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On Mordell's conjecture for algebraic curves over function fields

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Introduction.

In this paper, we are concerned with Mordell's conjecture on the set of rational points on algebraic curves in "relative case" (cf. [2] p. 139).

Let k be any field and K be a function field with k as constant field, i.e. a regular extension of finite type of k. Let C be a complete non-singular curve defined over K. We say that C is *trivially defined*, if there is a curve C_0 defined over k which is birationally equivalent to C over K. Then our main Theorem reads:

*If the genus g of C is $\geqq 2$, then the set of all rational points of C over K is a finite set or C is trivially defined**).

This was proved by Grauert [3] in the case where the characteristic of k is 0 and k is algebraically closed. Manin [4] obtained the same result with a transcendental method. We shall prove the above Theorem for the field k of any characteristic p (which may be $=0$ or $\neq 0$), without supposing k to be algebraically closed.

The proof is given in two cases (1) $p=0$ (§1), (2) $p\neq 0$ (§2)[1]. We shall use the results of [3] as formulated at the beginnings of §1 and §2, and the theory of abelian varieties (cf. [1], [6]). As to the terminology we follow generally the usage in [1].

More specifically, the method we shall use is that of descent. To explain

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*) For the case $p=0$, we shall prove another related proposition concerning the curve of genus 1. (See Proposition 1 below.)

1) To avoid misunderstanding we add here the following remark. Grauert [3] introduced the notion of "quasi-trivially defined curve" which implies that of "trivially defined curve" when $p=0$. He considered also a certain fibre variety X with C as fibre, such that when X becomes trivial (i.e. isomorphic to the direct product of fibre and base space), then C is trivially defined in our sense. He proved that in case $p=0$, X becomes trivial when C is quasi-trivial and used this to obtain his main Theorem. For the case $p\neq 0$, he constructed an example showing that X need not become trivial even if C is quasi-trivial. But this is of course in no contradiction with the validity of our Theorem for $p=0$.

it for the case $p=0$, the result is already obtained if the ground field k is $=\bar{k}$ (algebraic closure of k), so we have to "descend" from $\bar{k}$ to k. For this purpose, we have to consider different fields $k', k'', \cdots$ containing k and suppose the curve C as defined over k' or $k'' \cdots$. Let C be defined over k', and $k'' \supset k'$. We shall say C is *k'/k-trivially defined over k''*, if there is a curve C_0 defined over k which is birationally equivalent to C over k''. If $k'=k''$, we shall say simply C is *k'/k-trivially defined.* When k' contains a field of definition of C, we denote with $C_{k'}$ the set of all k'-rational points on C. These notations will be used throughout this paper.

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§1. Case $p=0$

In this paragraph, the characteristic p of the ground field k (and consequently of all fields considered) is always 0. K is a function field with constant field k. C will denote always a complete, non-singular curve defined over some field k' containing k, with genus g.

First we prove the following Proposition for the case $g=1$ (to complete our main Theorem concerning the case $g \geqq 2$).

PROPOSITION 1. *Let C be defined over K and $g=1$. Then C_K forms an abelian group in the well-known sense. Either this group is finitely generated or C is K/k-trivially defined.*

PROOF. First we notice that if $C_K=\phi$, our Proposition is trivial. Therefore we assume $C_K \neq \phi$. Then C turns out to be abelian variety defined over K. Let (B, τ) be the K/k -trace of C. Since K is a regular extension of k and k is of characteristic 0, τ is birational isomorphism from B into C, defined over K. Therefore B is either a point or one-dimensional. If B is a point, we have $C_K \approx C_K/\tau B_k$ and, by Mordell-Weil Theorem, C_K is finitely generated. When B is 1-dimensional, B is birationally isomorphic to C over K. Since B is defined over k, our Proposition was proved. Q. E. D.

Next we notice the following classical result for the later use.

