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On meromorphic and circumferentially mean univalent functions

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Introduction.

It is well known that the so-called one-quarter theorem plays an important role in the theory of regular and univalent functions in $|z|<1$. This theorem was extended to the case of circumferentially mean univalence (defined in § 1) by Hayman [6] and moreover to the case of areally mean univalence by Garabedian and Royden [5]. Their method was based on the fact that inner radius does not decrease by circular symmetrization (cf. [7]). On the other hand, corresponding to the one-quarter theorem, the following Montel-Bieberbach's theorem ([2], [3], [13], [14]) is well known in the case of meromorphic and univalent functions.

If $f(z)=z+a_2z^2+\cdots$ *is meromorphic and univalent in* $|z|<1$, *then at least one of the circles* $|w|<\delta$ *or* $|w|>\delta^{-1}$ $(\delta=\sqrt{5}-2)$ *is wholly covered by the image-domain under* $w=f(z)$.

In this paper we shall first prove a fundamental theorem on meromorphic and circumferentially mean univalent functions in $|z|<1$, by means of the fact that transfinite diameter does not increase by circular symmetrization and then generalized Montel-Bieberbach's theorem to the case of circumferentially mean univalence or p-valence.

Secondly we shall deal with values omitted by meromorphic and circumferentially mean univalent functions in $|z|<1$ also by means of the above mentioned property of transfinite diameter.

Thirdly we consider meromorphic and circumferentially mean univalent functions in $|z|<1$, whose Taylor expansions about the origin are given by $f(z)=z+a_2z^2+\cdots$ and whose poles are explicitly denoted by $z=z_\infty$, (as will be remarked in § 1, $f(z)$ has only one simple pole in $|z|<1$). By means of the pole $z=z_\infty$ we shall evaluate the values taken by $w=f(z)$ and its second Taylor coefficient a_2. Moreover a type of distortion theorem based on the pole $z=z_\infty$ will be derived.

§ 1. Preliminary.

Let $w=Re^{i\Phi}=f(z)$ be regular or meromorphic in $|z|<1$, and let $n(R, \Phi)$ denote the number of the roots of the equation $Re^{i\Phi}=f(z)$ in $|z|<1$.

If for a positive number p

$$\frac{1}{2\pi}\int_0^{2\pi} n(R, \Phi)d\Phi \leqq p \qquad (R>0),$$

then $f(z)$ is called "*circumferentially mean p-valent in* $|z|<1$". (Biernacki [**4**])

If $p=1$, $f(z)$ is also called "*circumferentially mean univalent in* $|z|<1$".

Let $f(z)=z+a_2z^2+\cdots$ be meromorphic and circumferentially mean univalent in $|z|<1$. These functions will be denoted by $\mathfrak{F}_1$. It is easily seen by the definition that any $f(z)\in\mathfrak{F}_1$ has at most only one simple pole in $|z|<1$.

Let $f(z)=z^p+a_{p+1}z^{p+1}+\cdots$ be circumferentially mean p-valent and regular except for a pole of order p in $|z|<1$. These functions will be denoted by $\mathfrak{F}_p$, which is a natural generalization of $\mathfrak{F}_1$.

Now we shall state the following lemma showing a closed relation between $\mathfrak{F}_1$ and $\mathfrak{F}_p$.

LEMMA 1. *Let $f(z)\in\mathfrak{F}_p$. Then*

$$(f(z))^{1/p}=z+\frac{a_{p+1}}{p}z^2+\cdots$$

belongs to $\mathfrak{F}_1$.

PROOF. It is clear by the definition that $(f(z))^{1/p}=z+\cdots$ is regular except for a simple pole in $|z|<1$. By means of the same method as in the case of regular functions by Hayman ([**6**] or [**7**]), we can prove

$$\int_0^{2\pi} n(R, \Phi)d\Phi = p\int_0^{2\pi} n(\rho, \varphi)d\varphi$$

and therefore

$$\frac{1}{2\pi}\int_0^{2\pi} n(\rho, \varphi)d\varphi \leqq 1,$$

where $(f(z))^{1/p}=\rho e^{i\varphi}$. Therefore we see $(f(z))^{1/p}\in\mathfrak{F}_1$.

§ 2. Values taken by $\mathfrak{F}_1$ or $\mathfrak{F}_p$.

We shall first quote the following Hayman's result [**8**].

