J. Math. Soc. Japan
Vol. 18, No. 1, 1966

Homomorphic images of Lie groups

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(Received July 29, 1965)

§ 0. Introduction.

In their suggestive paper [3], Gleason and Palais studied some fundamental properties of homeomorphism group $H(M)$ of a manifold M and then proposed a problem: *Does the closure of a homomorphic image of any connected Lie group in $H(M)$ necessarily become a Lie group?* The topology for $H(M)$ is of course the compact open topology, which is known to give an important example of non locally compact group. This problem, however, seems to be still far from the final answer.

Another open problem related to this is the one, raised by Montgomery and Zippin [6], that states: *Does a locally compact subgroup of $H(M)$ necessarily become a Lie group?*

Being suggested by these two problems, we are led to consider the following similar to but weaker than that of Gleason-Palais':

(1) *Is the closure of homomorphic image of any connected Lie group in $H(M)$ necessarily locally compact?*

Our main concern in this paper is to investigate this problem from several points of view and solve it in the case of one dimensional manifolds.

First we are interested in knowing to what extent the problem (1) reflects the characteristic properties of the homeomorphism group $H(M)$ instead of general topological groups, which primarily implies the following:

(2) *Are there a topological group H and a connected Lie group G such that the closure of a homomorphic image of G in H is not locally compact?*

An answer to this problem is seen in [7], namely we have shown the existence of such a topological group by giving an extraordinary topology to a real line.

Now the problem (1) naturally has two versions: (a) one is to characterize the class $\mathfrak{A}$ of Lie groups whose monomorphic images in $H(M)$ have locally compact closures, and (b) the other is to characterize the class $\mathfrak{B}$ of topological groups in which any homomorphic images of any connected Lie groups have the locally compact closures.

* He would like to acknowledge the financial support given by Sakko-kai Foundation.

In § 1, the problem (a) will be mainly treated. First, we see easily, a Lie group G belongs to the class $\mathfrak{A}$ whenever it enjoys the property that any monomorphism f of G into any topological group is necessarily an open map of G onto $f(G)$. This property is refered to as *absolute closedness* of a Lie group G in this paper. Obviously any compact Lie group falls under this category. In this section, the fundamental properties of absolute closedness of Lie groups will be studied. Though a necessary and sufficient condition for a Lie group to be absolutely closed is not known yet, it will be proved that *Lie groups of a certain type that is called* (*CA*)- *groups with finite center are absolutely closed* (Theorem 1.1). This is a generalization of a theorem of Est [2]. As a corollary of the above result, we see that any connected semi-simple Lie group with finite center and solvable Lie groups of some special types are absolutely closed.

In § 2, the problem (b) will be treated mainly. Though the problem (1) is still open in general case, the following reduction is possible: *a topological group with the first countability axiom comes within the class* $\mathfrak{B}$ *if and only if any homomorphic images of any vector groups* (*instead of any connected Lie groups in the original form*) *have the locally compact closures* (Theorem 2.1). This will be proved by induction on the length of the series of derived groups of a solvable group. As a starting point of the proof the Lie groups, which have already been proved to be absolutely closed, will play an important role.

In § 3, it will be shown that the homeomorphism groups of one dimensional manifolds belong to the class $\mathfrak{B}$, which amounts to the same that the problem (1) is affirmatively solved in the case of one-dimensional manifolds. Since the conjecture of Montgomery and Zippin is affirmative in this case [6], it turns out that the original conjecture of Gleason and Palais is also affirmative.

The author wishes to express his sincere thanks to Professors Morio Obata and Kôji Shiga for their kind advice.

§ 1. Absolute closedness.

Let $(G, \mathfrak{T}_0)$ be a connected Lie group, where G is the underlying group and $\mathfrak{T}_0$ be the topology for G. For a fixed underlying group G, by $T(G, \mathfrak{T}_0)$ is meant the pairs of the abstract group G and the topologies $\mathfrak{T}$ for G such that (1) $\mathfrak{T}$ is weaker than $\mathfrak{T}_0$ (2) $(G, \mathfrak{T})$ is a topological group with Hausdorff's seperation axiom and the first countability axiom.

A basis of neighborhoods of the identity e in $(G, \mathfrak{T}) \in T(G, \mathfrak{T}_0)$ is identified with a family $\mathfrak{F}$ of open subsets in $(G, \mathfrak{T})$ satisfying:

(a) $\cap \{V; V \in \mathfrak{F}\} = \{e\}$,

(b) if $U, V \in \mathfrak{F}$, there exists $W \in \mathfrak{F}$ such that $W \subset U \cap V$,

(c) for any $U \in \mathfrak{F}$, there exists $V \in \mathfrak{F}$ such that $VV^{-1} \subset U$,

(d) for any $U \in \mathfrak{F}$ and for any element $a \in U$, there exists $V \in \mathfrak{F}$ such that $U \supset Va$,

(e) for any $U \in \mathfrak{F}$ and for any element $a \in G$, there exists $V \in \mathfrak{F}$ such that $a^{-1}Va \subset U$.

Since $(G, \mathfrak{T})$ satisfies the first countability axom, we can find a base $\mathfrak{F}$ of neighborhoods of the identity e in $(G, \mathfrak{T})$ such that $\mathfrak{F}$ consists of countably many elements.

Conversely, if $\mathfrak{F}$ is a family of countably many open subsets of $(G, \mathfrak{T}_0)$ satisfying (a)$\sim$(e) then $\mathfrak{F}$ determines uniquely a topology $\mathfrak{T}$ for G such that $(G, \mathfrak{T}) \in T(G, \mathfrak{T}_0)$.

Denote by ρ a left invariant metric on $(G, \mathfrak{T}_0)$ such that $(G, \mathfrak{T}_0)$ is complete with respect to this metric. Then, putting

$$D(r) = \{x\,;\ \rho(x, e) \leqq r,\ 0 \leqq r < \infty\}\,,$$

$D(r)$ is a compact subset of $(G, \mathfrak{T}_0)$.

LEMMA 1.1. *Notations being as above, if* $(G, \mathfrak{T}) \neq (G, \mathfrak{T}_0)$ *and* $(G, \mathfrak{T}) \in T(G, \mathfrak{T}_0)$, *then* $V \cap (G - D(r)) \neq \emptyset$ *for any* $V \in \mathfrak{F}$ *and for any* $r \geqq 0$.

PROOF. Since $(G, \mathfrak{T}) \neq (G, \mathfrak{T}_0)$, $(G, \mathfrak{T}_0)$ is not a compact group.

If there exist $V \in \mathfrak{F}$ and r such that $V \subset D(r)$, then $Cl_0(V)$ is compact, where $Cl_0(A)$ is the closure of A in $(G, \mathfrak{T}_0)$. Since the identity mapping from $(G, \mathfrak{T}_0)$ onto $(G, \mathfrak{T})$ is continuous, $Cl_0(V)$ is closed in $(G, \mathfrak{T})$. It follows that $Cl(V)$ is compact, where $Cl(A)$ is the closure of A in $(G, \mathfrak{T})$. Thus, $(G, \mathfrak{T})$ is locally compact. Since a connected Lie group satisfies the second countability axiom, the identity mapping from $(G, \mathfrak{T}_0)$ onto $(G, \mathfrak{T})$ is bi-continuous and then $(G, \mathfrak{T}) = (G, \mathfrak{T}_0)$ contradicting the assumption.

LEMMA 1.2. *Notations and assumptions being as above, for any* $\varepsilon > 0$, *there exists* $V \in \mathfrak{F}$ *such that the diameter of any connected component of* V *is smaller than* ε.

PROOF. Since $(G, \mathfrak{T})$ is a regular space and satisfies the first countability axiom, there is a family $\mathfrak{F}'$ of open subsets of $(G, \mathfrak{T}_0)$ such that (1) $\mathfrak{F}'$ is cofinal to $\mathfrak{F}$ (2) $\mathfrak{F}' = \{V_i\,;\ i \in I$ (the integers)$\}$, (3) $V_i = V_i^{-1}$ and $Cl_0(V_i^2) \subset V_{i-1}$.

Assume that there is $\varepsilon > 0$ such that any V_i has a connected component V_i' with the diameter not less than ε. Since $(G, \mathfrak{T}_0)$ is locally arcwise connected, V_i' can be assumed arcwise connected. Let a, b be points in V_i' with $\rho(a, b) \geqq \varepsilon$ and C be an arc joining a and b in V_i'. Since the metric ρ is left invariant,

$$\rho(a, b) = \rho(e, a^{-1}b) \geqq \varepsilon$$

and $a^{-1}C$ is an arc in $V_i'^{-1}V_i'$ joining e and $a^{-1}b$.

