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A criterion for the existence of a non-trivial partition of a finite group with applications to finite reflection groups

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§ 0. Introduction

The purpose of this paper is to give a criterion for the existence of a non-trivial partition of a finite group G in terms of the existence of a certain permutation representation of G which we have considered in our previous papers [6] and [7]. We shall also give several applications of this criterion.

We refer to Baer [1], [2], [3]; Kegel [8], [9]; Kegel-Wall [10]; Suzuki [11], [13] as for basic concepts and theorems about the partitions of a group.

A group G was called of positive type in [6] if there exist a positive integer k and a G-space M (this means that G acts on M as a transformation group) with the following two properties:

(i) every element σ in $G-\{1\}$ has exactly k fixed points in M, and

(ii) no point in M is fixed by all elements in G. (We have called in [6] such a G-space M to be of type k. We shall also say that G is of type k on the G-space M.)

Now our criterion is stated as follows:

THEOREM 1. *A finite group G has a non-trivial partition if and only if G is of positive type.*

Although the proof of this theorem is quite elementary, it is divided into several steps and will be given in § 1.

In [6, Theorems IV and V], we have tried to distinguish the groups of positive type among Chevalley groups over a finite field. However our result there was not complete. Now by a profound result of Suzuki [13] and by the criterion above, this question is settled immediately. Let us state here Suzuki's theorem in a modified form for the convenience of the reader:

THEOREM 2. *Let G be a finite semi-simple group. (Recall that a finite group is called semi-simple if it has no nilpotent normal subgroups other than the unit group. Thus the semi-simplicity is equivalent to the non-existence of non-trivial abelian normal subgroups.) Then G is of positive type if and only if G is isomorphic with one of the following groups:*

$$PGL(2, q),\ PSL(2, q),\ Sz(q),$$

where $Sz(q)$ means the Suzuki group (denoted by $G(q)$ in [12]) over the finite field F_q (q being a power of 2 with odd exponent, $q \geqq 8$).

In § 2, we shall determine finite reflection groups on a Euclidean space which have non-trivial partitions using our criterion and a result in [6, Theorem VI] characterizing among the Weyl groups of complex simple Lie algebras those which are of positive type. We shall finally determine in § 3 by an elementary method among the alternating groups $\mathfrak{A}_n$ those which have non-trivial partitions, though this is immediate if one utilizes Suzuki's theorem cited above.

We wish to express our thanks to Professor N. Ito for the helpful discussions and suggestions for the content of § 1.

§ 1. The proof of Theorem 1.

Let us recall that a partition of a group G is a collection $\pi = \{H_i\}$ of subgroups H_i ($\neq 1$) of G such that $G - \{1\}$ is a disjoint union of the subsets $H_i - \{1\}$. Each subgroup H_i in the partition π is called a component of π. A partition $\pi = \{H_i\}$ of G is called non-trivial if any subgroup H_i in π is $\neq G$. Let $\pi = \{H_i\}$ be a partition of a group G. Then any automorphism θ of G induces a partition $\theta(\pi) = \{\theta(H_i)\}$ of G. A partition of G is called normal if $\theta(\pi) = \pi$ for every inner automorphism θ of G.

We shall divide the proof of our theorem into several steps. We shall denote by $|A|$ the cardinality of a set A.

LEMMA 1. *Let G be a finite group of positive type. Then G has a non-trivial normal partition.*

PROOF. Let M be a G-space of type k ($k > 0$) (cf. § 0). We shall call a non-empty subset D of M degenerate if there exists an element σ in $G - \{1\}$ such that $D \subset M_\sigma$, where M_σ means the set of fixed points of σ in M. Let Ω be the set of all degenerate subsets of M. Obviously $D \in \Omega$ and $\tau \in G$ imply $\tau D \in \Omega$. Thus G acts on Ω in a natural way. Let Ω_i ($i = 1, 2, \cdots$) be the subset of Ω defined by

$$\Omega_i = \{D \in \Omega\ ;\ |D| = i\}.$$

Obviously each Ω_i is stable under the action of G. Now Ω is a finite set. In fact, the subset P of M defined by

$$P = \bigcup_{\sigma \in G - \{1\}} M_\sigma$$

(we called P the pure part of M in [6]) is a finite subset of $M : |P| \leqq k(|G| - 1)$. Now if $D \in \Omega$, then $D \subset P$ by definition. Thus we get $|\Omega| \leqq 2^{|P|} < \infty$.