LEMMA 2. *Let $f(z)=1/z+a_0+a_1z+\cdots$ be meromorphic in $|z|<1$, and let τ_f denote the transfinite diameter of the complement E_f of the image-domain under $w=f(z)$. Then*

$$\tau_f \leqq 1.$$

Equality holds only when $f(z)$ is univalent.

Next we shall state the following lemma which is nothing but an application of Pólya-Szegö's result [**15**]. The proof can be easily given by means of Pólya-Szegö's idea (cf. [7], 81-83).

LEMMA 3. *Let E_f and τ_f be defined in Lemma 2. Moreover let E_f^* be the circular symmetrization of E_f with respect to the positive real axis and τ_f^* be the transfinite diameter of E_f^*. Then we have*

$$\tau_f \geqq \tau_f^* .$$

Here we shall state Darboux's theorem in a slightly generalized form.

LEMMA 4. *Let D be a simply connected domain enclosed by a rectifiable Jordan curve C. Let $f(z)$ be regular in the closed domain $\bar{D}=D+C$, or $f(z)$ be regular there except for a simple pole in D. Moreover if C is mapped univalently on a Jordan curve Γ by $w=f(z)$, then D is also univalently mapped into the interior or exterior domain with respect to Γ respectively.*

Now we shall prove the following fundamental theorem useful for the generalization of Montel-Bieberbach's theorem.

THEOREM 1. *Let $f(z)=1/z+a_0+a_1z+\cdots$ be meromorphic and circumferentially mean univalent in $|z|<1$. If we put $M=\max|w_c|$, $m=\min|w_c|$, where w_c denotes any point belonging to the complement E_f of the image-domain D_f under $w=f(z)$. Then*

$$M-m\leqq 4 .$$

Equality holds only when $f(z)=1/z+a_0+e^{i\varepsilon}z$ ($\varepsilon=2\arg a_0$, $|a_0|\geqq 2$).

PROOF. We make the circular symmetrization of the complement E_f of the image-domain D_f, with respect to the positive real axis. The intersection of the symmetrized set E_f^* and the positive real axis is denoted by S. Then S is contained in the closed interval $[m, M]$. Now we prove that S is truely the interval $[m, M]$. Suppose that $m\leqq r\leqq M$ and $r\notin S$. Then the circle $|w|=r$ must be wholly contained in D_f. Since $f(z)$ is circumferentially mean univalent in $|z|<1$, the circle $|w|=r$ is univalently covered by D_f, that is, a Jordan curve C in the z-plane is univalently mapped onto the circle $|w|=r$. On the other hand by the reason of circumferentially mean univalence in $|z|<1$, $f(z)$ has only one simple pole at $z=0$. Now we denote by D the domain enclosed by C and consider the following two cases:

(i) if D contains the simple pole $z=0$, then by means of Lemma 4 D is univalently mapped to the circle $|w|>r$. If it is so, the closed annulus $r\leqq|w|\leqq M$ is wholly contained in D_f. This is incompatible with the definition of M.

(ii) if D does not contain the pole $z=0$, then we see similarly by means of Lemma 4 that the closed annulus, $m\leqq|w|\leqq r$ is wholly contained in D_f. This is also absurd.

Therefore we see that $S=[m, M]$. Hence we have by the well known result on transfinite diameter (cf. Tsuji [**17**, p. 84]), $\tau(S)=(M-m)/4$ where $\tau(S)$ denotes the transfinite diameter of S.

On the other hand by Lemma 2 and Lemma 3 we have

$$1 \geqq \tau(E_f) \geqq \tau(E_f^*)$$

where $\tau(E_f)$ and $\tau(E_f^*)$ respectively denote the transfinite diameters of E_f and E_f^*. Since $E_f^* \supseteqq S$, we have also $\tau(E_f^*) \geqq \tau(S)$. Therefore we see the following inequality.

$$M-m \leqq 4 .$$

According to Lemma 2, equality holds only when $f(z)=1/z+a_0+a_1z+\cdots$ is univalent in $|z|<1$. $f(z)=1/z+a_0+e^{i\varepsilon}z$ ($\varepsilon=2\arg a_0$, $|a_0|\geqq 2$) maps the unit circle $|z|<1$ univalently onto the w-plane cut by a segment of length 4. On the other hand, the equality sign in the one-quarter theorem is attained only by the Koebe function $f(z)=z/(1-e^{i\varepsilon}z)^2$ (ε real). Therefore we see that the equality sign in Theorem 1 is attained only by the function $f(z)=1/z+a_0+e^{i\varepsilon}z$ ($\varepsilon=2\arg a_0$, $|a_0|\geqq 2$). This completes the proof.

