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Between topology for lattices

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E. S. Wolk [1], E. S. Northam [2], M. Kolibiar [3], and the author [4] have studied the problem finding conditions for a lattice to be a Hausdorff space in the interval topology. In papers [5-7], we have studied the concept of $B(B^*)$-covers in lattices, where

$$B(a, b)=\{x \mid axb, \text{ that is, } (a \cup x) \cap (b \cup x)=x=(a \cap x) \cup (b \cap x)\},$$

$$B^*(a, b)=\{x \mid abx\}.$$

We shall define the between-topology ($\mathscr{B}(\mathscr{B}^*)$-topology) on a lattice L by taking the between sets $B(a, b)$ ($B^*(a, b)$) as a sub-base for the closed sets. We denote by $\mathscr{I}$ the interval topology, and by $\mathscr{B}(\mathscr{B}^*)$ the $\mathscr{B}(\mathscr{B}^*)$-topology on L.

In this note, we shall first consider the relations between the $\mathscr{B}^*$-topology and the $\mathscr{I}$-topology, including the problem finding the conditions that $\mathscr{B}^*$ coincides with $\mathscr{I}$. Next we shall consider the $\mathscr{B}^*$-topology in lattices, and then we shall apply our results to the theorems in the $\mathscr{I}$-topology.

We can easily prove that the $\mathscr{B}$-topology coincides with the $\mathscr{I}$-topology in a distributive lattice with $0, I$ ([7]). We have $\mathscr{B}^* \geqq \mathscr{I}$ in the sence that every $\mathscr{I}$-closed set is $\mathscr{B}^*$-closed in any lattice with $0, I$. In a Boolean algebra we have $\mathscr{I}=\mathscr{B}^*$ [Th. 1.1]. In Theorems 1.2 and 1.3 we shall give the sufficient conditions for $\mathscr{I}=\mathscr{B}^*$ in some lattices.

Let $C(a)$ be the connected component containing a. Then we shall call that L is totally disconnected if and only if $C(a)=a$ for any a in L. We shall show in Theorem 2.1 that a modular lattice satisfying the ascending condition is totally disconnected in the $\mathscr{B}^*$-topology.

In Theorem 2.2 we shall find the sufficient condition for a modular lattice to be a Hausdorff space in the $\mathscr{B}^*$-topology.

We shall show the sufficient condition for an element of a lattice to be an isolated element in the $\mathscr{B}^*$-topology in Theorem 2.4 which is close connection with the Northam's theorem ([2]).

In a Boolean algebra L, we have $(\beta) \rightarrow$ T. D. ($\mathscr{I}$) and T. D. ($\mathscr{B}^*$), where

(β): every element is over an atom,

T. D. ($\mathcal{J}$): L is totally disconnected in the $\mathcal{J}$-topology,

T. D. ($\mathcal{B}^*$): L is totally disconnected in the $\mathcal{B}^*$-topology [Th. 3.1].

In a complemented modular lattice L satisfying (a), we have

(1). $(\beta)\rightarrow H(\mathcal{J})$ and $H(\mathcal{B}^*)$ (2). $H(\mathcal{J})\rightarrow H(\mathcal{B}^*)$,

where

(a): the number of the complements of any element is finite,

$H(\mathcal{J})$: L is a Hausdorff space in the $\mathcal{J}$-topology,

$H(\mathcal{B}^*)$: L is a Hausdorff space in the $\mathcal{B}^*$-topology [Th. 3.2].

In a complemented modular lattice L, satisfying (a), we have $(\beta)\rightarrow$ T. D. ($\mathcal{J}$) and T. D. ($\mathcal{B}^*$) [Th. 3.3].

§1. Relations between the $\mathcal{J}$-topology and the $\mathcal{B}^*$-topology.

LEMMA 1.1. *$\mathcal{J}\leqq\mathcal{B}^*$ in a lattice with* $0, I$.

PROOF. Since $B^*(0, a)=\{x \mid x\geqq a\}$, $B^*(I, a)=\{x \mid x\leqq a\}$, any $\mathcal{J}$-closed set is $\mathcal{B}^*$-closed.

THEOREM 1.1. *In a Boolean algebra, we have $\mathcal{J}=\mathcal{B}^*$.*

PROOF. Let L be a Boolean algebra, then we have the following equalities: $B^*(a, b)=B(b, a')=B(b\cup a', b\cap a')$, where a' is the complement of a. Indeed we can prove $abx\rightleftarrows bxa'$ as follows.

