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On existence of Green function and positive superharmonic functions for linear elliptic operators of second order

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§1. **Introduction.** Let D be a subdomain of an N-dimensional orientable C^∞-manifold $\boldsymbol{M}$ ($N \geqq 2$), and A be an elliptic differential operator of the following form:

$$(1.1)\qquad Au(x) = \frac{1}{\sqrt{a(x)}} \frac{\partial}{\partial x^i}\left[\sqrt{a(x)}\ a^{ij}(x)\frac{\partial u(x)}{\partial x^j}\right] + b^i(x)\frac{\partial u(x)}{\partial x^i} \quad ^{1)}$$

$$\text{for } u \in C^2(D)$$

where $\| a^{ij}(x) \|$ and $\| b^i(x) \|$ are contravariant tensors of class C^2 in D, $\| a^{ij}(x) \|$ is symmetric and strictly positive-definite for each $x \in D$ and $a(x) = \det \| a_{ij}(x) \| = \det \| a^{ij}(x) \|^{-1}$. We require neither regularity of the boundary of D, nor restriction on the behavior of $\| a^{ij}(x) \|$ and $\| b^i(x) \|$ near the boundary of D.

By definition, a function $u(x)$ is said to be *A-harmonic* in D if it satisfies $Au = 0$ in D, and is said to be *A-superharmonic* in D if it satisfies the following three conditions:

i) $-\infty < u(x) \leqq \infty$ and $u(x) \not\equiv \infty$ in D,

ii) $u(x)$ is lower semi-continuous in D,

iii) if Ω is a domain with its closure $\bar{\Omega} \subset D$, and if $w(x)$ is continuous on $\bar{\Omega}$, A-harmonic in Ω and satisfies $w(x) \leqq u(x)$ on $\partial\Omega$, then $w(x) \leqq u(x)$ holds in Ω.

The purpose of the present paper is to prove that there exists a Green function associated with the elliptic differential operator A in D if, and only if, there exists at least one non-constant positive A-superharmonic function in D. This fact is well known in the case of Riemann surfaces—see [**1**] and [**2**].

§2. **Preliminaries.** In this §, we shall state some properties of fundamental solutions of parabolic differential equations. The following facts 1°), 2°) and 3°) are implied in the results of the author's previous paper [**3**][2].

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1) We omit the summation sign $\sum$ according to the usual rule of tensor calculus.

2) Differential operators A and A^* in the present paper correspond to A^* and A in [**3**] respectively.

By definition, a subdomain Ω of $\boldsymbol{M}$ is called a *domain with property* (S) if the boundary of Ω consists of a finite number of simple closed hypersurfaces of class C^3.

1°) For any domain Ω with its closure $\bar{\Omega}\subset D$ and with property (S), there exists one and only one fundamental solution $U_\Omega(t, x, y)$ of the initial-boundary value problem for the parabolic equation:

$$\frac{\partial u}{\partial t}=Au+f \text{ in } (0, \infty)\times\Omega,\ u|_{t=0}=u_0,\ u|_{x\in\partial\Omega}=\varphi. \tag{2.1}$$

The function $U_\Omega(t, x, y)$ satisfies that

$$\begin{cases} U_\Omega(t, x, y)\geqq 0 \text{ for any } \langle t, x, y\rangle\in(0, \infty)\times\bar{\Omega}\times\bar{\Omega}\text{; the equality holds} \\ \text{if and only if at least one of } x \text{ and } y \text{ belongs to } \partial\Omega \end{cases} \tag{2.2}$$

and that

$$\frac{\partial U_\Omega(t, x, y)}{\partial \boldsymbol{n}_y}\leqq 0 \text{ for any } t>0,\ y\in\partial\Omega \text{ and } x\in\bar{\Omega}-\{y\} \tag{2.3}$$

where $\frac{\partial}{\partial \boldsymbol{n}}$ denotes the exterior normal derivative. Furthermore

$$G_\Omega(x, y)=\int_0^\infty U_\Omega(t, x, y)dt \tag{2.4}$$

is well-defined whenever $x, y\in\bar{\Omega}$ and $x\neq y$, and is the Green function of the boundary value problem for the elliptic equation:

$$Au=f \text{ in } \Omega,\ u|_{\partial\Omega}=\varphi. \tag{2.5}$$

2°) Assume that $u_0(x)$, $f(t, x)$ and $\varphi(t, x)$ are functions continuous on $\bar{\Omega}$, on $[0, \infty)\times\bar{\Omega}$ and on $[0, \infty)\times\partial\Omega$ respectively. Then, if $u(t, x)$ is a solution of (2.1), it is expressible by

