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An extension theorem on valuations

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In this paper, we shall prove

THEOREM. *Let* K_0 *be a field,* v_0 *a valuation on* K_0 *with the value group* Γ_0 *and the residue field* Δ_0. *Let* Γ_1 *be a linearly ordered abelian group containing* Γ_0, Δ_1 *a field containing* Δ_0. *Then* v_0 *can be extended to a valuation* v_1 *on some field* K_1 *containing* K_0 *with the value group* Γ_1 *and the residue field* Δ_1.

All fields considered are commutative. By a valuation v on a field K with the value group Γ (which is an additively written linearly ordered abelian group), we mean as usual a map of K onto $\Gamma \cup \{\infty\}$ with the properties: $v(xy) = v(x)+v(y)$, $v(x+y) \geqq \min(v(x), v(y))$, for any $x, y \in K$, (cf. e.g. Schilling [**1**], Zariski [2], Bourbaki [**3**]).

It suffices to prove the theorem in two cases:

$$(1) \qquad \Gamma_1 \supset \Gamma_0, \qquad \Delta_1 = \Delta_0$$

and

$$(2) \qquad \Gamma_1 = \Gamma_0, \qquad \Delta_1 \supset \Delta_0 .$$

We denote the valuation ring of v_0 by R_0, and the maximal ideal of R_0 by $\mathfrak{m}_0$. The same notations will be used for other valuations.

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PROOF. Case (1). (i) Assume first that Γ_1 is generated by Γ_0 and one element θ, where θ is free modulo Γ_0. Let $K_1 = K_0(t)$, where t is transcendental over K_0. We shall note that for any two monomials at^m, $bt^n \in K_0[t]$, we have $v_0(a)+m\theta = v_0(b)+n\theta$ only if $m=n$ and $v_0(a)=v_0(b)$, for if $m \neq n$, we would have $(m-n)\theta \in \Gamma_0$, contradicting the hypothesis that θ is free modulo Γ_0.

For any polynomial $F(t) = \sum_{i=0}^{n} a_i t^i$ in $K_0[t]$, define

$$v_1(F(t)) = \min_{0 \leqq i \leqq n} (v_0(a_i)+i\theta) .$$

In view of the above remark, we can easily verify the following relations for any $F(t)$, $G(t) \in K_0[t]$:

$$v_1(F(t)+G(t)) \geqq \min(v_1(F(t)), v_1(G(t))),$$
$$v_1(F(t)G(t)) = v_1(F(t)) + v_1(G(t)).$$

Thus v_1 defines a valuation on K_1, which has the value group Γ_1.

Let $x = \sum_{i=0}^{m} a_i t^i / \sum_{j=0}^{n} b_j t^j$ be any element of K_1 with $v_1(x) = 0$. Then there exists an index ν such that

$$v_1(a_\nu t^\nu) = v_1(b_\nu t^\nu),$$
$$v_1(a_i t^i) > v_1(a_\nu t^\nu) \quad \text{for } i \neq \nu,$$
$$v_1(b_j t^j) > v_1(b_\nu t^\nu) \quad \text{for } j \neq \nu.$$

Thus

$$x = \sum_{i=0}^{m} \frac{a_i}{b_\nu} t^{i-\nu} / \sum_{j=0}^{n} \frac{b_j}{b_\nu} t^{j-\nu} \equiv \frac{a_\nu}{b_\nu} \in R_0 \quad (\mathrm{mod}\ \mathfrak{m}_1),$$

so the residue field of v_1 is Δ_0.

(ii) Next assume that Γ_1 is generated by Γ_0 and one element θ, where θ is a torsion mod Γ_0. Let n be the minimum positive integer such that $n\theta \in \Gamma_0$ holds. Let $\tilde{\Gamma}_0$ be the rational completion of $\Gamma_0 : \Gamma_0 \otimes_Z Q$ considered as an ordered group in the canonical way. Then Γ_1 can be imbedded in $\tilde{\Gamma}_0$ in the unique way. Take $a \in K_0$ with $v_0(a) = n\theta$. Take a root t of $X^n - a$, and extend v_0 in any way to a valuation v_1 on $K_1 = K_0(t)$. This is a finite algebraic extension, and so the value group Γ_1' can be imbedded in $\tilde{\Gamma}_0$.