Let D be a non-empty subset of M. We denote by G_D the subgroup of G defined by

$$G_D=\{\sigma\in G\,;\,D\subset M_\sigma\}\,.$$

Now we claim that $\pi=\{G_D\,;\,D\in\Omega_k\}$ is a non-trivial normal partition of G.

To begin with, if D_1 and D_2 are two distinct degenerate subsets in Ω_k, then $G_{D_1}\cap G_{D_2}=\{1\}$. In fact, $G_{D_1}\cap G_{D_2}=G_{D_1\cup D_2}$ and $|D_1\cup D_2|>|D_1|=k$ implies that $G_{D_1}\cap G_{D_2}=\{1\}$ since $|M_\sigma|=k$ for every $\sigma\in G-\{1\}$ by our assumption. Let us show next that G is the union of the subgroups G_D, $D\in\Omega_k$. In fact if $\tau\in G-\{1\}$, then M_τ is a degenerate subset consisting of k points. Hence $M_\tau\in\Omega_k$. Then we have $\tau\in G_D$ for $D=M_\tau$.

Thus $\pi=\{G_D\,;\,D\in\Omega_k\}$ is a partition of G. This partition π is non-trivial, i.e. $|\Omega_k|>1$. In fact, if Ω_k consists of a single degenerate subset D of M, then we must have $M_\sigma=D$ for any element σ in $G-\{1\}$. Thus each point in D is fixed by all elements of G. However this is impossible by definition of M.

Finally π is a normal partition. In fact, it is easy to check $\sigma G_D\sigma^{-1}=G_{\sigma D}$. Since Ω_k is G-stable, the partition π is normal and the proof is now complete.

To show the converse of Lemma 1, we begin with the following

LEMMA 2. *Let G be a finite group and H a subgroup of G. Let 1_H be the unit character of H and 1_H^* the character of G induced by 1_H. Suppose that*

$$\sigma H\sigma^{-1}\cap H=H \text{ or } 1$$

for any $\sigma\in G$, i.e. $\sigma H\sigma^{-1}\cap H=1$ for any $\sigma\notin N_G(H)$ (= the normalizer of H in G). Let τ be any element in $G-\{1\}$. Then we have

$$1_H^*(\tau)=\begin{cases}[N_G(H):H] & \text{if } \ \mathfrak{K}_\tau\cap H\neq\phi\,,\\ 0 & \text{if } \ \mathfrak{K}_\tau\cap H=\phi\,,\end{cases}$$

where $\mathfrak{K}_\tau$ is the conjugate class in G containing τ.

PROOF. As is well-known, $1_H^*(\tau)$ is nothing but the number of fixed elements of τ under the action of G on the coset space $G/H=\{\sigma H\,;\,\sigma\in G\}$. Hence $1_H^*(\tau)=0$ if $\mathfrak{K}_\tau\cap H=\phi$. Suppose that $\mathfrak{K}_\tau\cap H\neq\phi$. Since 1_H^* is a class function on G, we may assume that $\tau\in H-\{1\}$. Then $\tau\sigma H=\sigma H\Leftrightarrow\sigma H\sigma^{-1}\cap H\ni\tau$. Therefore, by our assumption on H, we have $\tau\sigma H=\sigma H\Leftrightarrow\sigma\in N_G(H)$. Thus we get $l_H^*(\tau)=[N_G(H):H]$, Q.E.D.