$$u(t, x)=\int_\Omega U_\Omega(t, x, y)u_0(y)dy+\int_0^t d\tau\int_\Omega U_\Omega(t-\tau, x, y)f(\tau, y)dy -\int_0^t d\tau\int_{\partial\Omega}\frac{\partial U_\Omega(t-\tau, x, y)}{\partial \boldsymbol{n}_y}\varphi(\tau, y)dS_y \tag{2.6}$$

where dy and dS_y respectively denote the volume element and the hypersurface element with respect to the 'Riemannian metric' defined by $\|a_{ij}(x)\|$; conversely, the function $u(t, x)$ defined by (2.6) satisfies (2.1) provided that $f(t, x)$ and $\varphi(t, x)$ are Hölder-continuous on $[0, \infty)\times\Omega$ and on $[0, \infty)\times\partial\Omega$ respectively.

Next assume that $f(x)$ and $\varphi(x)$ are functions continuous on $\bar{\Omega}$ and on $\partial\Omega$ respectively. Then, if $u(x)$ is a solution of (2.5), it is expressible by

$$u(x)=-\int_\Omega G_\Omega(x, y)f(y)dy-\int_{\partial\Omega}\frac{\partial G_\Omega(x, y)}{\partial \boldsymbol{n}_y}\varphi(y)dS_y; \tag{2.7}$$

conversely the function $u(x)$ defined by (2.7) satisfies (2.5) provided that $f(x)$ and $\varphi(x)$ are Hölder-continuous on $\bar{\Omega}$ and $\partial\Omega$ respectively.

3°) Let $\{D_n; n=1,2,\cdots\}$ be a sequence of domains with property (S) such that $\bar{D}_n$ is compact and $\bar{D}_n \subset D_{n+1} \subset D$ for each n and that $\lim_{n\to\infty} D_n = D$. Then

$$U_{D_n}(t,x,y) \leqq U_{D_{n+1}}(t,x,y) \text{ for any } \langle t,x,y\rangle \in (0,\infty)\times\bar{D}_n\times\bar{D}_n \tag{2.8}$$

$(n=1,2,\cdots)$, and

$$U_D(t,x,y) = \lim_{n\to\infty} U_{D_n}(t,x,y) \tag{2.9}$$

is well-defined on $(0,\infty)\times D\times D$ and independent of the choice of sequence $\{D_n\}$, and $U_D(t,x,y)$ is a fundamental solution of the initial-boundary value problem for the parabolic equation $\partial u/\partial t = Au$ in $(0,\infty)\times D$. If a part of the boundary of D consists of a simple hypersurface S of class C^3 and if $\|a^{ij}(x)\|$ and $\|b^i(x)\|$ are of class C^2 in a domain containing $D\cup S$, then we can choose the sequence $\{D_n\}$ such that $\partial D_n \cap S$ contains a relatively open subregion of S for any $n\geqq 1$ and $\lim_{n\to\infty}\partial D_n\cap S = S$, and $U_D(t,x,y)$ is a fundamental solution of the initial-boundary value problem of the form (2.1) where Ω and $\partial\Omega$ are replaced by D and S. In this case, we have

$$\frac{\partial U_D(t,x,y)}{\partial \boldsymbol{n}_y} = \lim_{n\to\infty}\frac{\partial U_{D_n}(t,x,y)}{\partial \boldsymbol{n}_y} \tag{2.10}$$

for any $t>0$, $y\in S$ and $x\in D\cup S-\{y\}$.

4°) Let Ω be a domain with property (S) and with compact closure $\bar{\Omega}\subset D$. Then we can choose the sequence $\{D_n\}$ of domains stated in 3°) such that $\bar{\Omega}\subset D_1$. If we put $D'_n = D_n - \bar{\Omega}$ $(n=1,2,\cdots)$ and $D' = D-\bar{\Omega}$, then we may consider $U_{D'_n}(t,x,y)$ $(n=1,2,\cdots)$ and $U_{D'}(t,x,y)$ in the same way as in 2°) and 3°), and we have

$$U_{D'}(t,x,y) = \lim_{n\to\infty} U_{D'_n}(t,x,y) \quad (t>0,\ x\in D-\Omega,\ y\in D-\Omega) \tag{2.11}$$