If $X^n - a$ were reducible in $K_0[X]$, we would have the relation of the type

$$t^m + \sum_{i=0}^{m-1} a_i t^i = 0, \quad a_i \in K_0, \quad 1 \leqq m < n.$$

Then $v_1(a_i t^i) = v_1(a_j t^j)$ for some $i > j$, which leads to $(i-j)\theta \in \Gamma_0$, where $1 \leqq i - j \leqq m < n$, contradicting the hypothesis.

Thus $[K_1 : K_0] = n$. On the other hand, $v_1(t^n) = v_1(a) = n\theta$ shows $v_1(t) = \theta$, so $\Gamma_1' \supset \Gamma_1$. Therefore $n \geqq [\Gamma_1' : \Gamma_0] \geqq [\Gamma_1 : \Gamma_0] = n$, and so $\Gamma_1' = \Gamma_1$, and the well-known inequality " $\sum_{i=1}^{g} e_i f_i \leqq n$ " of the ramification theory shows that the residue field of v_1 is Δ_0.

REMARK 1. The same inequality also shows that v_1 is actually the only extension of v_0 to $K_1 = K_0(t)$.

REMARK 2. The direct construction of v_1 is described as follows:

$$v_1\left(\sum_{i=0}^{n-1} a_i t^i\right) = \min_{0 \leqq i \leqq n-1} (v_0(a_i) + i\theta) \quad \text{for } a_i \in K_0.$$

We can also verify directly that this v_1 has the required properties without using the above inequality.

(iii) The above discussion proves our theorem in case Γ_1 is finitely generated over Γ_0, and $\Delta_1=\Delta_0$. We shall proceed to the proof of the general case (still assuming $\Delta_1=\Delta_0$) by help of Zorn's lemma.

Let $\{\Gamma_\lambda \mid \lambda\in\Lambda\}$ be the set of all subgroups of Γ_1 containing Γ_0, and define the order in the indexing set Λ by $\lambda\geqq\mu\Leftrightarrow\Gamma_\lambda\supset\Gamma_\mu$. Then the index 0 is just the minimum element of Λ.

We consider the index 0 as an element of K_0, and other λ's as independent variables over K_0. Let Ω be the algebraic closure of the field obtained by adjoining all λ's to K_0.

Let $\mathfrak{X}$ be the set of all pairs (K, v), satisfying the following conditions.

1°. K is an intermediate field between K_0 and Ω.

2°. v is a valuation on K, extending v_0.

3°. The value group of v is some Γ_λ.

4°. The residue field of v is Δ_0.

5°. K is contained in the algebraic closure of the field obtained by adjoining to K_0 all μ's in Λ such that $\mu\leqq\lambda$ hold.

$\mathfrak{X}$ contains (K_0, v_0), so $\mathfrak{X}$ is not empty. The order in $\mathfrak{X}$ defined by

$$(K, v)\leqq(K', v') \Leftrightarrow K'\supset K,\quad v'\mid K=v$$

makes $\mathfrak{X}$ an inductive set. Any maximal element (K_1, v_1) in $\mathfrak{X}$, which exists by Zorn's lemma, has the value group Γ_1, since otherwise we could find some (K_2, v_2) in $\mathfrak{X}$ strictly greater than (K_1, v_1) by virtue of (i) and (ii).

Case (2).

(i) Assume $\Delta_1=\Delta_0(\xi)$, where ξ is transcendental over Δ_0. Let $K_1=K_0(t)$, where t is transcendental over K_0.

The canonical homomorphism $\pi_0: R_0\to R_0/\mathfrak{m}_0$ can be extended to the surjective homomorphism $R_0[t]\to\Delta_0[\xi]$ by $\pi_0(t)=\xi$, the kernel $\mathfrak{p}$ being $\mathfrak{m}_0[t]$. Since any element of K_1 can be denoted by $F(t)/G(t)$, where $F(t), G(t)\in R_0[t]$ and either $F(t)\notin\mathfrak{m}_0[t]$ or $G(t)\notin\mathfrak{m}_0[t]$, $R_1=R_0[t]_{\mathfrak{p}}$ is a valuation ring of K_1, and the residue field is $\Delta_0(\xi)=\Delta_1$.