THEOREM OF DE FRANCHIS. *If C is defined over k, then $C_K - C_k$ is a finite set.*

For the proof we refer to Lang's book [2] pp. 139-140.

We owe the following Theorem to Grauert [3].

THEOREM OF GRAUERT. *If C is defined over K, k is algebraically closed and $g \geqq 2$ then either C_K is a finite set or C is K/k-trivially defined.*

Moreover we use the following Lemma.

LEMMA 1. *Let k' be an algebraic extension of k, C be defined over k' and the jacobian variety J of C be defined over k. If C is birationally isomorphic over $k'K$ to a curve C^* on J defined over K such that the inclusion map $C^* \to J$ makes J the jacobian variety of C^*, then C is k'/k-trivially defined.*

PROOF. First we notice that if the genus g of C is $=1$, Lemma is trivial. Therefore we assume $g \geqq 2$. Let k'' be a Galois extension of k, containing k', with Galois group $G = G(k''/k)$ such that $C_{k''} \neq \phi$. Since K/k is a regular extension, $k''K$ is also a Galois extension of K and the Galois group $G(k''K/K)$ can be identified with G. For an element σ of G and an algebraic object V defined over $k''K$, we denote by V^σ the transform of V by σ. Since C and C^* are birationally isomorphic over $k''K$, there is an automorphism h of J defined over $k''K$ and a $k''K$-rational point a of J such that $h(C^*)+a=C$. We denote by f the birational isomorphism from C^* to C induced by $h+a$. Then we have $C = h^\sigma \circ h^{-1}(C) - h^\sigma h^{-1}(a) + a^\sigma$. Since $k''K$ is a regular extension of k'', h is defined over k'' by Chow's Theorem and since C^σ and $h^\sigma \circ h^{-1}(C)$ are defined over k'', $h \circ h^{-1}(a) - a^\sigma$ is rational over k''. If we define an isomorphism $f_{\tau,\sigma}$ defined over k'' from C to C by $f^\tau(f^\sigma)^{-1} = f_{\tau,\sigma}$, then $f_{\tau,\sigma}$ satisfies the cocycle conditions 1) $f_{\rho,\tau} \circ f_{\tau,\sigma} = f_{\rho,\sigma}$ and $(f_{\tau,\sigma})^\rho = f_{\tau\rho,\sigma\rho} (\sigma, \tau, \rho \in G)$. By Weil's Theorem ([1] p. 16) there exists a curve C_0 defined over k which is birationally isomorphic to C over k'. Thus we complete the proof of Lemma.

Now we prove our Theorem in case of characteristic 0.

THEOREM 1. *Let k be a field of characteristic 0 and K be a function field with constant field k. Let C be a complete non-singular curve defined over K with genus $\geqq 2$. Then either the set C_K of all rational points of C over K is finite or C is K/k-trivially defined. In the latter case there exists a birational isomorphism θ from a curve C_0 defined over k to C, such that $C_K - \theta((C_0)_k)$ is a finite set.*