Thus, on putting

$$S(\varepsilon)=\{x\in G;\ \rho(x,e)=\varepsilon\},$$

we have

$$S(\varepsilon)\cap Cl_0(V_{i-1})\supset S(\varepsilon)\cap Cl_0(V_i^2)\neq\phi$$

for all i. Since $S(\varepsilon)$ is compact and

$$\bigcap_i(Cl_0(V_{i-1})\cap S(\varepsilon))\subset(\bigcap_i V_{i-2})\cap S(\varepsilon)=\{e\}\cap S(\varepsilon)=\phi,$$

there is $i_0\in I$ such that $Cl_0(V_{i_0})\cap S(\varepsilon)=\phi$, contradicting the fact that $S(\varepsilon)$ is compact.

A Lie group $(G,\mathfrak{T}_0)$ is called to be absolutely closed, if $T(G,\mathfrak{T}_0)$ consists of only one element $(G,\mathfrak{T}_0)$. Obviously, compact Lie groups are absolutely closed.

If $(G,\mathfrak{T}_0)$ is absolutely closed, then Proposition 10 in [4] shows that $(G,\mathfrak{T}_0)$ has the compact center.

A connected Lie group $(G,\mathfrak{T}_0)$ is called a (CA)-group, if the image of the adjoint representation $Ad(G)$ of G on the Lie algebra $\mathfrak{g}$ is a closed subgroup of $GL(\mathfrak{g})$. Since the kernel of the adjoint representation is the center Z, absolute closedness of G/Z implies that $(G,\mathfrak{T}_0)$ is a (CA)-group.

THEOREM 1.1. *A connected (CA)-group with the compact center is absolutely closed.*

PROOF. Let $\mathfrak{g}$ be the Lie algebra of $(G,\mathfrak{T}_0)$ and $\|\ \|$ be an ordinary norm on $\mathfrak{g}$. On putting

$$D(s)=\{X\in\mathfrak{g};\ \|X\|\leqq s\},$$

there is $r>0$ such that the mapping

$$\exp;\quad D(2r)\to(G,\mathfrak{T}_0)$$

is a homeomorphism into. We fix such an r and denote $D'=\exp D(2r)$.

A metric d on $GL(\mathfrak{g})$ is defined by

$$d(A,A')=\sup\{\|(A-A')X\|;\ \|X\|\leqq r\}.$$

For any sufficiently small $\varepsilon>0$, the ε-neighborhood $V(I,\varepsilon)$ of the identity I in $GL(\mathfrak{g})$ has the compact closure. Since $Ad(G)$ is closed in $GL(\mathfrak{g})$, $U=Ad(G)\cap V(I,\varepsilon)$ has the compact closure in $Ad(G)$. By the assumption that the center of G is a compact subgroup of $(G,\mathfrak{T}_0)$, we see that the full inverse $Ad^{-1}(U)$ has the compact closure in $(G,\mathfrak{T}_0)$. Denote $K=Cl_0(Ad^{-1}(U))$.

Since D' is compact, the mapping

$$\exp^{-1};\quad D'\to D(2r)$$

is uniformly continuous. Thus, for the above ε, there is $\delta_1>0$ such that (i) the δ_1-neighborhood of $\exp D(r)$ is contained in D' (ii) for $X, Y\in D(X)$ if $\rho(\exp X,\exp Y)<\delta_1$, then $\|X-Y\|<\varepsilon$, where ρ is a left invariant metric with respect to which $(G, \mathfrak{T}_0)$ is complete.

Since D' is compact, for the above δ_1, there is δ_2 such that

$$kV(\delta_2)k^{-1} \subset V(\delta_1) \qquad \text{for all} \quad k \in D',$$

where $V(\delta)$ denotes the δ-neighborhood of the identity e in $(G, \mathfrak{T}_0)$.

Now assume that $(G, \mathfrak{T}_0)$ is not absolutely closed and there is $(G, \mathfrak{T}) \in T(G, \mathfrak{T}_0)$ such that $(G, \mathfrak{T}) \neq (G, \mathfrak{T}_0)$. Let $\mathfrak{F}$ be a family of open subsets of $(G, \mathfrak{T}_0)$ satisfying (a)~(e), which is identified with a base of the neighborhoods of e in $(G, \mathfrak{T})$. By Lemma 1.2, for the above δ_2, there is $U \in \mathfrak{F}$ such that the diameter of any connected component of U is smaller than δ_2. Since D' is also compact in $(G, \mathfrak{T})$, there is $W \in \mathfrak{F}$ such that $kWk^{-1} \subset U$ for all $k \in D'$. By Lemma 1.1, there is $g \in G$ such that $g, g^{-1} \in W \cap (G-K)$. We fix such an element g.

Since D' is connected, $\{kg^{-1}k^{-1};\ k \in D'\}$ is contained in the connected component of U containing g^{-1}. Since any connected component of U has the diameter smaller than δ_2, we see

$$\{kg^{-1}k^{-1};\ k \in D'\} \subset g^{-1}V(\delta_2).$$

It follows that

$$gkg^{-1} \in V(\delta_2)k \subset kV(\delta_1), \qquad k \in D'.$$

Put $k = \exp X$, and we have

$$\rho(\exp(-X) \exp Ad(g)X, e) < \delta_1.$$

Since the metric ρ is left invariant,

$$\rho(\exp Ad(g), \exp X) < \delta_1, \qquad X \in D(2r).$$

By our choice of δ_1, we see $\exp Ad(g)X \in D'$ for $X \in D(r)$. It follows that for any $X \in D(r)$

$$\rho(\exp Ad(g)X, \exp X) < \delta_1.$$

Then, $\| Ad(g)X - X \| < \varepsilon$ for all $X \in D(r)$ and then $d(Ad(g), I) < \varepsilon$. It follows $g \in K$, contradicting the fact that $g \in G-K$.

COROLLARY 1.1. *Connected semi-simple Lie group with finite center is absolutely closed.*

COROLLARY 1.2. *Let $(G, \mathfrak{T}_0)$ be a connected semi-simple Lie group with the finite center and φ a continuous homomorphism from $(G, \mathfrak{T}_0)$ into a topological group $\mathfrak{G}$ with the first countability axiom. Then, $\varphi(G)$ is locally compact with respect to the relative topology for $\varphi(G)$ in $\mathfrak{G}$.*

PROOF. For convenience we denote G instead of $(G, \mathfrak{T}_0)$. Let $\hat{G}$ be the simply connected covering group of G and π be the natural projection from $\hat{G}$ onto G. The kernel Z' of π is a subgroup of the center Z of $\hat{G}$. Since the center of G is finite, so also is Z/Z'. As is well-known, the kernel N of φ is a closed normal subgroup of G and so is $\hat{N} = \pi^{-1}(N)$ in $\hat{G}$ and then $G/N \cong \hat{G}/\hat{N}$.

Clearly, G/N is a connected semi-simple Lie group. Since the center of $\hat{G}/\hat{N}$ is $Z\hat{N}/\hat{N}\cong Z/Z\cap\hat{N}$ and $\hat{N}\supset Z'$, $Z/Z\cap\hat{N}$ is a homomorphic image of the finite group Z/Z'. It follows that G/N is a connected semi-simple Lie group with finite center.

Let π' be the natural projection $G\to G/N$. Defining $\hat{\varphi}=\varphi\pi^{-1}$, we see that $\hat{\varphi}$ is a monomorphism from G/N onto $\varphi(G)$. Since G/N is absolutely closed, $\hat{\varphi}$ is an open mapping, where the topology for $\varphi(G)$ is of course the relative topology. Thus, $\varphi(G)$ is locally compact and then a closed subgroup of $\mathfrak{G}$.

For convenience, we denote G instead of $(G, \mathfrak{T}_0)$. Let G be a connected solvable Lie group such that (i) the derived group V of G is a vector group and (ii) there is a closed subgroup H of G such that $G=HV$ and $H\cap V=\{e\}$. Since V is a normal subgroup of G, for any element h of H

$$A(h)\,;\quad v\to hvh^{-1}$$

is a linear transformation of V.

