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On compact complex analytic manifolds of complex dimension 3.

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The purpose of this paper is to prove some analogous propositions to the results of Kodaira [8] in three dimensional case. Terminologies and notations are the same as those in Kodaira [8]. We shall use the fundamental results of Hironaka [5].

Let M^n be a compact complex analytic manifold of complex dimension n. Let $\mathcal{F}(M^n)$ be the field of all meromorphic functions on M^n. Then by a theorem of Chow-Remmert [9] $\mathcal{F}(M^n)$ is an algebraic function field of complex dimension not greater than n. Hence there is a non-singular projective model V of $\mathcal{F}(M^n)$. We identify $\mathcal{F}(M^n)$ and the function field of V. Let $(1, x^1, \cdots, x^\nu)$ be a generic point of V. Then $x^i \in \mathcal{F}(M^n)$. Hence we obtain a mapping

$$\Phi: \quad M \ni z \rightarrow (1, x^1(z), \cdots, x^\nu(z)) \in V .$$

PROPOSITION. *Φ is a meromorphic mapping. That is, there exists an irreducible and locally irreducible complex subspace X of $M^n \times V$ which is the closure of the graph of Φ and the natural projection p of X to M^n is a proper modification.*

$$\begin{array}{l} \varphi: X \xrightarrow{\ \iota\ } M^n \times V \longrightarrow V \\ \qquad\ \ \searrow \quad \downarrow \\ \qquad\quad p \quad M^n \end{array}$$

Proof is parallel to Remmert [10] and we do not reproduce it here.

Let φ be the natural projection from X to the second component V.

Clearly the underlying continuous map of φ is surjective and φ induces an isomorphism of $\mathcal{F}(X)$ and $\mathcal{F}(V)$, where $\mathcal{F}(X)$ and $\mathcal{F}(V)$ are the function fields of X and V, respectively.

THEOREM 1. *Every fibre of φ is connected. Consequently, if* $\dim \mathcal{F}(M^n)=n$, *then M^n is bimeromorphically equivalent to a non-singular projective variety.*

COROLLARY. *If* $\dim \mathcal{F}(M^n)=n=3$, *then the first Betti number of M^3 is even.*

Let n be equal to 3 and $\rho: M' \rightarrow X$ be the resolution of singularities.

Then the underlying continuous map of $\psi=\varphi\circ\rho$ is surjective and ψ induces an isomorphism of the function fields of M' and V.

THEOREM 2. *If* $\dim \mathfrak{F}(M')=\dim V=2$, *then a general fibre of* ψ *is a non-singular elliptic curve. Consequently, if* $\dim \mathfrak{F}(M^3)=2$, *then* M^3 *is bimeromorphically equivalent to an elliptic fibre space over a projective surface.*

§ 1. Preliminaries.

PROPOSITION 1 (H. Cartan [2]). *If a morphism of complex spaces* $f: X\to Y$ *is finite and* Y *is compact algebraic. Then* X *is also algebraic.*

PROOF. By Houzel [7] we may assume that $X=\mathrm{Specan}(A)$ where A is a coherent algebra on Y. By a result due to Serre-Grothendieck (cf. Sém. H. Cartan 1956/57 Exp. 2) A is algebraic. Taking these into account, follow the construction of Specan(A) in Houzel [7]. Then the proof is immediate.

PROPOSITION 2. *Let* $f: M\to M'$ *be a morphism of compact complex manifolds of complex dimension* n *which is a modification. Then the induced homomorphism*

$$f_*: H_1(M, R)\to H_1(M', R)$$

is an isomorphism.

PROOF. By Grauert and Remmert [3] there is a proper analytic set A (resp. A') of M (resp. M') (where the codimension of A' is at least 2) and f induces an isomorphism of $M-A$ and $M'-A'$. Every 1-cycle in M' is homotopic to a 1-cycle in $M'-A'$. Hence f_* is surjective. On the other hand from the exact sequence

$$H^{2n-1}(M-A, R)\to H^{2n-1}(M, R)\to H^{2n-1}(A, R)$$

we have $\dim H^{2n-1}(M-A, R)\geqq \dim H^{2n-1}(M, R)$. By the excision theorem $\dim H^{2n-1}(M'-A', R)=\dim H^{2n-1}(M', R)$. Hence $\dim H^{2n-1}(M', R)=\dim H^{2n-1}(M, R)$. By Poincaré duality we have $\dim H_1(M', R)\geqq \dim H_1(M, R)$.

COROLLARY. *The first Betti number is invariant under bimeromorphic mappings of compact complex manifolds of complex dimension not greater than 3.*

PROPOSITION 3 (Bertini). *Let* D *be an effective divisor on a compact complex manifold. Then the singular point of a general member of* $|D|$ *is a fixed point of it.*

Proof is well-known.