LEMMA 3. *Let G be a finite group with a non-trivial partition. Then G is of positive type.*

PROOF. We may assume that G has a non-trivial normal partition $\pi=\{U_i\}$ (see the proof of [**1**, Satz 4.7] or [**13**, Lemma 1]). Then $\sigma U_i\sigma^{-1}\cap U_i=U_i$ or 1 for any $\sigma\in G$. Let $\pi^*=\{U_1,\cdots,U_n\}$ be a subset of π such that every subgroup U_i in π is conjugate to one and only one subgroup in π^*. Put m_i

$=[N_G(U_i):U_i]$ $(i=1,\cdots,n)$ and let k be the least common multiple of $m_1,\cdots,m_n$.

We claim that G admits a G-space of type k. In fact, put $\dfrac{k}{m_i}=\alpha_i(i=1,\cdots,n)$. Then we have

$$\alpha_1 1_{U_1}^* + \cdots + \alpha_n 1_{U_n}^* = k\cdot 1_G^* + l\cdot 1_{\{1\}}^* \tag{2.1}$$

where $k+l=\sum_i \alpha_i$. (Note that $1_G^*=1_G$ is the unit character of G and $1_{\{1\}}^*$ the character of the regular representation of G.) In fact, we have for any $\tau\in G-\{1\}$,

$$1_{U_i}^*(\tau)=\begin{cases} m_i & \text{if } \ \mathfrak{K}_\tau\cap U_i\neq\phi\,, \\ 0 & \text{if } \ \mathfrak{K}_\tau\cap U_i=\phi\,, \end{cases}$$

by Lemma 2. Thus $\sum_i \alpha_i 1_{U_i}^*(\tau)=k$ for any $\tau\in G-\{1\}$. Now for $\tau=1$, we have $\sum\alpha_i 1_{U_i}^*(1)=\sum\alpha_i[G:U_i]=k+l\,|\,G\,|$, since

$$\begin{aligned} |\,G\,|-1 &= \sum_{U\in\pi}(|\,U\,|-1)=\sum_{i=1}^{n}[G:N_G(U_i)]\ (|\,U_i\,|-1) \\ &= \sum_{i=1}^{n}\Big(\frac{|\,G\,|}{m_i}-[G:N_G(U_i)]\Big) \\ &= \frac{|\,G\,|}{k}\sum_{i=1}^{n}\alpha_i-\sum_{i=1}^{n}\frac{G:U_i]}{m_i} \\ &= |\,G\,|\frac{k+l}{k}-\frac{1}{k}\sum_{i=1}^{n}\alpha_i[G:U_i]\,. \end{aligned}$$

Thus (2.1) was proved. However (2.1) implies the existence of a G-space of type k (cf. [**6**, Theorem II]).

Now the proof of Theorem 1 is complete by Lemmas 1 and 3. Theorem 2 is then nothing but a re-formulation of Suzuki's theorem in [**13**].

§ 2. Finite reflection groups having non-trivial partitions.

Let E be a finite dimensional Euclidean space and G a finite group generated by reflections on E. Such a group G is called a finite reflection group. We refer to Coxeter [4], [5] and Witt [14] as for the basic properties of finite reflection groups and in particular the classification of irreducible finite reflection groups. (Recall that G is called irreducible if the G-module E is irreducible.)

Now let G be a finite reflection group on a Euclidean space E. To begin with let us consider the case where G is irreducible. When G is the Weyl group of a complex simple Lie algebra $\mathfrak{g}$, the question about the existence of a non-trivial partition of G is settled immediately by Theorem 1 and [6, Theorem VI]. Namely, among these Weyl groups, only the following 5 groups are of positive type: symmetric groups $\mathfrak{S}_3$, $\mathfrak{S}_4$, $\mathfrak{S}_5$; dihedral groups $\mathfrak{D}_4$, $\mathfrak{D}_6$ (of orders 8, 12 respectively).

Thus we have only to examine the remaining irreducible groups in the Coxeter's classification; they are as follows (we use the Coxeter's graph to represent these groups):

(a) $\circ\underset{r}{-}\circ$ G is then the dihedral group $\mathfrak{D}_r$ of order $2r$. In this case G is of positive type. In fact G admits a G-space of type 2 by [**6**, Theorem III].