$b=(a\cap b)\cup(x\cap b)=(a\cup x)\cap b$ is equivalent to $b\leqq a\cup x$, $a'\cap b\leqq x$ and $x=(a'\cap b)\cup x=(a'\cup x)\cap(b\cup x)$. Dually $b=(a\cup b)\cap(x\cup b)$ is equivalent to $x=(a'\cap x)\cup(b\cap x)$. $B(b, a')=B(b\cup a', b\cap a')$ is obtained in [**6**]. Now $B(b\cup a', b\cap a')$ is an $\mathcal{J}$-closed set, so that we have $\mathcal{B}^*\leqq\mathcal{J}$ in a Boolean algebra.

THEOREM 1.2. *In a complemented modular lattice L satisfying* (a), *we have $\mathcal{B}^*=\mathcal{J}$, where as above*

(a): *the number of complements of any element is finite.*

PROOF. We shall prove that $B^*(a, b)$ is an $\mathcal{J}$-closed set for any two elements a, b of L. Since it can be proved easily that $B^*(a, b)=B^*(a\cup b, b)\cap B^*(a\cap b, b)$, it suffices to prove in the case $b\geqq a$ or $a\geqq b$. Let $b\geqq a$, and a' be a complement of a. If we take $x\in[b\cap a', I]$, then we have $b\geqq(a\cap b)\cup(x\cap b)\geqq a\cup(b\cap a')=(a\cup a')\cap b=b$ by the modularity, and $(a\cup b)\cap(x\cup b)=b$. Thus $x\in B^*(a, b)$, and hence we have $[b\cap a', I]\subset B^*(a, b)$.

On the other hand we can prove that $B^*(a, b)\subset[b\cap a', I]\cup[b\cap a'', I]\cup\cdots$, where $a', a'', \cdots$ are the complements of a, as follows.

Assume $x\in B^*(a, b)$ and $a\leqq b$, that is, $b=a\cup(x\cap b)$. Let b' be a complement of b, y a complement of $a\cap x$, and put $a'=b'\cup(b\cap x\cap y)$.

Then we get $a\cup a'=a\cup(a\cap x)\cup(b\cap x\cap y)\cup b'=a\cup((b\cap x)\cap((a\cap x)\cup y))\cup b'$ $=a\cup(b\cap x)\cup b'=b\cup b'=I$, $b\cap a'=b\cap((b\cap x\cap y)\cup b')=(b\cap b')\cup(b\cap x\cap y)$ $=b\cap x\cap y\leqq x$, and $a\cap a'=a\cap x\cap y=0$. Thus $x\in[b\cap a', I]$ with a complement a' of a.

It follows that $B^*(a, b) \subset [b \cap a', I] \cup [b \cap a'', I] \cup \cdots$, where $a', a'', \cdots$ are the complements of a.

Consequently, we have $B^*(a, b) = [b \cap a', I] \cup [b \cap a'', I] \cup \cdots$, where $a', a'', \cdots$ are the complements of a. Thus, if L satisfies the condition (a), then $B^*(a, b)$ is an $\mathcal{J}$-closed set for $b \geqq a$. Similarly we can prove for $b \leqq a$. This completes the proof.

THEOREM 1.3. *Let L be any lattice with* 0, *I. If L satisfies the conditions* (b_1), (b_2), *then we have* $\mathcal{J} = \mathcal{B}^*$, *where*

(b_1): $a_i = \min\{x_{i,k}\}$ *such that* $a \cup x_{i,k} = b$, $x_{i,k} \geqq x_{i,k+1}$ *for* $b \geqq a$, $i = 1, 2, \cdots, n$, $k = 1, 2, \cdots$ *implies that n is finite,*

(b_2): $b_i = \max\{x_{i,k}\}$ *such that* $a \cap x_{i,k} = b$, $x_{i,k} \leqq x_{i,k+1}$ *for* $b \leqq a$, $i = 1, 2, \cdots, n$, $k = 1, 2, \cdots$ *implies that n is finite.*

PROOF. Suppose that abx for $b \geqq a$. Then we have $a \cup (b \cap x) = b$, so that $b \cap x \in [a_1, I] \cup [a_2, I] \cup \cdots$ from (b_1), and hence $x \in [a_1, I] \cup [a_2, I] \cup \cdots$.

On the other hand if we take any x such that $a_i \leqq x \leqq I$, then $a_i = b \cap a_i \leqq b \cap x \leqq b$. We have $a \cup (b \cap x) = b$ since $a \cup (b \cap a_i) = b$, that is, abx.

Consequently we have $B^*(a, b) = [a_1, I] \cup [a_2, I] \cup \cdots$ for $b \geqq a$.