THEOREM 1.2. *Denote by R and S_1 the real numbers and the real numbers modulo* 1 *respectively. The following connected solvable Lie groups are absolutely closed.*

(a) $H=R,\ V=R$ *and* $A(t)t'=e^{at}t',\ a\neq 0.\quad t\in H,\ t'\in V.$

(b) $H=S_1,\ V=R^2$ *and*

$$A(t)=\begin{pmatrix}\cos 2\pi mt, & -\sin 2\pi mt\\ \sin 2\pi mt, & \cos 2\pi mt\end{pmatrix},\qquad m=\textit{integer}\neq 0\,.$$

(c) $H=R,\ V=R^2$ *and*

$$A(t)=\begin{pmatrix}e^{at}\cos 2\pi bt, & -e^{at}\sin 2\pi bt\\ e^{at}\sin 2\pi bt, & e^{at}\cos 2\pi bt\end{pmatrix},$$

where a, b be real numbers with $ab\neq 0$.

(d) $H=R\times S_1,\ V=R^2$ *and*

$$A(t,t')=\begin{pmatrix}e^{at}\cos 2\pi mt', & -e^{at}\sin 2\pi mt'\\ e^{at}\sin 2\pi mt', & e^{at}\cos 2\pi mt'\end{pmatrix},\qquad am\neq 0\,,$$

where a, m be a real number, an integer respectively.

PROOF. Let $\mathfrak{h}$, $\mathfrak{v}$ be Lie algebras of H, V respectively. With respect to suitable bases $\{H_i\}$, $\{V_i\}$ of $\mathfrak{h}$, $\mathfrak{v}$ respectively, $[H_iV_i]$ take the following forms:

(a) $[HV]=aV$.

(b) $[HV_1]=-2\pi mV_2,\ [HV_2]=2\pi mV_1$.

(c) $[HV_1]=aV_1-2\pi bV_2,\ [HV_2]=aV_2-2\pi bV_1$.

(d) $[H_1V_1]=aV_1,\ [H_1V_2]=aV_2,\ [H_2V_1]=-2\pi mV_2,\ [H_2V_2]=2\pi mV_1$.

By the fact $Ad(\exp X)=\exp adX$, we see easily that the adjoint groups $Ad(G)$ are given as follows:

(a) $\begin{pmatrix} 1, & 0 \\ *, & A(H) \end{pmatrix}$

(b), (c) $\begin{pmatrix} 1, & 0, & 0 \\ *, & & \\ & & A(G) \\ *, & & \end{pmatrix}$

where $*$ is an arbitrary real number.

As for the case (d), the following group (d′) is the universal covering of the group (d):

(d′) $H=R^2$, $V=R^2$ and $A((t, t'))$ is the same as in (d).

Since the adjoint group coincides in the cases of (d) and (d′), we consider the case (d′).

Let C be the complex number field. We can identify H, V with C by the mapping

$$H \longrightarrow C \qquad\qquad V \longrightarrow C$$

$$(t, t') \rightarrow at+\sqrt{-1}\,2\pi mt', \qquad (t, t') \rightarrow t+\sqrt{-1}\,t'.$$

Under this identification, the group (d′) turns out to be the group

(d″) $H=C$, $V=C$ and $A(z)=e^z$ for $z \in H$.

Thus the adjoint group $Ad(G)$ consists of the matrices of the form

$$\begin{pmatrix} 1, & 0 \\ *, & A(z) \end{pmatrix} \qquad * = \text{an arbitrary complex number.}$$

It follows that the ajoint groups in any cases are closed subgroup of $GL(\mathfrak{g})$.

Let G, $\mathfrak{G}$ be a connected Lie group, a topological group with the first countability axiom and f a continuous homomorphism from G into $\mathfrak{G}$. A triple $\{G, f, \mathfrak{G}\}$ is called a *V-triple*, if for any closed vector subgroup V of G $Cl(f(V))$ is a locally compact subgroup of $\mathfrak{G}$.

THEOREM 1.3. *Let G be one of the groups* (a)~(d) *in Theorem* 1.2. *If $\{G, f, \mathfrak{G}\}$ is a V-triple, then $f(G)$ has a locally compact closure.*

PROOF. It is easy to see that G has the finite center Z and the group G/Z is the same type as G (i.e., if G is the group of the type (b) in Theorem 1.2 for example, then so is G/Z.) In fact, we see that $Z \subset H$ and $G/Z=H'V$, $H' \cap V=\{e\}$, $H' \cong H/Z$. Thus, G/Z is absolutely closed. Let Z' be a subgroup of Z. By the same reason as above, we see that G/Z' is absolutely closed.

Assume first that the kernel K of f is discrete. Then, $K \subset Z$. Let π be the natural projection $G \rightarrow G/K$ and define the monomorphism $\hat{f}: G/K \rightarrow \mathfrak{G}$ by $\hat{f}=f\pi^{-1}$. Since G/K is absolutely closed, $\hat{f}(G/K)=f(G)$ is locally compact and then closed in $\mathfrak{G}$.

Assume next that the kernel K of f is not discrete. Considering the Lie algebra $\mathfrak{g}$ of G case by case, we see that the connected component K_0 of

K containing e also contains V. It follows that $f(G)=f(H)$. Since $H=R$, S_1 or $R\times S_1$ and $\{G, f, \mathfrak{G}\}$ is a V-triple, $f(R)$ has the locally compact closure in $\mathfrak{G}$ and then $f(H)$ has the locally compact closure, completing the proof.

Remark that if $H=S_1$, we can prove this theorem without the assumption that $\{G, f, \mathfrak{G}\}$ is a V-triple.

Now, Est [3] showed that connected nilpotent Lie groups (Lie groups having nilpotent Lie algebras) are (CA)-groups. Thus, as a corollary of Theorem 1.1 we have the following:

COROLLARY 1.3. *Connected nilpotent Lie groups with compact center are absolutely closed.*

In the following part of this section we consider a nilpotent Lie group whose derived group is contained in the center. This will be used in the next section.

Let G be a connected nilpotent Lie group and G_1 the derived group of G.

LEMMA 1.3. *If G_1 is contained in the center Z of G, then Z is connected and G/Z is a vector group.*

PROOF. Let Z_0 be the connected component of Z containing e. Assume that there is an element $z\in Z-Z_0$. There is a one parameter group g_t such that $g_1=z$. In fact, denoting by π' the natural projection $G\to G/Z_0$, $\pi'(z) =\exp \hat{X}$, where $\hat{X}$ is an element of the Lie algebra of the abelian group G/Z_0. Let $\mathfrak{g}$ be the Lie algebra of G. There is $X\in\mathfrak{g}$ such that $d\pi'(X)=\hat{X}$. Thus, we see that $z\in\exp X\cdot Z_0$. It follows that $z=\exp X\exp Y$ for some Y in the Lie algebra of Z_0. Since $[XY]=0$, we have $z=\exp(X+Y)$. Trerefore G/Z contains a non-trivial circle subgroup. Since G/Z is abelian, $G/Z=V\times T$ and $T\neq\{0\}$, where V, T are vector group, toroidal group respectively.

Define the mapping $G\times G\to G_1$ by $\varphi(g, h)=ghg^{-1}h^{-1}$. We see that

$$\varphi(gg', h)=gg'hg'^{-1}g^{-1}h^{-1}=gg'hg'^{-1}h^{-1}hg^{-1}h^{-1}.$$

Since $g'hg'^{-1}h^{-1}$ is in the center, we have

$$\varphi(gg', h)=\varphi(g, h)\cdot\varphi(g', h).$$

By the same argument we see that $\varphi(g, hh')=\varphi(g, h)\cdot\varphi(g, h')$.

Putting $\varphi_g(g')=\varphi(g, g')$, φ_g is a homomorphism from G into G_1, whose kernel contains Z. Thus, denoting by π the natural projection from G onto G/Z, $\hat{\varphi}_g=\varphi_g\cdot\pi^{-1}$ is a homomorphism from G/Z into G_1. Let $G_1=V'\times T'$, where V', T' are vector group, toroidal group respectively. Since $\hat{\varphi}_g(T)\subset T'$, the restriction $\hat{\varphi}_g|T$ is a homomorphism from T into T'. Let $\mathrm{Hom}(T, T')$ be the space of the homomorphisms from T into T' with the compact open topology. Let $T'=S_1^r$ $(=\overbrace{S_1\times S_1\times\cdots\times S_1}^{r})$. Then, we see easily that

$$\operatorname{Hom}(T, T') = \overbrace{\operatorname{Hom}(T, S_1) \times \operatorname{Hom}(T, S_1) \times \cdots \times \operatorname{Hom}(T, S_1)}^{r}.$$

Since Hom (T, S_1) is discrete, so also is Hom (T, T').