(b) $\circ\underset{3}{-}\circ\underset{5}{-}\circ$ In this case G is of order 120 and is isomorphic with the direct product $\mathfrak{A}_5\times \boldsymbol{Z}_2$, where $\mathfrak{A}_5$ is the alternating group on 5 letters and $\boldsymbol{Z}_2$ is the cyclic group of order 2. Since $\mathfrak{A}_5$ is generated by elements $\sigma_1, \sigma_2, \cdots$ of order 3, it is easy to see that G is generated by elements $\sigma_1\tau, \sigma_2\tau, \cdots$ of of order 6, where τ is the generator of $\boldsymbol{Z}_2$. Let C_i be the cyclic subgroup of G generated by $\sigma_i\tau$ $(i=1, 2, \cdots)$. Then $C_i\cap C_j\ni\tau\neq 1$ for each pair i, j. Therefore G is of type zero by [**6**, Lemma 1.2], i.e. G has no non-trival partition.

(c) $\circ\underset{3}{-}\circ\underset{3}{-}\circ\underset{5}{-}\circ$ In this case G is of order 14400. G is generated by 4 elements a, b, c, d together with the following defining relations:

$$\begin{cases} a^2=b^2=c^2=d^2=1\,, \\ (ab)^3=(bc)^3=1\,, \\ (cd)^5=1\,, \\ ac=ca,\ ad=da,\ bd=db\,. \end{cases}$$

Let G_1 be the subgroup of G generated by ab and d. Then G_1 is a cyclic group of order 6. Next let G_2 be the subgroup of G generated by b, c and d. Then G_2 is isomorphic with the group $\mathfrak{A}_5\times\boldsymbol{Z}_2$ considered above in the case (b). Hence G_2 is of type zero. Finally let G_3 be the subgroup of G generated by a and cd. Then G_3 is a cyclic group of order 10. Now it is easy to see that

$$G_1\cap G_2\ni d,\quad G_2\cap G_3\ni cd\,.$$

Moreover, G_1, G_2 and G_3 obviously generate the whole group G. Therefore G is of type zero again by [**6**, Lemma 1.2]. Thus G has no non-trivial partition.

We have thus proved the following generalization of [**6**, Theorem VI].

THEOREM 3. *Let G be an irreducible, finite reflection group on a Euclidean space. Then G has a non-trivial partition if and only if G is isomorphic with one of the following groups.*

(i) $\mathfrak{S}_4$, (ii) $\mathfrak{S}_5$, (iii) $\mathfrak{D}_r$ (*the dihedral group of order* $2r$).

Let us consider now the case where G is a reducible, finite reflection group on a Euclidean space E.

Then one can decompose E into a direct sum of G-stable subspaces $E_1, \cdots, E_r$ which are mutually orthogonal. G is then decomposed into a direct product of subgroups $G_1, \cdots, G_r$ defined by

$$G_i = \{\sigma \in G\,;\ \sigma \mid E_j = \text{identity for all } 1 \leqq j \leqq r \text{ with } j \neq i\}\,.$$

It is well-known that each G_i is an irreducible finite reflection group on E_i. Now when we consider the question of the existence of a non-trivial partition of G, we may assume that $G_i \neq 1$ for all i.

Suppose now that G has a non-trivial partition. Then there exists a G-space M of type k for some $k > 0$ (Theorem 1).

If every element in $G_1 - \{1\}$ is of order 2, then G_1 is abelian and, as is seen easily, E_1 must be one-dimensional and G_1 is of order 2.

Let us assume now that G_1 contains an element σ of order > 2. Put $\bar{G}_2 = G_2 \times \cdots \times G_r$. Then $\bar{G}_2$ is a finite reflection group on $E_2 + \cdots + E_r$.

Then for any reflection τ in $\bar{G}_2$ we have $M_\sigma = M_\tau$ by [**6**, Lemma 1.3]. Therefore any point in M_σ is fixed by all elements in $\bar{G}_2$, since $\bar{G}_2$ is generated by reflections. Hence M_σ must coincide with the set $M_{\bar{G}_2} = \{x \in M\,;\ \rho x = x$ for all $\rho \in \bar{G}_2\}$, because $k = |M_\sigma| \leqq |M_{\bar{G}_2}| \leqq k$. Let Γ be the subgroup of G_1 generated by elements of order > 2 in G_1. Then Γ is a normal subgroup of G_1 and we have seen above that $M_\sigma = M_{\bar{G}_2}$ for every σ in Γ. Now if $G_1 = \Gamma$, then any point in $M_{\bar{G}_2}$ is fixed by all elements of G, which is however impossible. Thus we must have $G_1 \neq \Gamma$.