Dually we have $B^*(a, b) = [b_1, 0] \cup [b_2, 0] \cup \cdots$ for $b \leqq a$. Thus if L satisfies (b_1), (b_2), then we have $\mathcal{J} = \mathcal{B}^*$.

§ 2. Theorems in the $\mathcal{B}^*$-topology.

LEMMA 2.1. *Let L be a modular lattice. If* $a > b$, *then* $B^*(a, b) \cap B^*(b, a) = \phi$, *where* ϕ *is the null set.*

PROOF. Suppose that abx and bax for $a > b$. Then we get $a \cap (b \cup x) = b$ from abx, and $b \cup (a \cap x) = a$ from bax. This is impossible, since $a \cap (b \cup x) = b \cup (a \cap x)$ by the modularity.

LEMMA 2.2. *Let L be a modular lattice. If a covers b* ($a \succ b$), *that is, azb implies* $z = a$ *or* $z = b$, *then we have either* $x \in B^*(a, b)$ *or* $x \in B^*(b, a)$ *for any* $x \in L$.

PROOF. Suppose that neither abx nor bax. Then we have $(a \cup b) \cap (b \cup x) \neq b$ and hence $(a \cup b) \cap (b \cup x) > b$, similarly we have $(a \cap b) \cup (a \cap x) < a$.

From $a \succ b$ and $a \geqq a \cap (b \cup x) > b$, we have $a \cap (b \cup x) = a$, and also from $a \succ b$ and $a > b \cup (a \cap x) \geqq b$, we have $b \cup (a \cap x) = b$. This is a contradiction, since $a \cap (b \cup x) = b \cup (a \cap x)$ holds by the modularity. Moreover, $B^*(a, b) \cap B^*(b, a) = \phi$ from Lemma 2.1, and hence we have the assertion.

THEOREM 2.1. *Any modular lattice L satisfying the ascending chain condition* (α), *is* T. D. ($\mathcal{B}^*$), *that is, totally disconnected in the* $\mathcal{B}^*$-*topology.*

PROOF. For any two elements a, b of L, we have $a(a \cap b)b$. There exists c such that $a \succ c \geqq a \cap b$ from (α). $a(a \cap b)b$, $ac(a \cap b)$ imply acb by [**6**, Lemma 8]. It follows that $b \in B^*(a, c)$.

By Lemmas 2.1 and 2.2, we have $B^*(a, c) \cap B^*(c, a) = \phi$, $B^*(c, a) \cup B^*(a, c) = L$. Thus L is totally disconnected in the $\mathscr{B}^*$-topology.

COROLLARY 1. *A modular lattice satisfying* (γ) *is* $T.D.$ ($\mathscr{B}^*$), *where*

(γ): *every closed interval has a leap; that is, every closed interval has two elements* a, b *such that* $a \succ b$ *or* $b \succ a$.

REMARK. M. Kolibiar [3] has proved that (1) $H(\mathscr{I}) \to (\gamma)$ in a relatively complemented lattice, (2) $(\beta) \to (\gamma)$ in a semi-modular relatively complemented lattice with 0, and (3) $(\gamma) \to (\beta)$ in a complemented modular lattice with 0, where as above

(β): every element is over an atom.

LEMMA 2.3. *Let* L *be a modular lattice. For any three elements* a, b, c *of* L *such that* $a > c > b$, *if* c *has no non-comparable relative complement in any sub-interval of* $[a, b]$, *then we have* $B^*(a, c) \cup B^*(b, c) = L$, *where* $B^*(a, c) \not\ni a$, $B^*(b, c) \not\ni b$.

PROOF. Let $a > c > b$ in a modular lattice L. Then we have $B^*(a, c) \ni x$ if and only if $a \cap x = c \cap x$, and $B^*(b, c) \ni x$ if and only if $b \cup x = c \cup x$. Suppose that $a \cap x > c \cap x$, $c \cup x > b \cup x$ for $x \in L$. Then we should have a contradiction. Indeed, let $X = a \cap (b \cup x) = b \cup (a \cap x)$, $Y = c \cap (b \cup x) = b \cup (c \cap x)$. Then we have $b \leqq X \leqq a$, $b \leqq Y \leqq c$. Since $Y \cap (a \cap x) = c \cap x$, $Y \cup (a \cap x) = X$; $[a \cap x, c \cap x]$ is isomorphic to $[X, Y]$, and hence we have $X > Y$ from $a \cap x > c \cap x$. Since $c \cup (b \cup x) = c \cup x$, $c \cap (b \cup x) = Y$, we have $c > Y$ from $c \cup x > b \cup x$. From $c \cup X = c \cup (a \cap x) = a \cap (c \cup x) \leqq a$, $c \cap X = Y$ we have a relative complement X of c in $[a \cap (c \cup x), Y]$ which is a sub-interval of $[a, b]$. This is a contradiction. Thus we have $x \in B^*(a, c) \cup B^*(b, c)$ for any element x of L. It is easily proved that a does not belong to $B^*(a, c)$.