Since $\hat{\varphi}_e | T = 0$ and G is connected, we have $\hat{\varphi}_g T = 0$ for any $g \in G$. This means that for any element $g' \in \pi^{-1}(T)$ we have $\varphi_g(g') = 0$. It follows that $\pi^{-1}(T) \subset Z$, contradicting the fact that $\pi^{-1}(T)/Z = T \neq \{0\}$.

Thus, we also see that $G/Z = V$.

Define the mapping $\hat{\varphi}: G/Z \times G/Z \to G_1$ by $\hat{\varphi} = \varphi(\pi^{-1}, \pi^{-1})$ and let $G_1 = V' \times T'$. Since $\hat{\varphi}$ is homomorphism with respect to each variable, $\hat{\varphi}$ can be considered as a bi-linear mapping from $V \times V$ into $V' \times T'$. By $(y_1 \cdots y_q, y_{q+1} \cdots y_{q+r})$ we mean a point of $V' \times T'$, where $y_i \in R$ for $1 \leqq i \leqq q$ and $y_{q+j} \in S_1$ for $1 \leqq j \leqq r$. The bi-linear mapping $\hat{\varphi}$ is then

$$\hat{\varphi} = (\hat{\varphi}_1, \hat{\varphi}_2, \cdots, \hat{\varphi}_q, \hat{\varphi}_{q+1}, \cdots \hat{\varphi}_{q+r})$$

$$\hat{\varphi}_i(\boldsymbol{x}, \boldsymbol{x}') = \sum a^i_{k,s} x_k x'_s \qquad \text{for } 1 \leqq i \leqq q$$

$$\hat{\varphi}_j(\boldsymbol{x}, \boldsymbol{x}') \equiv \sum a^j_{k,s} x_k x'_s \qquad \text{mod } 1 \qquad \text{for } 1 \leqq j \leqq r,$$

where $\boldsymbol{x} = (x_1, \cdots, x_n)$, $\boldsymbol{x}' = (x'_1, \cdots, x'_n)$ respectively.

Now, since we need, in the following part, to discuss the topologies for G, we denote $(G, \mathfrak{T}_0)$ instead of G. Let $(G, \mathfrak{T}_0)$ be a Lie group such that the derived group of $(G, \mathfrak{T}_0)$ is contained in the center Z of G. Let $(G, \mathfrak{T}) \in T(G, \mathfrak{T}_0)$. By $(Z, \mathfrak{T})$ we mean the subgroup Z with the relative topology with respect to $(G, \mathfrak{T})$. For any $(G, \mathfrak{T}) \in T(G, \mathfrak{T}_0)$ we have that $(Z, \mathfrak{T})$ is a closed subroup of $(G, \mathfrak{T})$ because Z is the center of G. Thus, $(G, \mathfrak{T})/(Z, \mathfrak{T})$ is a topological group. For convenience we use the notation $(G/Z, \mathfrak{T})$ for $(G, \mathfrak{T})/(Z, \mathfrak{T})$.

LEMMA 1.4. *Notations and assumptions being as above, we have* $(G/Z, \mathfrak{T}) = (G/Z, \mathfrak{T}_0)$ *for any* $(G, \mathfrak{T}) \in T(G, \mathfrak{T}_0)$.

PROOF. We see easily that $(G/Z, \mathfrak{T}) \in T(G/Z, \mathfrak{T}_0)$. Let $\mathfrak{F}$ be a basis of the neighborhoods of the identity e in $(G, \mathfrak{T})$ and this family $\mathfrak{F}$ can naturally be identified with a family of open subsets of $(G, \mathfrak{T}_0)$ satisfying (a)$\sim$(e) in the first part of this section. Let K be a compact connected subset of $(G, \mathfrak{T}_0)$ containing e as an interior point. By ρ is meant a left invariant metric on $(G, \mathfrak{T}_0)$ such that $(G, \mathfrak{T}_0)$ is complete with respect to this metric. By Lemma 1.2, for any $\varepsilon > 0$ there is $W \in \mathfrak{F}$ such that the diameter of any connected component of W is smaller than ε. Since K is compact in $(G, \mathfrak{T})$, there is $U \in \mathfrak{F}$ such that

$$\varphi(k, U) \subset W \cap G_1$$

for any $k \in K$. Since K is connected, for every $g \in U$ we have

$$\varphi(K, g) \subset V(\varepsilon),$$

where $V(\varepsilon)$ is the ε-neighborhood of the identity e of $(G_1, \mathfrak{T}_0)$.

Let π be the natural projection from G onto $G/Z=V$. Since $\pi(K)$ contains the identity 0 as an interior point, denoting by $(x_1, \cdots, x_n)$ a point of G/Z, $\pi(K)$ contains a compact set $D(a)$ of the form

$$D(a)=\{(x_1, \cdots, x_n)\in G/Z\,;\ \sum_{i=1}^{n} x_i^2 \leqq a\}$$

for some $a>0$.

By the argument above, for $\varepsilon=\frac{1}{k}$ $(k>2)$ there is $U_k\in\mathfrak{F}$ such that

$$\hat{\varphi}(\boldsymbol{x}, \pi(U_k))\subset V\left(\frac{1}{k}\right)$$

for any $\boldsymbol{x}\in D(a)$.

Assume that $(G/Z, \mathfrak{T})\neq(G/Z, \mathfrak{T}_0)$. Then, by Lemma 1.1, there is $g_k\in U_k$ such that denoting $\pi(g_k)=(x_1^{(k)}, \cdots, x_n^{(k)})$,

$$\|\pi(g_k)\|^2=\sum_{i=1}^{n}(x_i^{(k)})^2$$

tends to ∞ if $k\to\infty$. Since

$$\hat{\varphi}_i(\boldsymbol{b}, \pi(g_k))=\sum a_{l,s}^i b_l x_s^{(k)}\,, \qquad 1\leqq i\leqq q$$
$$\hat{\varphi}_j(\boldsymbol{b}, \pi(g_k))=\sum a_{l,s}^j b_l x_s^{(k)} \mod 1\,, \qquad q+1\leqq j\leqq q+r\,.$$

the fact that $\hat{\varphi}(\boldsymbol{b}, \pi(g_k))\in V\left(\frac{1}{k}\right)$ means that

$$|\sum a_{l,s}^i b_l x_s^{(k)}|<\frac{1}{k}\,, \qquad 1\leqq i\leqq q+r\,,$$

for any $\boldsymbol{b}=(b_1, \cdots, b_n)\in D(a)$. In fact, if

$$|\sum a_{l,s}^i b_l x_s^{(k)}-m|<\frac{1}{k}\,, \qquad m=\text{non zero integer}$$

for some j, $q+1\leqq j\leqq q+r$, then there is $\lambda\in[0, 1]$ such that

$$|\sum a_{l,s}^i \lambda b_l x_s^{(k)}|=\frac{1}{2}\,.$$

If $\boldsymbol{b}\in D(a)$, then $\lambda\boldsymbol{b}\in D(a)$ for any $\lambda\in[0, 1]$. This contradicts the fact that

$$\hat{\varphi}(\boldsymbol{b}, \pi(g_k))\in V\left(\frac{1}{k}\right)$$

for any $\boldsymbol{b}\in D(a)$.

Let $t_k=\|\pi(g_k)\|$, $\boldsymbol{c}_k=\frac{1}{t_k}\pi(g_k)$ and $\boldsymbol{y}_j=(0, \cdots, \overset{j}{y}, 0, \cdots, 0)\in D(a)$. Then,

$$\hat{\varphi}_i(\boldsymbol{y}_j, \pi(g_k))=t_k y\sum_{s=1}^{n} a_{j,s}^i c_s^{(k)}\,,$$

where $\boldsymbol{c}_k=(c_1^{(k)}, \cdots, c_n^{(k)})$. By $\boldsymbol{a}_j^i$ we mean the vector

$$(a_{j,1}^i, a_{j,2}^i, \cdots, a_{j,n}^i)$$

and by $\langle \boldsymbol{x}, \boldsymbol{y}\rangle$ the ordinary inner product of $\boldsymbol{x}, \boldsymbol{y}$. Thus, we see that

$$|\hat{\varphi}_i(\boldsymbol{y}_j, \pi(g_k))| = |t_k \cdot y \cdot \langle \boldsymbol{a}_j^i, \boldsymbol{c}_k\rangle| = t_k|y| \cdot \|\boldsymbol{a}_j^i\| \cdot |\cos(\widehat{\boldsymbol{a}_j^i, \boldsymbol{c}_k})| < \frac{1}{k}.$$

Thus,

$$\|\boldsymbol{a}_j^i\| \cdot |\cos(\widehat{\boldsymbol{a}_j^i, \boldsymbol{c}_k})| < \frac{1}{k}\frac{1}{t_k|y|}.$$

Since $t_k \to \infty$, we have that if $\boldsymbol{a}_j^i \neq 0$, then for any accumulate point $\boldsymbol{c}_0$ of $\boldsymbol{c}_k$, $\boldsymbol{a}_j^i \perp \boldsymbol{c}_0$. This means that $\hat{\varphi}_i(\boldsymbol{y}_j, \boldsymbol{c}_0) = 0$ for any i and j.