COROLLARY 1. *Let $f(z)=1/z+a_0+a_1z+\cdots$ be meromorphic and circumferentially mean univalent in $|z|<1$. Then the image-domain under $w=f(z)$ covers wholly and univalently at least one of the circles $|w|<\delta$ or $|w|>\delta^{-1}$ ($\delta=\sqrt{5}-2$). This result is best possible as is shown by*

$$f(z)=\frac{1}{z}+\sqrt{5}\,e^{i\varepsilon}+e^{i2\varepsilon}z \quad (\varepsilon \text{ real}).$$

PROOF. Considering the relation $\delta^{-1}-\delta=4$ ($\delta=\sqrt{5}-2$) it is easily seen by means of Theorem 1 that the circles $|w|<\delta$ or $|w|>\delta^{-1}$ are wholly covered by the image-domain. The univalency of the covering of these circles by the image-domain is seen similarly as in the proof of Theorem 1. Here the proof is completed.

From Corollary 1 we can extend Montel-Bieberbach's theorem to the case of circumferentially mean univalence as follows.

THEOREM 2. *Let $f(z)\in\mathfrak{F}_1$. Then the image-domain under $w=f(z)$ covers wholly and univalently at least one of the circles, $|w|<\delta$ or $|w|>\delta^{-1}$ ($\delta=\sqrt{5}-2$). This result is best possible as is shown by*

$$f(z)=\frac{z}{1+\sqrt{5}\,e^{i\varepsilon}z+e^{i2\varepsilon}z^2} \quad (\varepsilon \text{ real}).$$

PROOF. Since $g(z)=1/f(z)$ satisfies the same conditions as in Corollary 1, we can apply Theorem 1 for $g(z)$. This completes the proof.

THEOREM 3. *Let $f(z)\in\mathfrak{F}_p$. Then the image-domain under $w=f(z)$ covers exactly p times at least one of the circles $|w|<\delta^p$ or $|w|>\delta^{-p}$ ($\delta=\sqrt{5}-2$).*

This result is best possible as is shown by

$$f(z)=\frac{z^p}{(1+\sqrt{5}\,e^{i\varepsilon}z+e^{i2\varepsilon}z^2)^p}\quad (\varepsilon \text{ real}).$$

PROOF. We put

$$g(z)=(f(z))^{1/p}=z+\frac{a_{p+1}}{p}z^2+\cdots.$$

Then since $g(z)\in\mathfrak{F}_1$ by means of Lemma 1, we see that Theorem 2 holds for $g(z)$. Therefore we have Theorem 3.

REMARK. Generalization of Montel-Bieberbach's theorem to the case of p-valent functions was done by the author [1].

§ 3. Values omitted by $\mathfrak{F}_1$ or the related functions.

By means of symmetrization and inner radius Jenkins [9] has dealt with values omitted by regular and univalent functions in $|z|<1$. Here we shall first study the related problem on meromorphic functions in $|z|<1$, by means of transfinite diameter and symmetrization similarly as in § 2. Next we shall remark that we can deal more precisely with the same problem on meromorphic and circumferentially mean univalent functions in $|z|<1$.

We consider a family of meromorphic functions $f(z)=1/z+a_0+a_1z+\cdots$ in $|z|<1$. Let E_f be the complement of the image-domain under each of these functions. Among these functions there exists such a function that the circle $|w|=R$ $(R\leqq 1)$ is wholly contained in E_f. For example $f(z)=1/z$. Now, considering this fact, we shall state the following theorem.

THEOREM. 4. *Let $f(z)=1/z+a_0+a_1z+\cdots$ be meromorphic in $|z|<1$. Let the intersection of the complement E_f of the image-domain under $w=f(z)$ and the circle $|w|=R$ be denoted by S_R whose angular measure with respect to the origin is denoted by $\theta(S_R)$. Then*

$$\theta(S_R)\leqq 4\sin^{-1}(R^{-1})\quad (R>1).$$

This result is best possible as is shown by

$$f(z)=\frac{R(1-Rz)}{z(R-z)}\quad (R>1).$$

PROOF. Let E_f^* be denoted by the circular symmetrization of E_f with respect to the positive real axis. Moreover let S_R^* be the intersection of E_f^* and the circle $|w|=R$. Then we see

$$\theta(S_R)=\theta(S_R^*)$$

where $\theta(S_R^*)$ denotes the angular measure of the single arc S_R^* with respect to the origin.