THEOREM 2.2. *A modular lattice* L *is* $H(\mathscr{B}^*)$ *if*

(δ): *every closed interval of* L *contains a chain as a sub-interval.*

PROOF. Let a, b be any two distinct elements of L and assume $a \not\leqq b$ without loss of generality. Then $[a \cap b, a]$ contains a sub-interval $[c, d]$ which is a chain. If an element e exists with $c < e < d$, then it follows from Lemma 2.3 that $B^*(c, e) \cup B^*(d, e) = L$, $B^*(c, e) \not\ni c$ and $B^*(d, e) \not\ni d$, whence $B^*(c, e) \not\ni a \cap b$ and $B^*(d, e) \not\ni a$. We have $B^*(c, e) \not\ni b$, since ceb and $e(e \cap b)b$ imply $ce(e \cap b)$; namely $B^*(c, e) \ni e \cap b = a \cap b$, which is a contradiction.

If $c \prec d$, then we get $B^*(c, d) \cup B^*(d, c) = L$ from Lemma 2.2 and it can be deduced in the same way as above that $B^*(c, d) \not\ni b$ and $B^*(d, c) \not\ni a$.

COROLLARY. *When* L *is the direct product of a finite number of chains, then* L *is* $H(\mathscr{B}^*)$.

LEMMA 2.4. *Let* x *be an element of a modular lattice* L *such that* $[x, I]$ *satisfies* (β) *and* $[0, x]$ *satisfies the dual of* (β). *Then any* $y \in L$ *different from* x *belongs to some* $B^*(x, a)$ *with* $a \succ x$ *or* $x \succ a$.

PROOF. If $x \geqq y$ or $y \geqq x$, then $y \in B^*(x, a)$ since $y \leqq a \prec x$ or $y \geqq a \succ x$. If y is non-comparable with x, then $x(x \cup y)y$ and $xa(x \cup y)$ imply xay [**6**, Lemma 8].

THEOREM 2.3. *Let be an element of a modular lattice L such that $[x, I]$ satisfies (β) and $[0, x]$ satisfies the dual of (β). If the number of elements a_i, b_j such that $a_i \succ x$, $x \succ b_j$ is finite, then x is an isolated element in the $\mathscr{B}^*$-topology.*

PROOF. By Lemma 2.4, any element y belongs to some $B^*(x, a)$ such that $a \succ x$ or $x \succ a$. Hence we have $L - x = B^*(x, a_1) \cup B^*(x, a_2) \cup \cdots \cup B^*(x, b_1) \cup \cdots$. Then if the number of elements a_i and b_j is finite, x is an isolated element in the $\mathscr{B}^*$-topology.

LEMMA 2.5. *In a modular lattice $B^*(a \cap b, b) \subseteqq B^*(a, a \cup b)$ and $B^*(a \cup b, b) \subseteqq B^*(a, a \cap b)$.*

PROOF. $x \in B^*(a \cap b, b)$ implies $(a \cap b) \cup (x \cap b) = b$, $((a \cap b) \cup x) \cap b = b$, $a \cup x \geqq (a \cap b) \cup x \geqq b$, $(a \cup x) \cap (a \cup b) = a \cup b$, $a \cup (x \cap (a \cup b)) = a \cup b$ and $x \in B^*(a, a \cup b)$.

THEOREM 2.4. *Let x be an element of a modular lattice L such that $[x, I]$ satisfies (β) and $[0, x]$ satisfies the dual of (β), and $\{a_i\}$, $\{b_j\}$ the sets of elements satisfying $a_i \succ x$, $x \succ b_j$ respectively. If there exist c and d satisfying that $c \geqq a_i$ and $d \leqq b_j$ for all i, j and the interval $[d, c]$ has a finite length, then x is an isolated element in the $\mathscr{B}^*$-topology.*

PROOF. We can find e and f such that $c \geqq e = \bigvee a_i$ and $d \leqq f = \bigwedge b_j$. Since $[x, e]$ has a finite length, we can choose a finite subsets $\{a_1, a_2, \cdots, a_n\}$ of $\{a_i\}$ so that $a_1 \prec a_1 \cup a_2 \prec \cdots \prec a_1 \cup a_2 \cup \cdots \cup a_n = e$. Put $c_0 = x$ and $c_\nu = a_1 \cup a_2 \cup \cdots \cup a_\nu$.