Let $V' = \{\lambda \boldsymbol{c}_0;\ \lambda \in R\}$. Then V' is a closed subgroup of G/Z. Let $G' = \pi^{-1}(V')$. Then, for any $g' \in G'$, $\hat{\varphi}(\boldsymbol{b}, \pi(g')) \equiv 0$, $\boldsymbol{b} \in D(a)$. It follows that G' is contained in the center of G, contradicting the fact that $G'/Z = V' \neq \{0\}$.

Remark that $\hat{\varphi}$ is skew-symmetric. In fact

$$\varphi(h, g) = gg^{-1}hgh^{-1}g^{-1} = g \cdot \varphi(g^{-1}, h) \cdot g^{-1} = \varphi(g^{-1}, h) = (\varphi(g, h))^{-1}.$$

THEOREM 1.4. *Let $(G, \mathfrak{T}_0)$ be a connected Lie group such that the derived group $(G_1, \mathfrak{T}_0)$ of $(G, \mathfrak{T}_0)$ is contained in the center Z of G. Let f be a continuous monomorphism from $(G, \mathfrak{T}_0)$ into a topological group $\mathfrak{G}$ with the first countability axiom. Then $Cl(f(G))$ is locally compact, if and only if $Cl(f(Z))$ is locally compact.*

PROOF. Since $Cl(f(Z))$ is contained in the center of $Cl(f(G))$, we see $f^{-1}(Cl(f(Z)) \cap f(G)) = Z$. Let π, π' be the natural projections $G \to G/Z$, $Cl(f(G)) \to Cl(f(G))/Cl(f(Z))$ respectively. By defining $\hat{f} = \pi' f \pi^{-1}$, we see that

$$\hat{f}: (G/Z, \mathfrak{T}_0) \to Cl(f(G))/Cl(f(Z))$$

is a continuous monomorphism because f is a monomorphism and $f^{-1}(Cl(f(Z)) \cap f(G)) = Z$.

Let $(f(G), \mathfrak{T})$ be the topological group with the relative topology for $f(G)$ in $Cl(f(G))$. Since f is monomorphism, $\mathfrak{T}$ determines uniquely the topology for G under which f is a homeomorphism. This topology for G is denoted by the same notation $\mathfrak{T}$. We see easily that $(G, \mathfrak{T}) \in T(G, \mathfrak{T}_0)$. By the same way, we determine the topology $\mathfrak{T}'$ for G/Z from the relative topology for $\hat{f}(G/Z)$ in $Cl(f(G))/Cl(f(Z))$. It will be shown below that $(G/Z, \mathfrak{T}) = (G/Z, \mathfrak{T}')$.

Let $\{V_k\}$ be a basis of the neighborhoods of the identity e in $Cl(G)$ satisfying $V_k^{-1} = V_k$ and $V_k^2 \subset V_{k-1}$. Then, $\pi f^{-1}(V_k)$, $\hat{f}^{-1}\pi'(V_k)$ are bases of the neighborhoods of the identity of $(G/Z, \mathfrak{T})$, $(G/Z, \mathfrak{T}')$ respectively. So we have only to prove that

$$f^{-1}(V_k)Z \subset f^{-1}(V_k Cl(f(Z))) \subset f^{-1}(V_{k-1})Z.$$

It is clear that $f^{-1}(V_k)Z \subset f^{-1}(V_k Cl(f(Z)))$. Let $a \in f^{-1}(V_k Cl(f(Z)))$. Then $f(a) \in V_k Cl(f(Z))$. Since $ff^{-1}(V_k)$ is dense in V_k, denoting $f(a) = vz$, $v \in V_k$, $z \in Cl(f(Z))$ there are sequences $\{v_n\} \subset f^{-1}(V_k)$ and $\{z_n\} \subset Z$ such that

$\lim f(v_n)=v$ and $\lim f(z_n)=z$. It follows that $f(a)=\lim f(v_n z_n)$ and then $\lim f(a^{-1}v_n z_n)=e$. Therefore for any sufficiently large $n, f(a^{-1}v_n z_n)$ is contained in V_k. Thus, we have $a^{-1}v_n z_n \in f^{-1}(V_k)$. It follows

$$a \in f^{-1}(V_k^{-1})v_n z_n \subset f^{-1}(V_k^2)Z \subset f^{-1}(V_{k-1})Z .$$

By the above result and Lemma 1.4, we see that $\hat{f}(G/Z, \mathfrak{T}_0)$ is locally compact and then

$$\hat{f}(G/Z, \mathfrak{T}_0)=Cl(f(G))/Cl(f(Z)) .$$

Thus, $Cl(f(G))/Cl(f(Z))$ is locally compact. This fact, together with Theorem 2.2 in [6], implies that $Cl(f(G))$ is locally compact.

§ 2. A reduction of the problem (b).

A topological group $\mathfrak{G}$ with the first countability axiom is called a *V-group* if any homomorphic image of any vector group into $\mathfrak{G}$ has the locally compact closure. In this section, the following will be proved.

THEOREM 2.1. *If a topological group $\mathfrak{G}$ is a V-group, then any monomorphic image of any connected Lie group into $\mathfrak{G}$ has the locally compact closure.*

The proof of the theorem will be given by a series of reductions and lemmas below.

We devide the proof of this theorem as follows:

i) reduce the problem to the case of a connected solvable Lie group.

ii) reduce the problem to the case of a connected solvable Lie group with discrete center.

iii) reduce the problem to the case of a connected solvable Lie group with discrete center whose derived group is an abelian group.

iv) reduce the problem to the case of S_0-group.

v) prove Theorem 2.1 in the case of S_0-group.

i) Let G be a connected Lie group with radical N. The group $G'=G/N$ is a semi-simple Lie group. Let Z be the center of G', which is discrete. Then G'/Z is isomorphic to the adjoint group of G'. Let K' be a maximal compact subgroup of G'/Z. There exists a subgroup M in G' such that, denoting by K a maximal compact subgroup of G', (i) $M=K$ or $K\times V$ with a vector group V, (ii) M contains Z and $M/Z=K'$ ([5] Lemma 3.12).

For any $z\in Z$, we see that $z=(k, v)$, $k\in K$, $v\in V$. Since $gzg^{-1}=z$ means $gkg^{-1}=k$, $gvg^{-1}=v$, we see that $k\in Z$, $v\in Z$. It follows that

$$Z=Z\cap K\times Z\cap V .$$

Denoting $Z'=Z\cap V$, G'/Z' has the finite center.

Let f be a continuous monomorphism from G into a topological group $\mathfrak{G}$ with the first countability axiom. Denote by π the natural projection from G onto G/N.

LEMMA 2.1. *Notations and assumptions being as above, if* $Cl(f(\pi^{-1}(V))$ *is locally compact, then so is* $Cl(f(G))$.

PROOF. Since $V\supset Z'$, we see that $Cl(f(\pi^{-1}(Z')))$ is locally compact. Let $\hat{f}$ be the homomorphism from $G/\pi^{-1}(Z')$ into $Cl(f(G))/Cl(f(\pi^{-1}(Z')))$ defined by $\hat{f}\pi=\pi'f$, where π' is the natural projection $Cl(f(G))\rightarrow Cl(f(G))/Cl(f(\pi^{-1}(Z')))$.

Since $G/\pi^{-1}(Z')=G'/Z'$, we see by Corollary 1.2 that $\hat{f}(G/\pi^{-1}(Z'))$ is a closed subgroup of $Cl(f(G))/Cl(f(\pi^{-1}(Z')))$. In fact, since $Cl(f(G))$ satisfies the first countability axiom, so does $Cl(f(G))/Cl(f(\pi^{-1}(Z')))$. Since $G/\pi^{-1}(Z')$ has the finite center, we see that the assumptions of Corollary 1.2 are satisfied.