Quite similarly as in the proof of Theorem 1, we have by Lemma 2 and Lemma 3

$$1 \geqq \tau(E_f) \geqq \tau(E_f^*).$$

Since $\tau(E_f^*) \geqq \tau(S_R^*)$, we have

$$\tau(S_R^*) \leqq 1.$$

On the other hand it is easily verified (cf. Komatu [12]) that

$$f(z) = \frac{R(1-Rz)}{z(R-z)} \quad (R > 1)$$

maps the unit circle $|z| < 1$ univalently onto the w-plane cut by a single arc A_R on the circle $|w| = R$ whose angular measure is equal to

$$4 \sin^{-1}(R^{-1}).$$

Now, considering that $\tau(A_R) = 1$ (cf. Tsuji [16, p. 84]) and $\tau(S_R^*) \leqq 1$, we have

$$\theta(S_R^*) \leqq 4 \sin^{-1}(R^{-1}).$$

Since $\theta(S_R) = \theta(S_R^*)$, the proof is completed.

From Theorem 4 we can directly prove the following.

COROLLARY 2. *Let $f(z) = z + a_2 z^2 + \cdots$ be meromorphic in $|z| < 1$. Let the intersection of the complement E_f of the image-domain under $w = f(z)$ and the circle $|w| = R$ be denoted by S_R whose angular measure with respect to the origin is denoted by $\theta(S_R)$. Then*

$$\theta(S_R) \leqq 4 \sin^{-1}(R) \quad (R < 1).$$

This result is best possible as is shown by

$$f(z) = \frac{Rz(1-Rz)}{R-z} \quad (R < 1).$$

PROOF. Applying Theorem 4 for $g(z) = 1/f(z)$, Corollary 2 is easily derived. The condition $R < 1$ means that the circle $|w| \geqq 1$ is not covered by the function $f(z) = z$ which is also one of meromorphic functions $f(z) = z + a_2 z^2 + \cdots$ in $|z| < 1$.

Adding the condition of circumferentially mean univalence to Theorem 4, we can prove the following, since we have by Theorem 1 bounds on values omitted by meromorphic and circumferentially mean univalent functions $f(z) = 1/z + a_0 + a_1 z + \cdots$ in $|z| < 1$.

THEOREM 5. *Let $f(z) = 1/z + a_0 + a_1 z + \cdots$ be meromorphic and circumferentially mean univalent in $|z| < 1$. Let the intersection of the complement of the image-domain under $w = f(z)$ and the circle $|w| = R$ be denoted by S_R whose angular measure with respect to the origin is denoted by $\theta(S_R)$. Suppose that $m > 1$, where $m = \min |w_c|$ $(w_c \in E_f)$. Then*

(i) $\qquad \theta(S_R) \leqq 4 \sin^{-1}(R^{-1}) \qquad ((1 <) m \leqq R \leqq m+4).$

(ii) $\qquad \theta(S_R) = 0 \qquad (m+4 < R).$

This result is best possible as is shown by

$$f(z)=\frac{R(1-Rz)}{z(R-z)}\quad (R>1).$$

Proof. If the circle $|w|=R(\geqq m)$ is not wholly contained in the image-domain under $|w|=f(z)$, then R is equal to $m+4$ at the largest. Therefore (ii) in Theorem 5 is clear. Moreover it is evident that (i) in Theorem 5 holds quite similarly as in Theorem 4. This completes the proof.

Here we can also deal by means of Corollary 2 with $\mathfrak{F}_1$ similarly as in Theorem 5. The details will be omitted.

§ 4. Some evaluations based on a pole.

As was remarked before, meromorphic and circumferentially mean univalent functions $f(z)=z+a_2z^2+\cdots$, in $|z|<1$, have at most only one simple pole in $|z|<1$. We shall derive some results by giving this simple pole explicitly.

We shall first state the following theorem closely related to Theorem 2.