Let v_1 be the valuation associated to R_1. It is an extension of v_0 by the construction. Any element $F(t)$ in $K_0[t]$ can be denoted as $F(t)=a\sum b_i t^i$, with $a, b_i\in K_0$, $v_0(b_i)\geqq 0$, with some $b_\nu=1$. Since $\sum b_i t^i$ mod $\mathfrak{m}_1$ is not 0, $\sum b_i t^i$ is a unit in R_1, and so we have $v_1(F(t))=v_1(a)\in\Gamma_0$, which proves that the value group of v_1 is Γ_0.

(ii) Next assume $\Delta_1=\Delta_0(\xi)$, where ξ is algebraic over Δ_0. Let $\bar{F}(X)$ be the monic irreducible polynomial in $\Delta_0[X]$ satisfied by ξ, and let $F(X)$ be a monic polynomial in $R_0[X]$ such that $\pi_0(F(X))=\bar{F}(X)$.

It is well-known that

$$v(\sum a_i X^i)=\min_i v_0(a_i)$$

defines a valuation on $K_0(X)$. If we have a decomposition

$$F(X)=F_1(X)F_2(X)$$

into monic factors in $K_0[X]$, we have, applying the above valuation v,

$$0=v(F(X))=v(F_1(X))+v(F_2(X)), \qquad v(F_i(X))\leqq 0,$$

which leads to $v(F_i(X))=0$, and so

$$F_i(X)\in R_0[X], \qquad \bar{F}(X)=\bar{F}_1(X)\bar{F}_2(X), \qquad \text{degree } \bar{F}_i(X)=\text{degree } F_i(X).$$

Since $\bar{F}(X)$ is irreducible in $\Delta_0[X]$, one of $F_i(X)$ must be of degree 0, which shows that $F(X)$ is irreducible in $K_0[X]$.

Let t be a root of $F(X)$, and set $K_1=K_0(t)$. v_0 can be extended to a valuation v_1 on K_1. Then the residue field of v_1 is obviously isomorphic to $\Delta_1=\Delta_0(\xi)$. Since $[\Delta_1:\Delta_0]=[K_1:K_0]$, the inequality of the ramification theory shows that v_1 has the value group Γ_0.

REMARK 3. v_1 is the unique extension of v_0 on K_1.

REMARK 4. The valuation ring of v_1 is $R_0[t]_{\mathfrak{p}}$, where $\mathfrak{p}=\mathfrak{m}_0+\mathfrak{m}_0t+\cdots+\mathfrak{m}_0t^{n-1}$ ($n=[K_1:K_0]=$ degree of $F(X)$). We could proceed as in (i) without using the ramification theory.

(iii) The same idea as in case 1 (iii) proves our theorem in case Δ_1 is any extension of Δ_0 and $\Gamma_1=\Gamma_0$. Thus the proof of our theorem is completed.

We have

COROLLARY. *Let Γ be an arbitrary non-trivial linearly ordered abelian group, and Δ be an arbitrary field. Then there exists a field K and a valuation v on K, which has Γ as the value group and Δ as the residue field.*

Moreover, if the characteristic of Δ is $p\neq 0$, we can preassign the characteristic of K as 0 or as p.

PROOF. It is enough to extend the trivial valuation on the prime field of Δ, or the p-adic valuation on the field of rational numbers.

NOTE. We have another proof of the corollary in equicharacteristic case as follows: Let K be the set of all the maps x of Γ to Δ whose supports are well-ordered, where the support of x means the set $\{\gamma\in\Gamma\mid x(\gamma)\neq 0\}$. We define $x+y$ and xy by

$$(x+y)(\gamma)=x(\gamma)+y(\gamma)$$
$$(xy)(\gamma)=\sum_{\alpha+\beta=\gamma}x(\alpha)y(\beta),$$

which are shown to be well-defined, and make K a field. Then $v(x)=\min\operatorname{supp}(x)$ defines a valuation on K satisfying the required conditions. For the details, cf. Neumann [4]. This field has obviously the same characteristic as Δ, and has complete uniform structure.

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