PROOF. Let J be the jacobian variety of C defined over K and (B, τ) be the K/k-trace of J. Since K is regular extension of k and k is of characteristic 0, τ is a birational isomorphism defined over K from B into J by Cor. 2 of Theorem 9 Chap. VIII of [1]. By the Cor. 1 of the same Theorem (B, τ) is also $\bar{k}K/\bar{k}$-trace of J, where $\bar{k}$ is the algebraic closure of k. We assume that C_K is not a finite set. By the Theorem of Grauert cited above, there exists a curve C_1 defined over $\bar{k}$ which is birationally isomorphic to C over $\bar{k}K$. Then there exists a finite Galois extension k' of k over which C_1 is defined and has a rational point such that C is birationally isomorphic to C_1 over $k'K$. Let J_1 be the jacobian variety of C_1 defined over k'. Then J_1 and B are birationally isomorphic over k'. In fact, there is a birational isomorphism β from J_1 to J defined over $k'K$. Therefore, by the property of trace, there exists a rational homomorphism β' from J_1 to B defined over k' such that $\beta = \tau \cdot \beta'$. Since τ is an into birational isomorphism, τ and also β' is surjective birational isomorphism. Thus β' maps J_1 onto B isomorphically and birationally over k'. So we can identify the curve C_1 on J_1 with a curve C_2 on B defined over k'. Since $C_K \neq \phi$, we can identify C with a curve on J defined over K. $\tau^{-1}(C)$ is also defined over K and is birationally isomorphic to C_2 over $k'K$. By Lemma 1, there exists a curve C_0 defined over k which is birationally isomorphic to C_2 over k'. If we show that C and C_0 are birationally isomorphic over K, then our proof will be completed. Let M be a generic point of C_0 over $\bar{k}K$. Then we can identify the curve C_0 with a curve C_0^M on B, defined over $k(M)$ by the canonical mapping defined over $k(M)$. Since C and C_0^M are birationally isomorphic over $k'K(M)$, there exists an automorphism f of B defined over $k'K(M)$ and a $k'K(M)$-rational point a of B such that $C+a=f(C_0^M)$. Let σ be an automorphism of $k'K(M)$ over $K(M)$. Then we have $C+a^\sigma=f^\sigma(C_0^M)$. Therefore we have $f(C_0^M)=f^\sigma(C_0^M)+a-a^\sigma$ and $a=a^\sigma$, $f=f^\sigma$ *). Consequently a is rational over $K(M)$ and f is defined over $K(M)$. Thus C and C_0^M are birationally isomorphic over $K(M)$. Since we assumed that C_K is infinite, $C_{K(M)}$ is also infinite. Hence $(C_0^M)_{K(M)}$ and $(C_0)_{K(M)}$ are infinite sets. Since we have taken M as a generic point of C_0 over K, $K(M)$ is also a function field with constant field k. Therefore by Theorem of de Franchis, $(C_0)_{K(M)}-(C_0)_k$ is a finite set and $(C_0)_k$ must be an infinite set. Thus we can take a canonical mapping from C_0 to B defined over k. If we identify C_0, by this canonical mapping, with a curve on B defined over k, then there exists a rational point a of B over $k'K$ and an automorphism f of B defined over $k'K$ such that $C+a=f(C_0)$, because C and C_0 are birationally isomorphic over $k'K$. By the same arguments as above we see that a is rational over K and f is defined over K. Hence C and C_0 are birationally equivalent over K. The fact that $C_K-\theta((C_0)_k)$ is a finite set, is clear by the Theorem of de Franchis. Thus we have completed the proof of our Theorem.

§ 2. Case $p \neq 0$.

In this paragraph, we assume the characteristic p of k to be $\neq 0$ and the genus g of C to be $\geqq 2$. We assume that C_K is an infinite set. Other notations are as in § 1. We cite the following results from [3] (§ 4 Satz 2 and its Corollary).

PROPOSITION 2 (Grauert). *There is an unramified Galois extension L of*

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*) Here we notice the following fact. ($g \geqq 2$).

"Let J be the jacobian variety of a curve C. If we consider C as a curve on J, then we have $\{a \in J \mid C+a=C\}=\{0\}$. Because if $C+a=C$ we have $\Theta=\overbrace{C+\cdots+C}^{(g-1)}=\overbrace{C+\cdots+C}^{(g-1)}+a=\Theta_a$. By Corollary 2 of Theorem 32 of [**6**] we have $a=0$.

$\bar{k}K$ and a curve Γ defined over L such that Γ is an unramified Galois covering of C and Γ is $L/\bar{k}$-trivially defined.