Theorem 6. *Let $f(z)$ be meromorphic and circumferentially mean univalent in $|z|<1$ and let $f(z)$ be expanded about its pole $z=z_\infty$ as follows.*

$$f(z)=\frac{1}{z-z_\infty}+\sum_{n=0}^{\infty}a_n(z-z_\infty)^n.$$

The image-domain under $w=f(z)$ covers at least one of the circles $|w|<\delta$ or $|w|>\delta^{-1}$, where

$$\delta=\frac{-2}{1-|z_\infty|^2}+\sqrt{\frac{4}{(1-|z_\infty|^2)^2}+1}\,.$$

This result is best possible as is shown by

$$f(z)=\frac{1}{1-|z_\infty|^2}\left(\frac{1-\bar{z}_\infty z}{z-z_\infty}+\frac{z-z_\infty}{1-\bar{z}_\infty z}\right)+\sqrt{\frac{4}{(1-|z_\infty|^2)^2}+1}\,.$$

Proof. By a linear transformation

$$\frac{z-z_\infty}{1-\bar{z}_\infty z}=\zeta,\quad \text{that is,}\quad z=\frac{z_\infty+\zeta}{1+\bar{z}_\infty\zeta},$$

we have

$$f(z)=f\left(\frac{z_\infty+\zeta}{1+\bar{z}_\infty\zeta}\right)=\frac{1}{1-|z_\infty|^2}\frac{1}{\zeta}+\left(\frac{\bar{z}_\infty}{1-|z_\infty|^2}+a_0\right)+\cdots\quad (|\zeta|<1).$$

Here $g(\zeta)=(1-|z_\infty|^2)f(z)=1/\zeta+\bar{z}_\infty+a_0(1-|z_\infty|^2)+\cdots$ satisfies the same conditions as in Corollary 1. Therefore if we put $M=\max|w_c|$, $m=\min|w_c|$ $(w_c\in E_f)$, where E_f denotes the complement of the image-domain under $w=f(z)$, then

$$M-m\leqq\frac{4}{1-|z_\infty|^2}\,.$$

Since $\delta=-2/(1-|z_\infty|^2)+(4/(1-|z_\infty|^2)^2+1)^{1/2}$ satifies the following relation

$$\delta^{-1}-\delta=\frac{4}{1-|z_\infty|^2},$$

we see that Theorem 6 holds.

Now we shall quote Hayman's result [**6**].

LEMMA 4. *Let $f(z)=z+a_2z^2+\cdots$ be regular and circumferentially mean univalent in $|z|<1$. Then*

(i) *the image-domain under $w=f(z)$ contains the circle $|w|<1/4$.*

(ii) $$|a_2|\leqq 2.$$

THEOREM 7. *Let $f(z)=z+a_2z^2+\cdots$ belong to $\mathfrak{F}_1$ and its pole be denoted by $z=z_\infty$. Then*

(i) *the image-domain under $w=f(z)$ wholly covers the circle $|w|<|z_\infty|/(1+|z_\infty|)^2$.*

(ii) $$|a_2|\leqq|z_\infty|+\frac{1}{|z_\infty|}.$$

These results are best possible as is shown by

$$f(z)=\frac{z}{(1-(|z_\infty|+|z_\infty|^{-1})e^{i\varepsilon}z+e^{i2\varepsilon}z^2)}\quad(\varepsilon=-\arg z_\infty).$$

PROOF. Without loss of generality we may suppose $z_\infty<0$. Otherwise we may make a rotation $z'=ze^{i\alpha}$, $(\alpha=\pi-\arg z_\infty)$. We consider the following Löwner mapping $z=z(\zeta)$ by which the unit circle $|\zeta|<1$ is mapped univalently and conformally onto the circle $|z|<1$ cut by a segment $[-1, z_\infty]$.

$$\frac{z}{(1-z)^2}=q\frac{\zeta}{(1-\zeta)^2},\quad q=\frac{4|z_\infty|}{(1+|z_\infty|)^2}.$$

Then

$$q^{-1}f(z(\zeta))=g(\zeta)=\zeta+(2(1-q)+qa_2)\zeta^2+\cdots\quad(|\zeta|<1).$$

Since $g(\zeta)$ satisfies the same conditions as in Lemma 4, the image-domain under $w=g(\zeta)$ covers the circle $|w|<1/4$. Therefore (i) in Theorem 7 holds. Next by means of Lemma 4 we have also

$$|2(1-q)+qa_2|\leqq 2.$$

From this inequality we have directly

$$|a_2|\leqq|z_\infty|+\frac{1}{|z_\infty|}.$$

This completes the proof.