Since $\hat{f}(G/\pi^{-1}(Z'))$ is dense in $Cl(f(G))/Cl(f(\pi^{-1}(Z')))$, we see

$$\hat{f}(G/\pi^{-1}(Z'))=Cl(f(G))/Cl(f(\pi^{-1}(Z'))).$$

If follows $Cl(f(G))/Cl(f(\pi^{-1}(Z')))$ is locally compact. Since $Cl(f(\pi^{-1}(Z')))$ is locally compact, so also is $Cl(f(G))$.

It is easy to see, in the above lemma, that $\pi^{-1}(V)$ is a connected solvable Lie group.

COROLLARY 2.1. *Let G be a connected Lie group and $\mathfrak{G}$ a topological group with the first countability axiom. Let f be a continuous monomorphism from G into $\mathfrak{G}$. Then, $Cl(f(G))$ is locally compact if and only if for any closed and connected solvable Lie subgroup S, $Cl(f(S))$ is locally compact.*

ii) When we consider a triple $\{G,f,\mathfrak{G}\}$ of a connected solvable Lie group, a topological group with the first countability axoim and a continuous homomorphism from G into $\mathfrak{G}$, we can assume that the kernel of f is discrete because in Lemma 2.1, $f:\pi^{-1}(V)\rightarrow\mathfrak{G}$ is a monomorphism.

For convenience we introduce the following definition.

A triple $\{G,f,\mathfrak{G}\}$ of a connected Lie group G, a topological group $\mathfrak{G}$ and a continuous homomorphism f from G into $\mathfrak{G}$ is called a *V'-triple* if $\{G,f,\mathfrak{G}\}$ is a V-triple and f has the discrete kernel.

Let $\{G,f,\mathfrak{G}\}$ be a V'-triple and let Z, Z_0 be the center of G and its connected component containing the identity e respectively. Define the homomorphism $\hat{f}$ from G/Z_0 into $Cl(f(G))/Cl(f(Z_0))$ by $\hat{f}=\pi'f\pi^{-1}$, where π, π' be the natural projections $G\rightarrow G/Z_0$, $Cl(f(G))\rightarrow Cl(f(G))/Cl(f(Z_0))$ respectively. By the assumption we see that $Cl(f(Z_0))$ is locally compact. In fact, since $Z_0=V\times T$ where V, T are a vector group, a toroidal group respectively, we have that $Cl(f(V))$ is locally compact. Since T is compact, $Cl(f(Z_0))$ is locally compact.

LEMMA 2.2. *Notations and assumptions being as above,* $\{G/Z_0,\hat{f},Cl(f(G))$

$/Cl(f(Z_0))\}$ *is a* V'*-triple.*

PROOF. Since $\mathfrak{G}$ satisfies the first countability axiom, so does $Cl(f(G))/Cl(f(Z_0))$.

Let D be the discrete kernel of f and π'' the natural projection $G\rightarrow G/D$. We have $D\subset Z$. Let Z' be the center of G/D. Since $Cl(f(Z_0))$ is contained in the center of $Cl(f(G))$, $f'^{-1}(Cl(f(Z_0)))\subset Z'$ where f' is the monomorphism from G/D into $\mathfrak{G}$ defined by $f'=f\pi''^{-1}$. Since D is discrete and G is connected, we see that $\pi''^{-1}(Z')=Z$. This means $\pi''^{-1}f'^{-1}(Cl(f(Z_0)))\subset Z$ and then $f^{-1}(Cl(f(Z_0)))\subset Z$. It follows that $\hat{f}$ has the discrete kernel, which is contained in Z/Z_0.

Let V be a closed vector subgroup of G/Z_0. Then $G'=\pi^{-1}(V)$ is a connected and closed subgroup of G. Since Z_0 is contained in the center C' of G' and G'/C' is abelian, we have that G' is a nilpotent Lie group such that the derived group G_1' of G' is contained in C'. It follows from Lemma 1.3 that C' is connected. Since D is discrete, $\pi''(G')=G''$ is also a connected nilpotent Lie group such that the derived group G_1'' of G'' is contained in the center C'' of G''. It follows that C'' is connected and $\pi''(C')=C''$ because D is discrete.

Since $\{G,f,\mathfrak{G}\}$ is a V-triple, $Cl(f(C'))=Cl(f'(C''))$ is locally compact. By Theorem 1.4, we obtain that $Cl(f(G'))=Cl(f'(G''))$ is locally compact. Since $\hat{f}(V)\subset Cl(f(G'))/Cl(f(Z_0))$, $Cl(\hat{f}(V))$ is locally compact. This means that

$$\{G/Z_0,\hat{f},\ Cl(f(G))/Cl(f(Z_0))\}$$

is a V-triple and then V'-triple.

Remark that if $Cl(f(G))/Cl(f(Z_0))$ is locally compact, then so is $Cl(f(G))$.

COROLLARY 2.2. *Let* $\{G,f,\mathfrak{G}\}$ *be a* V'*-triple. If* G *is a connected nilpotent Lie group, then* $Cl(f(G))$ *is locally compact.*

iii) Now, to prove Theorem 2.1, on account of Lemmas 2.1 and 2.2, we consider the following problem:

(A) Let $\{G,f,\mathfrak{G}\}$ be a V'-triple. Assume that G is a connected solvable Lie group with discrete center. Then, is $Cl(f(G))$ locally compact?

It is easy to see by Lemma 2.1, 2.2, that if (A) is affirmative, then Theorem 2.1 is true.

Thus, we have only to consider (A) in the following part of this section.

Let $\{G,f,\mathfrak{G}\}$ be a V'-triple and D be the discrete kernel of f. Let π be the natural projection $G\rightarrow G/D$. Defining $\hat{f}=f\pi^{-1}$, we see that $\{G/D,\hat{f},\mathfrak{G}\}$ is a V-triple. In fact, let V be a closed vector subgroup of G/D and $\pi^{-1}(V)=G'$. Since D is discrete and V is simply connected, we see that $G'=G_0'\times D$, where G_0' is the connected component of G' containing e. Thus,

$$\hat{f}(V)=f\pi^{-1}(V)=f(G_0\times D)=f(G_0)\,.$$

Since G_0 is connected abelian and $\{G, f, \mathfrak{G}\}$ is a V-triple, we have that $Cl(\hat{f}(V))$ is locally compact. Thus, when we consider (A), we can assume that f is a monomorphism.

LEMMA 2.3. *Let V, A and f be a vector group, a locally compact abelian group with the first countability axiom and a continuous homomorphism from V into A such that $Cl(f(V))=A$. Let K be a maximal compact subgroup of A. Assume that $K\neq\{0\}$. Then there is a non-trivial closed vector subgroup V' of V such that $f(V')\subset K$.*

PROOF. Obviously A is connected. By 4.6 Theorem and 4.7.1 Lemma in [6], we can find a sequence of compact normal subgroups of A:

$$N_1\supset N_2\supset\cdots$$

such that $\cap N_i=\{e\}$ and that $A/N_i=A_i$ is a connected Lie group.

Let $A_i=V_i\times T_i$, where V_i, T_i are a vector group, a toroidal group respectively. Since $K\neq\{0\}$, we have $T_i\neq\{0\}$. Let π_i be the natural projection from A onto A_i. Since $\pi_i f(V)$ is dense in A_i, there is a closed and non-trivial subgroup V_i' of V such that $(f^{-1}\pi_i^{-1}(T_i))_0=V_i'$. Since $V_{i+1}'\subset V_i'$ and $V_i'\neq\{0\}$ for all i, we see that $V'=\cap V_i'$ is a closed non-trivial vector subgroup of V. Thus, $\pi_i f(V')\subset T_i$ for all i. It follows that $f(V')\subset K$.

Let $\{G, f, \mathfrak{G}\}$ be a V-triple. Assume that G is a connected solvable Lie group with the discrete center and f is a monomorphism. Denote by G_i the derived group of G_{i-1} $(G=G_0)$. Then

$$G=G_0\supset G_1\supset G_2\supset\cdots\supset G_k\supset G_{k+1}=\{0\}$$

for some k.