LEMMA 4. *Let* D *be a non-singular divisor on a compact complex manifold* M^n *such that the restriction of* $[D]$ *to* D *contains an effective divisor. Then for every positive integer* m, $\dim H^{n-1}(M, \Omega(F+mD))$ *is bounded, where* F *is an arbitrary complex line bundle on* M^n.

PROOF. From the exact sequence

$$0\to\Omega(F+(m-1)[D])\to\Omega(F+m[D])\to\hat{\Omega}_D(F+m[D])\to 0\,,$$

where $\hat{\Omega}_D(F+m[D])=\Omega(F+m[D])/\Omega(F+(m-1)[D])$, we have the exact sequence

$$H^{n-1}(M,\Omega(F+(m-1)[D]))\to H^{n-1}(M,\Omega(F+m[D]))$$
$$\to H^{n-1}(D,\Omega((F+m[D])_D)).$$

Let K be the canonical line bundle of D, by the duality theorem we obtain

$$H^{n-1}(D,\Omega((F+m[D])_D))=H^0(D,\Omega(K-F_D-m([D]_D)))\,.$$

The latter is 0 for sufficiently large m by Kodaira [8]. Hence $\dim H^{n-1}(M, \Omega(F+m[D]))$ is a non-increasing function for sufficiently large m, which proves the proposition.

§ 2. Proof of Theorem 1.

Let $\varphi: X\to X'\xrightarrow{f} V$ be the factorization of Stein. That is, $X'=$(Specan $(\varphi_*(O_X)))_{red}$. Clearly X' is irreducible and f induces an isomorphism of the function fields. By Proposition 1 X' is algebraic. Hence by the connectedness theorem of Zariski (cf. [4] (4.3.7)) every fibre of f is connected. Therefore every fibre of φ is also connected.

§ 3. Proof of Theorem 2.

We denote by S the set of degeneracy points of the jacobian of ψ. Then S is a proper analytic set of M' and the restriction of ψ to $M'-S$ is a simple morphism. Therefore the fibre space $\psi\,|\,M'-\psi^{-1}(\psi(S)): M'-\psi^{-1}(\psi(S))\to V-\psi(S)$ is differentiably locally trivial. Hence general fibres of ψ are diffeomorphic and homotopic to each other.

Let C be the divisor on V by a hyperplane section. We set $D=\psi^{-1}(C)$. For a given complex line bundle F on M', if $|F+mD|$ contains no effective divisor, then $\dim|F+mD|=-1$. If $|F+mD|$ contains an effective divisor D', then $F=D''$, where $D''=D'-mD$. Clearly

$$\dim|F+lD|=\dim|D'+(l-m)D|,\qquad \text{for}\quad l\geqq m\,.$$

For every effective divisor E on M' we denote by $\mathfrak{a}(E)$ the effective divisor on V defined in the following way. Each component of $\mathfrak{a}(E)$ appears in E by ψ^{-1} and its multiplicity in $\mathfrak{a}(E)$ is the same as in E. From the fact that ψ induces an isomorphism of the function fields we have

$$\dim|E|=\dim|\mathfrak{a}(E)|\,.$$

Therefore $\dim|F+lD|=\dim|\mathfrak{a}(D')+(l-m)C|$. For sufficiently large l $|\mathfrak{a}(D')$

$+(l-m)C|$ is ample and by the theorem of $R.$-$R.$-Hirzebruch we have

$$\dim|\mathfrak{a}(D')+(l-m)C|=\frac{1}{2}l^2C^2+\alpha_1 l+\alpha_0$$

where α_i is a constant. Consequently we obtain

$$\dim|F+lD|\leqq\frac{1}{2}l^2C^2+\alpha_1 l+\alpha_0 \cdots\cdots\cdots\cdots\cdots\cdots(1)$$

Let K be the canonical line bundle of M' and c_1 (resp. d) be the first Chern class of M' (resp. $[D]$). Clearly we have

$$d^3[M']=D^3=0\,.$$

Hence by the theorem of R.-R.-Hirzebruch ([**1**])

$$\dim|nK+lD|=\frac{1}{4}(1-2n)l^2d^2c_1[M']+\alpha_1'l+\alpha_0'$$

$$+\sum_{i=1}^{3}(-1)^{i-1}\dim H^i(M', \Omega(nK+lD)) \cdots\cdots(2)$$

where n is an arbitrary integer aud α_1' is some constant. Considering Proposition 3 and Lemma 4, we have from (1) and (2)

$$(1-2n)d^2c_1[M']\leqq 2C^2\,. \cdots\cdots\cdots\cdots\cdots\cdots(3)$$

By a theorem of Hirzebruch [**6**] the arithmetic genus $\alpha(D^2)$ of D^2 is $-[d^3-\frac{1}{2}c_1d^2][M']$. Hence if the the genus of a general fibre of ϕ is g, we have

$$\frac{1}{2}c_1d^2[M']=C^2(1-g)\,.$$

Inserting this into (3) we obtain

$$(1-2n)(1-g)\leqq 2\,,$$

from which we infer immediately that

$$g=1\,.$$

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