),$$

(3) *there is a dense set* $\mathfrak{M}$ *in* X *on which*

$$\| A_n x \| \leqq K(x) < \infty$$

where $K(x)$ *is independent of* n.

Then the limit operator $A = \lim A_n$ *exists on* $\mathfrak{M}$ *and has a closed extension* $\hat{A}$ *which generates a strongly continuous semi-group of positive contraction operators.*

PROOF. Since X is weakly complete, (3) implies the existence of the limit $A = \lim A_n$ and the assertion readily follows from Theorem 4.

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