Assume that $G_k\neq\{0\}$. G_k is a connected, abelian and normal subgroup of G. Let $\hat{f}$: $G/G_k\rightarrow Cl(f(G))/Cl(f(G_k))$ be the homomorphism defined by $\hat{f}=\pi' f\pi^{-1}$, where π, π' be the natural projections $G\rightarrow G/G_k$, $Cl(f(G))\rightarrow Cl(f(G))/Cl(f(G_k))$ respectively.

LEMMA 2.4. *Notations and assumptions being as above, $\hat{f}$ is a monomorphism.*

PROOF. Since G_k is connected, $Cl(f(G_k))$ is a connected locally compact subgroup of $\mathfrak{G}$. Let K be a maximal compact subgroup of $Cl(f(G_k))$. Since K is also a normal subgroup of $Cl(f(G))$, we see by Theorem 4, [5], that K is contained in the center of $Cl(f(G))$. If $K\neq\{0\}$, then by Lemma 2.3, we have that $G\cap f^{-1}(K)$ contains a vector subgroup. Since $f^{-1}(K)$ is contained in the center of G, this contradicts the assumption that G has the discrete center. Thus, we see $K=\{0\}$.

It follows from Theorem 6 [5] that $Cl(f(G_k))$ is a connected abelian Lie group without any compact subgroup, i. e., a vector group. So we see easily that $Cl(f(G_k))=f(G_k)$. Thus, $f^{-1}(Cl(f(G_k)))=G_k$, because f is a monomorphism. This means that $\hat{f}$ is a monomorphism.

Now, if (A) is affirmative, then especially the following is true:
(B) Let $\{G, f, \mathfrak{G}\}$ be a V-triple. Assume that G is a solvable Lie group with the discrete center such that the derived group G_1 of G is abelian and that f is a monomorphism. Then $Cl(f(G))$ is locally compact.

LEMMA 2.5. (A) *is affirmative, if and only if* (B) *is true.*

PROOF. We have only to prove that if (B) is true, then (A) is affirmative.

Assume that (B) is true. We use the induction on the length k of the series of the derived groups

$$G=G_0\supset G_1\supset \cdots \supset G_k\supset G_{k+1}=\{0\}\,, \qquad G_k\neq\{0\}\,.$$

Assume that (A) is affirmative for any connected solvable Lie group with the discrete center such that the length of the series of the derived groups is smaller than k ($k\geqq 2$).

Let G be a connected solvable Lie group with the discrete center such that the length of the series of the derived groups is k and let $\{G, f, \mathfrak{G}\}$ be a V'-triple. Denote by D the discrete kernel of f and π'' the natural projection from G onto G/D. By defining $f'=f\pi''^{-1}$, we see that $\{G/D, f', \mathfrak{G}\}$ is a V-triple and f' is a monomorphism. So we may assume that f is monomorphism.

By Lemma 2.4, the mapping

$$\hat{f}:\ G/G_k\to Cl(f(G))/Cl(f(G_k))$$

is a monomorphism. Let V be a closed vector subgroup of G/G_k and $G'=\pi^{-1}(V)$ where π is the natural projection $G\to G/G_k$. Since G_k is a vector subgroup, G' is a closed and connected solvable Lie group such that the derived group G'_1 of G' is abelian.

Let Z'_1 be the connected component of the center of G', containing the identity e and let Z'_i/Z'_{i-1} be the connected component of the center of G'/Z'_{i-1} containing the identity. Then we see that

$$Z'_1\subset Z'_2\subset \cdots \subset Z'_r=Z'_{r+1}$$

for some r and G'/Z'_r is a connected solvable group with discrete center such that the derived group of G'/Z'_r is abelian. By using Lemma 2.2 repeatedly, we see that $Cl(f(G'))$ is locally compact if and only if (A) is true for V'-triple $\{G'', h, \mathfrak{G}'\}$ where G'' is a connected solvable Lie group whose center is discrete and whose derived group is abelian. Since we can assume that h is a monomorphism, we have by the assumption that (B) is true that $Cl(f(G''))$ is locally compact.

It follows that $Cl(f(G'))$ is locally compact and then

$$\{G/G_k, \hat{f}, Cl(f(G))/Cl(f(G_k))\}$$

is a V-triple and $\hat{f}$ is a monomorphism.

The length of the derived groups of G/G_k is smaller than k. Again by using Lemma 2.2 repeatedly, and by the induction assumption, we have consequently that $Cl(\hat{f}(G/G_k))$ is locally compact. Since $\hat{f}(G/G_k)$ is dense in $Cl(f(G))/Cl(f(G_k))$, we have $Cl(f(G))/Cl(f(G_k))$ is locally compact. Since $Cl(f(G_k))$ is locally compact, so also is $Cl(f(G))$.

iv) By Lemma 2.5, we have only to consider (B).

Let G be a connected solvable Lie group with discrete center such that the derived group G_1 of G is abelian. Since the center Z of G is discrete, we see that G_1 is a vector group (that is, G_1 contains no toroidal subgroup).

Let N be a closed normal subgroup of G and $G'=G/N$. Then, the derived group G_1' of G' is contained in G_1N/N. Thus we see that

$$\dim G_1' \leqq \dim G_1 .$$

If N is a non-trivial closed vector subgroup of G_1, then

$$\dim G_1' < \dim G_1 .$$

For convenience, a connected Lie group G is called a S_0-*group* if (i) G is solvable and the derived group G_1 of G is abelian (ii) G has the discrete center and (iii) G_1 contains no non-trivial connected closed normal subgroup of G.

LEMMA 2.6. *Notations being as above,* (B) *is true, if and only if* (B) *is true for S_0-groups.*

PROOF. The necessity is trivial, so we show the sufficiency. We use the induction on $\dim G_1$. If $\dim G_1=1$, then G is an S_0-group. Thus (B) is true. Let G be a connected solvable Lie group such that $\dim G_1=n$. Assume that (B) is true for any connected solvable Lie group G' such that $\dim G_1'<n$.

Assume that G is not an S_0-group. Then, there is a vector subgroup N of G_1 such that N is a normal subgroup of G. By defining $\hat{f}=\pi' f\pi^{-1}$, where π, π' be the natural projection from $G\rightarrow G/N$, $Cl(f(G))\rightarrow Cl(f(G))/Cl(f(N))$ respectively, we have by Lemma 2.3 and in the same way as in Lemma 2.4 that $\hat{f}$ is a monomorphism from G/N into $Cl(f(G))/Cl(f(N))$. It will be shown below that

$$\{G/N, \hat{f}, Cl(f(G))/Cl(f(N))\}$$

is a V-triple.

Let G' be a closed subgroup of G containing N such that G'/N is a vector subgroup of G/N. Then, G' is connected. The derived group G_1' of G' is contained in $G_1\cap N$. Thus,

$$\dim G_1' \leqq \dim N < \dim G_1 .$$

Then, by using Lemma 2.2 repeatedly and by the induction assumption, we see that $Cl(f(G'))$ is locally compact. Since $\hat{f}(G'/N)$ is dense in $Cl(f(G'))/Cl(f(N))$, we see that $Cl(\hat{f}(G'/N))$ is locally compact. Thus,

$$\{G/N, \hat{f}, Cl(f(G))/Cl(f(N))\}$$

is a V-triple.

Since the dimension of the derived group of G/N is smaller than $\dim G_1$, we see, by the induction assumtion and by using Lemma 2.2 repeatedly, that $Cl(\hat{f}(G/N))$ is locally compact. Since $Cl(f(N))$ is locally compact, we have that $Cl(f(G))$ is locally compact.

LEMMA 2.7. *Notations being as above, if G is an S_0-group, then there is a subgroup H of G such that*

$$G = H \cdot G_1 , \qquad H \cap G_1 = \{e\} .$$

The proof of this lemma is the same as the last part of the proof of Lemma 3.4 in [5].

Since G_1 is a vector group, we denote V instead of G_1. Eor any $h \in H$, the transformation from V onto V defined by $v \rightarrow hvh^{-1}$ induces naturally a linear transformation $A(h)$ on V. The correspondence $h \rightarrow A(h)$ gives obviously a representation of the abelian group H.

LEMMA 2.8. *Notations being as above, if G is an S_0-group, then* $\dim V$ *is one or two.*

PROOF. For any $h \in H$, the eigen values of $A(h)$ are equal or conjugate each other because H is abelian and $A(h)$ is an irreducible representation of H. Moreover, since $A(h)$ is a real matrix, by Shur's lemma we see that $\dim V = 1$ or 2.

Since H is connected abelian Lie group, letting $\mathfrak{h}$ be the Lie algebra of H, we see $\exp \mathfrak{h} = H$.