Now we claim that G_1 is of positive type. To show this, it is enough to prove that the set M_{G_1} of G_1-fixed points has less than k points (see [**7**, Lemma 1.1]). Suppose for a moment that $|M_{G_1}| = k$. Then, since $M_{G_1} \subset M_\Gamma = M_{\bar{G}_2}$, $|M_\Gamma| = k$, we have $M_{G_1} = M_{\bar{G}_2}$. Thus $M_G = M_{G_1} = M_{\bar{G}_2}$ is not empty, which is impossible. Thus $|M_{G_1}| < k$ and G_1 is of positive type.

Thus we can apply Theorem 3 to our group G_1. Since the symmetric group $\mathfrak{S}_4$ or $\mathfrak{S}_5$ is generated by elements of order > 2, $G_1 \neq \Gamma$ implies that G_1 is isomorphic with a dihedral group $\mathfrak{D}_s$ of order $2s$, where $s > 2$.

We have proved thus the following

LEMMA 4. *Let G be a reducible, finite reflection group on a Euclidean space E and $G = G_1 \times \cdots \times G_r$ be the decomposition of G into irreducible components $G_1, \cdots, G_r$. Assume that G has a non-trivial partition. Then each G_i is isomorphic with $\boldsymbol{Z}_2$ or $\mathfrak{D}_s$ for some $s > 2$.*

Let us consider to begin with the case where

$$G_1 \cong G_2 \cong \cdots \cong G_r \cong \boldsymbol{Z}_2$$

Then G is an elementary abelian group; hence G is of positive type by [**6**, Theorem I].

To settle the case where some G_i is of dihedral type, we prove now the following

LEMMA 5. *Let G be a finite group isomorphic with $\mathfrak{D}_s\times \boldsymbol{Z}_2\times\cdots\times\boldsymbol{Z}_2$. Then G is of positive type.*

PROOF. Since $\mathfrak{D}_s$ contains a normal cyclic subgroup C of order s and an involution σ such that $\mathfrak{D}_s=C\cup C\sigma$, $\sigma\xi\sigma^{-1}=\xi^{-1}$ (for all $\xi\in C$), it is easy to see that there exists an abelian normal subgroup Γ of G such that

$$[G:\Gamma]=2,\ G=\Gamma\cup\Gamma\sigma,\ \sigma\xi\sigma^{-1}=\xi^{-1}\ (\text{for all } \xi\in\Gamma).$$

Then every element of $G-\Gamma$ is of order 2. Let π be the family of subgroups of G consisting of Γ and of all cyclic subgroups C_i of order 2 with $C_i\cap\Gamma=1$. Then obviously π is a normal non-trivial partition of G, Q. E. D.

LEMMA 6. *Let $G=\mathfrak{D}_s\times\mathfrak{D}_t$ with $s>2$ and $t>2$. Then G is of type zero.*

PROOF. G is generated by a, b, c, d together with the following defining relations:

$$\begin{cases} a^2=b^2=c^2=d^2=1\,, \\ (ab)^s=1,\ (cd)^t=1\,, \\ ac=ca,\ ad=da,\ bc=cb,\ bd=db\,. \end{cases}$$

Since ab is of order $s>2$, the abelian subgroup G_1 of G generated by ab and c is of type zero ([**6**, Theorem 1]). Similarly the abelian subgroup G_2 of G generated by ab and d is of type zero.

Moreover, since $G_1\cap G_2\ni ab\neq 1$, the subgroup G_3 of G generated by G_1 and G_2 is of type zero by [**6**, Lemma 1.2]. G_3 is generated by ab, c and d. In the same manner, the subgroup G_4 of G generated by a, b and cd is also of type zero. Moreover we have $G_3\cap G_4\ni ab\neq 1$ and G is generated by G_3 and G_4. Thus G is of type zero, Q. E. D.