Then for any a_i we can find ν such that $c_{\nu-1} \ngeqq a_i$ and $c_\nu \geqq a_i$, and it follows from Lemma 2.5 that $B^*(x, a_i) \subseteqq B^*(c_{\nu-1}, c_\nu)$. Similarly we can find $d_0, d_1, \cdots, d_m$, where $d_0 = x$, $d_\mu = b_1 \cup b_2 \cup \cdots \cup b_\mu$, such that, for any b_j $B^*(x, b_j) \subseteqq B^*(d_{\mu-1}, d_\mu)$ holds for some μ. From Lemma 2.4 we obtain $L - x = \bigvee B^*(x, a_i) \cup \bigvee B^*(x, b_j) \subseteqq B^*(c_0, c_1) \cup B^*(c_1, c_2) \cup \cdots \cup B^*(c_{n-1}, c_n) \cup B^*(d_0, d_1) \cup \cdots \cup B^*(d_{m-1}, d_m)$. It is evident that $B^*(c_{\nu-1}, c_\nu) \not\ni x$ and $B^*(d_{\mu-1}, d_\mu) \not\ni x$.

§ 3. Applications.

We shall apply our results in **1** and **2** to known results in the interval topology.

EXAMPLE. Let L be a lattice containing countably many element $0, I, x_1, x_2, \cdots$, such that $I \succ x_i \succ 0$ for all i. 0 is not an isolated element in the interval topology, but it is an isolated element in the $\mathscr{B}^*$-topology by Theorem 2.4. L is not $H(\mathscr{J})$ but $H(\mathscr{B}^*)$, moreover it is T. D. ($\mathscr{B}^*$). Indeed, if we take two distinct elements a, b of L, and if they are non-comparable, then we have $B^*(b, I) \cup B^*(I, b) = L$, $B^*(b, I) \cap B^*(I, b) = \phi$, $b \in B^*(I, b)$, $a \in B^*(b, I)$. Similarly we have the assertion in the case a, b are comparable.

M. Katetov and E. S. Northam [2] have proved that (β) is equivalent to $H(\mathcal{J})$ in a Boolean algebra, where

(β): every element is over an atom.

THEOREM 3.1. *In a Boolean algebra L, (β) implies* T. D.($\mathcal{J}$) *and* T. D.($\mathcal{B}^*$).

PROOF. For x, y ($x \not\leqq y$) of L, let z be the relative complement of $x \cap y$ in $[x, 0]$. Then we have an atom p such that $z \geqq p \succ 0$. Since $[z, 0]$ is isomorphic to $[x, x \cap y]$, there exists an element w such that $x \geqq w \succ x \cap y$. Hence we have $B^*(w, x \cap y) \cup B^*(x \cap y, w) = L$, $B^*(w, x \cap y) \cap B^*(x \cap y, w) = \phi$, $x \in B^*(x \cap y, w)$, $y \in B^*(w, x \cap y)$ by Lemma 2.2. Thus L is T. D. ($\mathcal{J}$) and T. D. ($\mathcal{B}^*$) by Theorem 1.1.

M. Kolibiar [3] has proved that in any complemented modular lattice L satisfying (c), (β) is equivalent to $H(\mathcal{J})$, where

(c): if L has an atom, then the number of its complements is finite.

We have proved in Theorem 1.2 that, in any complemented modular lattice L satisfying (a), $\mathcal{B}^* = \mathcal{J}$.

THEOREM 3.2. *In any complemented modular lattice L satisfying* (a) *we have the following:*

(1). $(\beta) \rightarrow H(\mathcal{J})$ *and* $H(\mathcal{B}^*)$,

(2). $H(\mathcal{J}) \rightarrow H(\mathcal{B}^*)$.

PROOF. (1). Since it is easily seen that (a) implies (c), we have the assertion from M. Kolibiar [3] and Theorem 1.2.

(2). From M. Kolibiar [3] and Theorem 1.2 we have $H(\mathcal{J}) \rightarrow (\beta) \rightarrow (a) \rightarrow H(\mathcal{J})$ and $H(\mathcal{B}^*)$.

THEOREM 3.3. *In any complemented modular lattice L satisfying* (a), (β) *implies* T. D. ($\mathcal{J}$) *and* T. D. ($\mathcal{B}^*$).

PROOF. From the remark of Corollary 1 of Theorem 2.1 we have $(\beta) \rightleftarrows (\gamma)$ in L. From Corollary 1 and Theorem 1.2, we have the assertion.

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