Let $\dim V = 1$. Then, $A(\exp X) = e^{\lambda(X)}$ for $X \in \mathfrak{h}$, where λ is a linear mapping from $\mathfrak{h}$ into R. If the kernel of λ, $\mathfrak{h}'$ would be non-trivial then $\exp \mathfrak{h}'$ must be contained in the center of G, contradicting the fact that G is an S_0-group. It follows that

$$\dim H = 1$$

and $\lambda(X) \neq 0$.

Let $\dim V = 2$. Then the eigen values of $A(\exp X)$ are $e^{\lambda(X)}$ and $e^{\overline{\lambda(X)}}$, where λ is a linear mapping from $\mathfrak{h}$ into the complex number field. Let $\lambda(X) = a(X) + \sqrt{-1}\, b(X)$. Then

$$A(\exp X)=\begin{pmatrix} e^{a(X)}\cos b(X), & -e^{a(X)}\sin b(X) \\ e^{a(X)}\sin b(X), & e^{a(X)}\cos b(X) \end{pmatrix}.$$

It follows that $\dim H \leqq 2$.

Thus, we have the following:

THEOREM 2.2. *Simply connected S_0-groups are the following:*

a) $H=R$, $V=R$ *and* $A(t)=e^{at}$, $a\neq 0$,

b) $H=R$, $V=R^2$ *and*

$$A(t)=\begin{pmatrix} \cos bt, & -\sin bt \\ \sin bt, & \cos bt \end{pmatrix}, \qquad b\neq 0,$$

c) $H=R$, $V=R^2$ *and*

$$A(t)=\begin{pmatrix} e^{at}\cos bt, & -e^{at}\sin bt \\ e^{at}\sin bt, & e^{at}\cos bt \end{pmatrix}, \qquad a\cdot b\neq 0,$$

d) $H=R^2$, $V=R^2$ *and*

$$A(t,t')=\begin{pmatrix} e^{at}\cos bt', & -e^{at}\sin bt' \\ e^{at}\sin bt', & e^{at}\cos bt' \end{pmatrix}, \qquad a\cdot b\neq 0.$$

We see easily that the groups of a), c) are without center and the groups of b), d) have the centers isomorphic to the additive group of the integers. It follows immediately that if an S_0-group G is not simply connected, then G has the finite center and then G is absolutely closed by Theorem 1.2.

v) On account of Lemmas 2.1, 2.2, 2.5 and 2.6, we have only to show that the proposition (B) is true for the S_0-groups whose universal covering groups are listed in Theorem 2.2 above.

Assume that $\{G, f, \mathfrak{G}\}$ be a V-triple and f a monomorphism. If the S_0-group G has the finite center, then $f(G)=Cl(f(G))$ and $Cl(f(G))$ is locally compact. So we assume that G has the infinite center. Then G is the group b) or d). Let Z be the center of G. Since $Z\subset H$ and $f(H)$ has the locally compact closure, we see that $Cl(f(Z))$ is locally compact.

Let π, π' be the natural projections $G\rightarrow G/Z$, $Cl(f(G))\rightarrow Cl(f(G))/Cl(f(Z))$ respectively. Since $Cl(f(Z))$ is contained in the center of $Cl(f(G))$, we see that

$$f^{-1}(Cl(f(Z))=Z.$$

Thus, by defining $\hat{f}=\pi' f\pi^{-1}$, $\hat{f}$ is a monomorphism from G/Z into $Cl(f(G))/Cl(f(Z))$. Since G/Z is without center, G/Z is absolutely closed. It follows that $Cl(\hat{f}(G/Z))=\hat{f}(G/Z)$ is locally compact. Since $Cl(\hat{f}(G/Z))=Cl(f(G))/Cl(f(Z))$, we have that $Cl(f(G))$ is locally compact.

Thus, we get the complete proof of Theorem 2.1.

Moreover, we have the following theorem which is slightly stronger than Theorem 2.1.

THEOREM 2.3. *Let $\{G, f, \mathfrak{G}\}$ be a V-triple of a connected Lie group G and a topological group $\mathfrak{G}$ with the first countability axiom and a continuous monomorphism f from G into $\mathfrak{G}$. Then $f(G)$ has the locally compact closure.*

§3. The group of homeomorphism of a one dimensional manifold.

By theorem 2.1 we obtained a fairly simple criterion for a topological group $\mathfrak{G}$ to belong to the class $\mathfrak{B}$ stated in the introduction. Namely we have only to consider the vector groups instead of the general Lie groups.

It might then be natural to consider the problem of reducing the dimension of the vector group in question to a smaller one. It seems, however, to be difficult, because we have an example of a connected abelian topological group $\mathfrak{G}$ which is the closure of a homomorphic image of a vector group V and which is not locally compact though the homomorphic images of any proper vector subgroups of V have locally compact closures [7].

This seems to mean that to solve the original conjecture of Gleason-Palais we have naturally to analyse the structures of the homeomorphism group itself.

In the case of one-dimensional manifolds, however, the homeomorphism group is an example of a V-group.

Let M be a connected manifolds with the second countability axiom. Then it is known [3] that $H(M)$ endowed with the compact open topology satisfies the second countability axiom and $H(M)$ is complete under the bilateral uniform structure, i.e., the uniform structure generated by uniformities of the form

$$\{(g, h) \in H(M)\times H(M);\ gh^{-1} \text{ and } g^{-1}h \in V\}$$

for some neighborhood V of the identity. It follows that $H(M)$ is a space of the second category.

Let G be a Lie group and φ a continuous homomorphism from G into $H(M)$ such that $\varphi(G)$ is closed. Then, φ is an open mapping from G onto $\varphi(G)$, since $\varphi(G)$ is a topological group of the second category as a subgroup of $H(M)$.

LEMMA 3.1. *Let f be a continuous monomorphism from $(R^r, \mathfrak{T}_0)$ into $H(R)$, where $\mathfrak{T}_0$ is the ordinary topology on R^r. Then, $f(R^r)$ is closed.*

PROOF. Assume $f(R^r)$ is not closed. Then, the relative topology for $f(R^r)$ determines a topology $\mathfrak{T}$ for R^r such that $(R^r, \mathfrak{T}) \neq (R^r, \mathfrak{T}_0)$ and $(R^r, \mathfrak{T})$ is contained in $T(R^r, \mathfrak{T}_0)$. By Lemma 1.1 we see

$$f^{-1}(U \cap f(R^r)) \cap (R^r - D(k)) \neq \phi$$

for any neighborhood U of the identity of $H(R)$.

Let $W(K, V)=\{h \in H(R);\ h(K) \subset V\}$, where K is a compact subset and V is an open subset containing K. By the definition of the compact open topology, there are K_i and V_i, $i=1, 2, \cdots, m$ such that $U \supset \bigcap_{i=1}^{m} W(K_i, V_i)=W'$. Take sufficiently small U such that the diameter of any connected component of $f^{-1}(U \cap f(R^r))$ is smaller than ε and $\varepsilon<k$.

Let $y \in f^{-1}(W' \cap f(R^r)) \cap (R^r-D(k))$. Since $\varepsilon<k$, we have

$$\cup \{ty;\ t \in [0, 1]\} \not\subset f^{-1}W'.$$

Thus, there are $t \in [0, 1]$ and some j such that $f(ty)(K_j) \not\subset U_j$. On the other hand, $f(y)(K_j) \subset U_j$. This contradicts the fact that $\{f(ty);\ t \in R\}$ is a one parameter transformation group on a line R.

LEMMA 3.2. *Let f be a continuous monomorphism from R^r into $H(S_1)$. Then $f(R^r)$ is closed in $H(S_1)$.*

PROOF. Assume that $f(R^r)$ is not closed in $H(S_1)$. Then, we see that $f(R^r)(t)=S_1$ for all $t \in S_1$. In fact, if there is $t_0 \in S_1$ such that $f(R^r)(t_0) \neq S_1$, then $f(R^r)(t) \neq S_1$ for all $t \in S_1$. By the same argument as in Lemma 3.1, we see in this case that $f(R^r)$ is closed, contradicting the assumption. Thus, $f(R^r)(t)=S_1$. Let K_t be the isotropy group of R^r. Since R^r is abelian and operates transitively on S_1, we see that $K_t=K_0$. This means that f is not a monomorphism, contradicting the assumption.

By the above two lemmas, we have the following:

THEOREM 3.1. *The group of the homeomorphisms on a one dimensional connected manifold is a V-group.*

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