By a similar argument as in the proof of Lemma 6, we get the following

LEMMA 7. *Let A be any finite group. Then, if $s>2$ and $t>2$, $G=\mathfrak{D}_s\times\mathfrak{D}_t\times A$ is of type zero.*

PROOF. Taking the generators a, b, c, d of $\mathfrak{D}_s\times\mathfrak{D}_t$ as in the proof of Lemma 6, we have now only to consider the abelian subgroups generated by, say, ab, c and an element σ in A, Q. E. D.

Combining Lemmas 4, 5, 6 and 7, we have proved the following

THEOREM 4. *Let G be a reducible, finite reflection group on a Euclidean space E such that no non-zero vector in E is fixed by all elements in G. Then G has a non-trivial partition if and only if G is isomorphic with one of the following groups:*

$$\boldsymbol{Z}_2 \times \cdots \times \boldsymbol{Z}_2 \ or \ \mathfrak{D}_s \times \boldsymbol{Z}_2 \times \cdots \times \boldsymbol{Z}_2 \ (s > 2).$$

§ 3. Alternating groups having non-trivial partitions.

Our purpose here is to determine the alternating group $\mathfrak{A}_n$ having non-trivial partitions.

LEMMA 8. *$\mathfrak{A}_7$ is of type zero.*

PROOF. Put $a = (12)(34)$. Then the centralizer $C(a)$ of a in $\mathfrak{A}_7$ is of type zero. In fact, it is easy to see that $C(a)$ is generated by $\sigma = (13)(24)(567)$ and $\tau = (12)(56)$. Moreover $|C(a)| = 24$. Let us denote by $\langle \xi \rangle$ the cyclic subgroup generated by ξ. Then we have $\langle \sigma \rangle \cap \langle \sigma^2 a \rangle \ni \sigma^4 \neq 1$ and $\langle \sigma\tau \rangle \cap \langle \sigma^2 a \rangle \ni a \neq 1$ since $(\sigma\tau)^2 = a$. Therefore the subgroup $C(a)$ generated by $\langle \sigma \rangle$, $\langle \sigma^2 a \rangle$, $\langle \sigma\tau \rangle$ is of type zero by [**6**, Lemma 1.2]. Similarly we see that the centralizers $C(\tau)$, $C(b)$ and $C(c)$, where $b = (12)(67)$, $c = (45)(67)$ are all of type zero. Furthermore, we have $C(a) \cap C(\tau) \ni \tau \neq 1$, $C(a) \cap C(b) \ni b \neq 1$, $C(b) \cap C(c) \ni b \neq 1$. Hence the subgroup Γ of $\mathfrak{A}_7$ generated by $C(\tau)$, $C(a)$, $C(b)$ and $C(c)$ is of type zero. Now Γ coincides with $\mathfrak{A}_7$, because Γ contains (12)(23), (12)(34), (12)(45), (12)(56), (12)(67), Q. E. D.

LEMMA 9. *$\mathfrak{A}_n$ $(n \geqq 7)$ are all of type zero.*

PROOF. It is easy to see that $\mathfrak{A}_8$ is generated by two subgroups X and Y such that $X \cong Y \cong \mathfrak{A}_7$, $X \cap Y \neq 1$. Hence $\mathfrak{A}_8$ is of typo zero. Similarly we see that $\mathfrak{A}_n$ $(n \geqq 7)$ are all of type zero by induction, Q. E. D.

Now $\mathfrak{A}_4$, $\mathfrak{A}_5$ are of positive type ([**6**, Theorem III]). $\mathfrak{A}_6 \cong PSL(2, 3^2)$ is also of positive type ([**6**, Theorem IV]). $\mathfrak{A}_2 = 1$, $\mathfrak{A}_3 \cong Z_3$ are both of type zero. Thus we have proved the following

THEOREM 5. *The alternating $\mathfrak{A}_n$ has a non-trivial partition if and only if $n = 4$ or 5 or 6.*

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