REMARK 1. (ii) in Theorem 7 was proved by Komatu [**11**] under the condition of univalency.

REMARK 2. We note that (i) and (ii) in Theorem 7 can be proved under a weak condition of areally mean univalence by means of Spencer's result [**16**], and Garabedian-Royden's one [**5**].

Now we can directly derive the following result from Theorem 7.

COROLLARY 3. *Let* $f(z)=z^p+a_{p+1}z^{p+1}+\cdots$ *belong to* $\mathfrak{F}_p$ *and let its pole be denoted by* $z=z_\infty$.

(i) *The image-domain under* $w=f(z)$ *covers exactly* p *times the circle*

$$|w|<\frac{|z_\infty|^p}{(1+|z_\infty|)^{2p}}.$$

(ii) $$|a_{p+1}|\leqq p\left(|z_\infty|+\frac{1}{|z_\infty|}\right).$$

These results are best possible as is shown by

$$f(z)=\frac{z^p}{(1-(|z_\infty|+|z_\infty|^{-1})e^{i\varepsilon}z+e^{i2\varepsilon}z^2)^p}\quad(\varepsilon=-\arg z_\infty).$$

PROOF. Since $g(z)=(f(z))^{1/p}=z+\frac{a_{p+1}}{p}z^2+\cdots$ satisfies the same conditions as in Theorem 7 and has only one simple pole at $z=z_\infty$, we see that Corollary 3 holds.

Here we shall derive a type of distortion theorems on $\mathfrak{F}_1$ or $\mathfrak{F}_p$, as an application of Theorem 7. But these estimates are not sharp.

THEOREM 8. *Let* $f(z)\in\mathfrak{F}_1$ *and let its pole be denoted by* $z=z_\infty$. *Then*

$$|f(z)|\geqq\frac{4r}{(1+r)^2}\frac{|\zeta_\infty|}{(1+|\zeta_\infty|)^2}\quad(|z|=r<1),$$

where ζ_∞ *is such the root of the following equation as satisfies the condition* $|\zeta_\infty|<1$.

$$\frac{z_\infty e^{i\varepsilon}}{(1-z_\infty e^{i\varepsilon})^2}=\frac{4r}{(1+r)^2}\frac{\zeta_\infty}{(1-\zeta_\infty)^2}\quad(\varepsilon=\pi-\arg z).$$

PROOF. We suppose $z=-|z|=-r<0$. Otherwise we may consider $z'=ze^{i\varepsilon}$ $(\varepsilon=\pi-\arg z)$. Similarly as in the proof of Theorem 7, we consider the following Löwner mapping

$$\frac{z}{(1-z)^2}=q\frac{\zeta}{(1-\zeta)^2},\quad q=\frac{4r}{(1+r)^2},$$

where $z=z_\infty$ is mapped to $\zeta=\zeta_\infty$.

$g(\zeta)=f(z(\zeta))/q=\zeta+\cdots$ satisfies the same conditions as in Theorem 7 and has only one simple pole at $\zeta=\zeta_\infty$. Therefore the image-domain under $w=g(\zeta)$ contains the circle $|w|<|\zeta_\infty|/(1+|\zeta_\infty|)^2$. Hence $f(-r)=qg(-1)$ is not covered by $w=f(z)$, that is,

$$f(-r)\geqq q\frac{|\zeta_\infty|}{(1+|\zeta_\infty|)^2}.$$

This completes the proof.

Now from Theorem 8 we can directly derive the following.

COROLLARY 4. *Let $f(z)\in\mathfrak{F}_p$ and let its pole be denoted by $z=z_\infty$. Then*

$$|f(z)|\geqq\frac{(4r)^p}{(1+r)^{2p}}\frac{|\zeta_\infty|^p}{(1+|\zeta_\infty|)^{2p}}\quad(|z|=r<1),$$

where ζ_∞ satisfies the same conditions as in Theorem 8.

REMARK. Under the condition of p-valence some distortion theorems on $\mathfrak{F}_p$ were derived from another point of view by Kobori [**10**] and the author [**1**].

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