and

$$\frac{\partial U_{D'}(t,x,y)}{\partial \boldsymbol{n}_y} = \lim_{n\to\infty}\frac{\partial U_{D'_n}(t,x,y)}{\partial \boldsymbol{n}_y} \quad (t>0,\ x\in D-\bar{\Omega},\ y\in\partial\Omega) \tag{2.12}$$

where $\partial/\partial\boldsymbol{n}_y$ denotes the exterior normal derivative at the point y of $\partial\Omega$ as a boundary of D' $(=D-\bar{\Omega})$. We put

$$U_{D'}(t,x,y) = 0 \text{ for any } t>0,\ x\in D' \text{ and any } y\in\bar{\Omega}. \tag{2.13}$$

Then ;—

LEMMA 2.1. *For any $t>0$, $x\in D'$ and $y\in D$, it holds that*

$$U_D(t,x,y) = U_{D'}(t,x,y) - \int_0^t d\tau\int_{\partial D'}\frac{\partial U_{D'}(t-\tau,x,z)}{\partial\boldsymbol{n}_z}U_D(\tau,z,y)dS_z \tag{2.14}$$

PROOF. For any fixed $\varepsilon>0$, $y\in D$ and $n\geqq 1$, the function $u(t,x) = U_{D_n}(t+\varepsilon,x,y)$ satisfies (2.1) where Ω is replaced by D'_n and

$$\begin{cases} f(t, x)=0,\ u_0(x)=U_{D_n}(\varepsilon, x, y)\ (x\in D_n-\Omega) \text{ and} \\ \varphi(t, x)=U_{D_n}(t+\varepsilon, x, y)\ (x\in\partial\Omega),\ =0\ (x\in\partial D_n). \end{cases}$$

Hence, by 2°), we have

$$U_{D_n}(t+\varepsilon, x, y)=\int_{D'_n} U_{D'_n}(t, x, z)U_{D_n}(\varepsilon, z, y)dz$$
$$-\int_0^t d\tau\int_{\partial\Omega}\frac{\partial U_{D'_n}(t-\tau, x, z)}{\partial \boldsymbol{n}_z}U_{D_n}(\tau, z, y)dS_z$$

for any $t>0$ and $x\in D'_n$. Letting $\varepsilon\to 0$ and then $n\to\infty$, we obtain (2.14) by means of (2.9), (2.11) and (2.12).

LEMMA 2.2. *If $u(x)$ is positive A-superharmonic in D, then*

$$(2.15)\qquad 0\leqq -\int_0^\infty d\tau\int_{\partial D'}\frac{\partial U_{D'_n}(\tau, x, \xi)}{\partial \boldsymbol{n}_\xi}u(\xi)dS_\xi\leqq u(x)\qquad \textit{for any } x\in D'.$$

PROOF. By lower semi-continuity of $u(x)$, there exists a monotone increasing sequence $\{\varphi_n(x)\}$ of continuous functions on $\partial\Omega$ such that $\varphi_1(x)\geqq 0$ and $\lim_{n\to\infty}\varphi_n(x)=u(x)$ on $\partial\Omega$. Let $w_n(x)$ be the solution of the boundary value problem:

$$Aw_n(x)=0 \text{ in } D'_n,\quad w_n(x)=\begin{cases}\varphi_n(x) & \text{on } \partial\Omega,\\ 0 & \text{on } \partial D_n\end{cases}$$

— see [3; §10]. Then, by means of A-superharmonicity of u and by the same argument as in the proof of the preceding lemma, we get

$$u(x)\geqq w_n(x)=\int_{D'_n}U_{D'_n}(t, x, y)w_n(y)dy-\int_0^t d\tau\int_{\partial\Omega}\frac{\partial U_{D'_n}(t-\tau, x, y)}{\partial \boldsymbol{n}_y}\varphi_n(y)dS_y$$
$$\geqq -\int_0^t d\tau\int_{\partial\Omega}\frac{\partial U_{D'_n}(\tau, x, y)}{\partial \boldsymbol{n}_y}\varphi_n(y)dS_y\geqq 0\qquad\text{for any } x\in D'.$$

Letting $n\to\infty$, we obtain (2.15).

§3. **Superharmonic functions and Green function.** We first notice that the domain Ω in the condition iii) in the definition of A-superharmonicity (in §1) can be restricted to domains with property (S); this may easily be seen from i), ii) and iii) in the definition.