Now we prove:

PROPOSITION 3. *The notations being as above, C is $\bar{k}K/\bar{k}$-trivially defined.*

PROOF. Let J be the jacobian variety of C defined over K and (B', τ') be the $L/\bar{k}$-trace of J. Let J_0 be the jacobian variety of the curve Γ_0 defined over $\bar{k}$ which is birationally isomorphic to Γ over L. Then there exists a separable homomorphism α from J_0 to J, which is surjective and defined over L. We identify the curve C and Γ_0 with the curve on J and J_0 respectively, the former being defined over K and the latter over $\bar{k}$. Then we have $\alpha(\Gamma_0) = C+a$ for a suitable rational point a of J over L. By the property of trace, there is a homomorphism α', defined over $\bar{k}$, from J_0 to B' such that $\alpha = \tau' \circ \alpha'$. Therefore τ' must be a surjective separable homomorphism. On the other hand, by Cor. 2 of Theorem 9, VIII of [1], τ' is a purely inseparable homomorphism. Hence τ' is a surjective birational isomorphism, defined over L, from B' to J. Let C_1 be the image of Γ_0 by α'. Then C_1 is defined over $\bar{k}$ and we have $\tau'(C_1) = C+a$. Since J and B' are defined over $\bar{k}K$ and are birationally isomorphic over a Galois extension L of $\bar{k}K$, they are birationally isomorphic over $\bar{k}K$. Let τ'' be the birational isomorphism over $\bar{k}K$ from B' to J such that $\tau''(C_1) = C+a$ for some $\bar{k}K$-rational point a of J. Then $\tau''+a$ defines a birational isomorphism defined over $\bar{k}K$. This completes the proof of our Proposition. Q. E. D.

The following Lemma 2 is an analogue of the Lemma 1 in the case of characteristic $p \neq 0$.

LEMMA 2. *Let J be the jacobian variety of C, which is defined over k. If C can be identified with a curve on J, which is defined over a purely inseparable extension k' of k and C is birationally isomorphic over $k'K$ to a curve C^* on J defined over K such that the inclusion map $C^* \to J$ makes J the jacobian variety of C^*, then C is k'/k-trivially defined.*

PROOF. Let g be the genus of C and $M_1, M_2, \cdots, M_g$ be a set of independent generic points of C over k'. We put $M = M_1 + \cdots + M_g$ where the summation is taken on J. Then we have $k'(M) = k'(M_1, M_2, \cdots, M_g)_s$, where $k'(M_1, M_2, \cdots, M_g)_s$ is the sub-field of $k'(M_1, \cdots, M_g)$ which is elementwise invariant by the symmetric group $S(g)$ permuting the g points $M_1, \cdots, M_g$. M is a generic point of J over k'. Since J is defined over k, $k(M)$ is also regular extension of k. Let L be the separable algebraic closure of $k(M)$ in $k'(M_1, M_2, \cdots, M_g)$. Then L is separably generated over k. Since $k'(M_1, M_2, \cdots, M_g) \cap \bar{k} = k'$ and k'/k is purely inseparable extension, we have $L \cap \bar{k} = k$. Hence L is a regular extension of k. L and $k'(M)$ are linearly disjoint over $k(M)$ and $L \cdot k'(M) = k'(M_1, M_2, \cdots, M_g)$. It follows that L is a Galois extension of $k(M)$ and the Galois group of L over $k(M)$ can be identified with $S(g)$. Let σ_i $(i=1, 2, \cdots, g)$ be elements of $S(g)$ such that $\sigma_i(M_1)=M_i$. Let $L\cap k'(M_1)=K'$. Then K' is a 1-dimensional finite type regular extension of k. Let C_0 be the complete non-singular model of K' over k*). Then for a generic point N_1 of C_0 over k, we have $k(N_1)=K'$. If we put $\sigma_i(N_1)=N_i$ $(i=1, 2, \cdots, g)$ we can easily see that $N_1, N_2, \cdots, N_g$ are independent generic points of C_0 over k and we have $k(N_1, N_2, \cdots, N_g)=L$, $k(N_1, N_2, \cdots, N_g)_s=k(M)$. On the other hand we have $k'(N_1)=k'(M_1)$. Therefore C and C_0 are birationally isomorphic over k'. Thus we have completed the proof of Lemma.