THEOREM 1. *Assume that $u(x)$ is of class C^2 in D. Then $u(x)$ is A-superharmonic in D if and only if $Au(x)\leqq 0$ holds in D.*

PROOF. We first assume that $Au(x)\leqq 0$ in D, and let Ω be a domain with property (S) and such that $\bar{\Omega}\subset D$ and $w(x)$ be a function continuous on $\bar{\Omega}$, A-harmonic in Ω and satisfying $w(x)\leqq u(x)$ on $\partial\Omega$. Then, by means of 2°) in §2, we have (see also (2.2), (2.3) and (2.4))

$$(3.1) \qquad u(x)=-\int_{\Omega}G_{\Omega}(x, y)\cdot Au(y)dy-\int_{\partial\Omega}\frac{\partial G_{\Omega}(x, y)}{\partial \boldsymbol{n}_y}u(y)dS_y$$
$$\geqq -\int_{\partial\Omega}\frac{\partial G_{\Omega}(x, y)}{\partial \boldsymbol{n}_y}w(y)dS_y=w(x) \quad \text{for any } x\in\Omega .$$

Hence $u(x)$ is A-superharmonic in D. Next assume that $Au(x)>0$ at some point $x\in D$. Then there exists a domain Ω with property (S) and such that $\bar{\Omega}\subset D$ and that $Au(x)>0$ in Ω. Let $w(x)$ be the solution of the boundary value problem:

$$Aw=0 \text{ in } \Omega,\ w=u \text{ on } \partial\Omega .$$

Then, by the similar argument to above (see (3.1)), we may obtain that $u(x)<w(x)$ in Ω. Hence $u(x)$ is not A-superharmonic, q. e. d.

LEMMA 3.1. *If $u(x)$ is A-superharmonic in D and takes its minimum at an inner point of D, then $u(x)$ is constant in D.*

PROOF. We may assume that the minimum of $u(x)$ in D is zero. Suppose that $E=\{x;u(x)=0\}$ is a proper subset of D. Then there exists a point $x_0\in E$ and an domain Ω with property (S) such that $x_0\in\Omega\subset\bar{\Omega}\subset D$ and that $\Omega-E$ is a non-empty open set. Hence, by the similar arguments to proofs of Lemmas 2.2 and 2.1, we may obtain

$$u(x_0)\geqq\int_{\Omega-E}U_{\Omega}(t, x_0, y)u(y)dy>0 \qquad \text{(see (2.2))};$$

this contradicts to the fact: $x_0\in E=\{x;u(x)=0\}$.

LEMMA 3.2. *Let y be a fixed point in D, and assume that $u(x)$ is A-harmonic in $D-\{y\}$ and satisfies $\lim_{\rho\to 0}\inf_{r(x,y)<\rho}u(x)=u(y)=\infty$ where $r(x,y)$ denotes the 'Riemannian distance' defined by $\|a_{ij}(x)\|$. Then $u(x)$ is A-superharmonic in D.*

PROOF. $u(x)$ clearly satisfies i) and ii) in § 1. Let Ω be a domain with property (S) and with its closure $\bar{\Omega}\subset D$, and $w(x)$ be a function continuous on $\bar{\Omega}$, A-harmonic in Ω and satisfying $w(x)\leqq u(x)$ on $\partial\Omega$. We consider the following three cases: 1) $y\notin\bar{\Omega}$, 2) $y\in\partial\Omega$, 3) $y\in\Omega$. In case 1), $u(x)-w(x)\geqq 0$ in Ω by means of Theorem 1 and Lemma 3.1. We may reduce case 2) to case 1) by considering a monotone increasing sequence $\{\Omega_n\}$ of domains with property (S) such that $y\notin\bar{\Omega}_n$ for any n, $\lim_{n\to\infty}\Omega_n=\Omega$ and $\lim_{n\to\infty}\partial\Omega\cap\partial\Omega_n=\partial\Omega-\{y\}$, since $w(x)$ is bounded on $\bar{\Omega}$. In case 3), there exists $\rho_0>0$ such that $\inf_{r(x,y)<\rho_0}u(x)>\max_{x\in\bar{\Omega}}w(x)$. Hence, by Theorem 1 and Lemma 3.1, $u(x)-w(x)\geqq 0$ in $\Omega-\{x;r(x,y)<\rho\}$ for any $\rho<\rho_0$, and accordingly $u(x)\geqq w(x)$ in Ω (since $u(y)=\infty$ is assumed). Thus $u(x)$ satisfies iii) in § 1, q. e. d.

THEOREM 2. *The function*

$$(3.2)\qquad G(x,y)=\int_0^{\infty}U_D(t,x,y)dt \qquad (x,y\in D,\ x\neq y)$$

is well-defined and is a Green function of the elliptic differential operator A if, and only if, there exists a non-constant positive A-superharmonic function in D.

PROOF. If $G(x,y)$ $(x\neq y)$ is well-defined by (3.2), then we may show by the similar argument to that in [**3**; § 10] that $G(\cdot,y)$ is A-harmonic in $D-\{y\}$ for any fixed y. It is also clear from the construction of fundamental solutions in [**3**; §§ 3–5] that (see 1°) and 3°) in § 2 of the present paper)

$$\lim_{\rho\to 0}\inf_{r(x,y)<\rho}G(x,y)\geqq \lim_{\rho\to 0}\inf_{r(x,y)<\rho}G_{D_n}(x,y)$$

$$=\lim_{\rho\to 0}\inf_{r(x,y)<\rho}\int_0^{\infty}U_{D_n}(t,x,y)dt=\infty\,.$$

Hence, by Lemma 3.2, $G(x,y)$ is A-superharmonic in $x\in D$ for any fixed y. The 'only if' part of Theorem 2 is thus proved.

To prove the 'if' part, it is sufficient to show, under the assumption of the existence of a non-constant positive A-superharmonic function $u(x)$ in D, that

$$(3.3)\qquad \int_0^{\infty}dt\int_E U(t,x_0,y)dy<\infty$$

for any $x_0\in D$ and any compact set $E\subset D$, since, if it be proved, the existence of Green function may be shown in the entirely same way as the proof of Theorem 8 in [**3**, § 10]. By virtue of Lemma 3.1, there exist positive numbers α and β such that

$$0<\alpha<\beta<\inf_{x\in\{x_0\}\cup E}u(x)\,.$$

Let Ω_1 and Ω_2 be subdomains of D with compact closures such that

$$\bar{\Omega}_1\subset\{x\in D\,;\,u(x)<\alpha\}\quad\text{and}\quad \{x\in D\,;\,u(x)>\beta\}\supset\bar{\Omega}_2\supset\{x_0\}\cup E$$

and that $D'=D-\bar{\Omega}_1$ and $D''=D-\bar{\Omega}_2$ are domains with property (S). Then, for any $z\in\bar{\Omega}_1$,

$$\alpha>u(z)\geqq-\int_0^{\infty}d\tau\int_{\partial D''}\frac{\partial U_{D''}(\tau,z,\xi)}{\partial \boldsymbol{n}_\xi}\cdot\beta dS_\xi\geqq 0$$

by Lemma 2.2, and hence

$$(3.4)\qquad 0\leqq-\int_0^{\infty}d\tau\int_{\partial D''}\frac{\partial U_{D''}(\tau,z,\xi)}{\partial \boldsymbol{n}_\xi}dS_\xi<\frac{\alpha}{\beta}\quad\text{for any}\quad z\in\bar{\Omega}_1\,.$$

Since $u_0(x)\equiv 1$ is also A-superharmonic, we may similarly show that

$$(3.5)\qquad 0\leqq-\int_0^{\infty}d\tau\int_{\partial D'}\frac{\partial U_{D'}(\tau,x,z)}{\partial \boldsymbol{n}_z}dS_z\leqq 1\quad\text{for any}\quad x\in\bar{\Omega}_2\,.$$

Since $-\int_{\partial D'}\dfrac{\partial U_{D'}(1,y,z)}{\partial \boldsymbol{n}_z}dS_z$ is positive and continuous in $y\in D'$, we see that

$$\gamma \equiv \min_{y\in E}\left\{-\int_{\partial D'}\frac{\partial U_{D'}(1, y, z)}{\partial \boldsymbol{n}_z}dS_z\right\}$$

is positive, and hence

$$(3.6)\qquad \gamma\int_0^\infty dt\int_E U_{D'}(t, x, y)dy \leqq -\int_0^\infty dt\int_E U_{D'}(t, x, y)dy\int_{\partial D'}\frac{\partial U_{D'}(1, y, z)}{\partial \boldsymbol{n}_z}dS_z$$

$$\leqq -\int_0^\infty dt\int_{\partial D'}dS_z\int_{D'}U_{D'}(t, x, y)\frac{\partial U_{D'}(1, y, z)}{\partial \boldsymbol{n}_z}dy$$

$$\overset{3)}{=} -\int_0^\infty dt\int_{\partial D'}\frac{\partial U_{D'}(t+1, x, z)}{\partial \boldsymbol{n}_z}dS_z \leqq 1 \qquad \text{(by (3.5))}.$$