Q. E. D.

Combining Lemma 1 and Lemma 2 we prove

LEMMA 3. *Let C be a curve defined over an algebraic extension k' of k and J be the jacobian variety of C. If we can take J defined over k and if C is birationally isomorphic over $k'K$ to a curve C^* on J defined over K suct that the inclusion map $C^*\to J$ makes J the jacobian variety of C^*, then C is k'/k-trivially defined.*

PROOF. Let k_0 be the separable closure of k in k'. By Lemma 2, there exists a curve C' defined over k_0 which is birationally equivalent to C over k'. Let k_1 be a suitable separable extension of k_0 over which C' has a rational point. Then we can identify the curve C' with a curve C'' on J which is defined over k_1, by a canonical mapping defined over k_1. Since C'' is birationally isomorphic to C^* over $k'\cdot k_1\cdot K$, there exists an automorphism and a point a of J such that $f(C^*)+a=C''$. Since $k'k_1K$ is primary extension of k_1, f is defined over k_1, and since $f(C^*)$ and C'' are defined over k_1K, a is rational over k_1K. By Lemma 1 there exists a curve C_0 defined over k which is birationally equivalent to C'' over k_1. Since C' and C_0 are defined over k_0, and k_1 is separably algebraic over k_0, C' and C_0 are isomorphic over k_0 by the unicity of descent (see [1] p. 15, Theorem 2). Thus C is birationally equivalent to C_0 over k'. Q. E. D.

Now we can prove:

THEOREM 2. *Let k be a field of characteristic $p\neq 0$, and K be a function field with constant field k. Let C be a complete non-singular curve of genus $g\geqq 2$ defined over K. Then either C_K is a finite set or C is K/k-trivially defined.*

PROOF. We shall take over the notations in the proof of Proposition 3. Let (B, τ) be the K/k-trace of J. (B, τ) is also $\bar{k}K/\bar{k}$-trace of J. Since K/k is regular extension, τ is a purely inseparable isomorphism from B into J by Cor. 2 of Theorem 9, Chap. VIII of [1]. Since τ'' is a surjective birational

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*) Let ψ be the birational isomorphism from C to C^* defined over $k'K$ and let $\psi(M_i)=M'_i$. Then we have $K(N_i)=K(M'_i)$. Therefore we can take a complete non-singular model of K' defined over k even if k is not a perfect field.

isomorphism from B' to J defined over $\bar{k}K$ there exists a surjective birational isomorphism β defined over $\bar{k}$ from B' to B such that $\tau''=\tau\cdot\beta$, τ being a surjective birational isomorphism defined over K. We denote by C_2 the image of C_1 by β. Then C_2 is defined over an algebraic extension k' of k. We also have $\tau^{-1}(C)=C_2+a$ for a point a on B. Since $\tau^{-1}(C)$ and C_2 are defined over $k'K$, by Cor. 2 of Theorem 3.2 of [6] a is rational over $k'K$. $\tau^{-1}(C)$ and C_2 are birationally equivalent over $k'K$. By Lemma 3 there exists a curve C_0 defined over k, which is birationally isomorphic to C_2 over k'. Let k_0 be a suitable algebraic extension of k with finite degree over which C_0 has a rational point. Then C_0 can be identified with a curve C_3 on J, defined over k_0 by a canonical mapping defined over k_0. By the Cor. 2 of Theorem 32 of [6] there exists a rational point a of J over k_0K such that $C_3+a=\tau^{-1}(C)$. Therefore C_0 and C are birationally isomorphic over k_0K. Since C and C_0 are defined over K, by the unicity of descent (see [1] p. 16, Theorem 2) C and C_0 are birationally isomorphic over K. This completes the proof of our Theorem. Q. E. D.

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