On the other hand, by Lemma 2.1, we have

$$U_D(t, x, y) = U_{D'}(t, x, y) - \int_0^t d\tau\int_{\partial D'}\frac{\partial U_{D'}(t-\tau, x, z)}{\partial \boldsymbol{n}_z}U_D(\tau, z, y)dS_z$$

for any $x, y\in\bar{\Omega}_2$ and any $t>0$, and

$$U_D(\tau, z, y) = -\int_0^\tau d\sigma\int_{\partial D''}\frac{\partial U_{D''}(\tau-\sigma, z, \xi)}{\partial \boldsymbol{n}_\xi}U_D(\sigma, \xi, y)dS_\xi$$

for any $z\in\bar{\Omega}_1$, $y\in\bar{\Omega}_2$ and any $\tau>0$. Combining these two equalities, we have

$$U_D(t, x, y) = U_{D'}(t, x, y)$$
$$+\int_0^t d\tau\int_{\partial D'}\frac{\partial U_{D'}(t-\tau, x, z)}{\partial \boldsymbol{n}_z}dS_z\int_0^\tau d\sigma\int_{\partial D''}\frac{\partial U_{D''}(\tau-\sigma, z, \xi)}{\partial \boldsymbol{n}_\xi}U_D(\sigma, \xi, y)dS_\xi$$

for any $x, y\in\bar{\Omega}_2$ and any $t>0$. Integrating both sides in y over E and then in t over $(0, T)$, and changing the order of integration, we get

$$\int_0^T dt\int_E U_D(t, x, y)dy = \int_0^T dt\int_E U_{D'}(t, x, y)dy$$
$$+\int_0^T dt\int_0^t d\tau\int_{\partial D'}\frac{\partial U_{D'}(t-\tau, x, z)}{\partial \boldsymbol{n}_z}dS_z\int_0^\tau d\sigma\int_{\partial D''}\frac{\partial U_{D''}(\tau-\sigma, z, \xi)}{\partial \boldsymbol{n}_\xi}dS_\xi\int_E U_D(\sigma,\xi,y)dy$$
$$\leqq\int_0^\infty dt\int_E U_{D'}(t, x, y)dy$$
$$+\int_0^\infty dt'\int_{\partial D'}\frac{\partial U_{D'}(t', x, z)}{\partial \boldsymbol{n}_z}dS_z\int_0^\infty d\tau'\int_{\partial D''}\frac{\partial U_{D''}(\tau', z, \xi)}{\partial \boldsymbol{n}_\xi}dS_\xi\int_0^T d\sigma\int_E U_D(\sigma, \xi, y)dy$$

for any $T>0$. If we put $\chi_T = \sup_{x\in\bar{\Omega}_2}\int_0^T dt\int_E U_D(t, x, y)dy$, then the above inequality, together with (3.4), (3.5) and (3.6), implies that $\chi_T \leqq \frac{1}{\gamma}+\frac{\alpha}{\beta}\chi_T$, and accordingly $\chi_T \leqq \frac{\beta}{\gamma(\beta-\alpha)} < \infty$; here α, β and γ are independent of T. Hence

3) This equality holds by virtue of the following property of the fundamental solution: $\int_{D'}U_{D'}(t, x, y)U_{D'}(s, y, z)dy = U_{D'}(t+s, x, z)$.

$\lim_{T\to\infty} \chi_T \leqq \frac{\beta}{\gamma(\beta-\alpha)} < \infty$; which implies (3.3).

Remark. The existence of the Green function defined by (3.2) does not necessarily imply the existence of non-constant positive *A-harmonic* function. For example, consider the case: $D=R^N$ with $N\geqq 3$ and $A=\Delta$ (Laplacian in usual sense). Then the fundamental solution $U_D(t, x, y)$ constructed with the method in 3°) of § 1 is identical with the 'Gaussian kernel' $(4\pi t)^{-N/2}\exp(-|x-y|^2/4t)$, and (3.2) and (3.3) clearly hold. However, it is well known that a positive (Δ-) harmonic function in the whole space R^N is always constant.

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