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A variational method for functions with positive real part

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§ 1. Introduction.

Variational formulas for a class of analytic functions are generally useful in solving extremal problems for the class of functions. Recently Robertson [1, 2] has investigated some extremal problems for the class of analytic functions $p(z)$ normalized so that $p(0)=1$ and with positive real part in the unit circle by making use of the variational formula

$$\begin{aligned}\frac{\delta p(z)}{\rho(1-|\alpha|^2)} &= \left(\frac{\alpha p'(z)}{p(\alpha)}-1\right)\frac{\varepsilon z}{\alpha(\alpha-z)}+\left(\frac{\alpha p(z)}{p(\alpha)}-z\right)\frac{\varepsilon z}{\alpha(\alpha-z)^2} \\ &\quad +\frac{p'(z)}{\overline{p(\alpha)}}\frac{\bar{\varepsilon}z^2}{1-\bar{\alpha}z}+\left(\frac{p(z)}{\overline{p(\alpha)}}+1\right)\frac{\bar{\varepsilon}z}{(1-\bar{\alpha}z)^2}+o(1),\end{aligned} \tag{1.1}$$

where α, ε are arbitrary complex numbers such that $|\alpha|<1$, $|\varepsilon|=1$, and ρ is a sufficiently small positive number.

This formula has been derived from the following one established by Hummel [3, 4] for the class of analytic functions $f(z)$ normalized so that $f(0)=0$, $f'(0)=1$ and starlike with respect to the origin in the unit circle:

$$\begin{aligned}\frac{\delta f(z)}{\rho(1-|\alpha|^2)} &= \varepsilon\frac{zf(z)}{\alpha(z-\alpha)}+\bar{\varepsilon}\frac{zf(z)}{1-\bar{\alpha}z}-\varepsilon\frac{f(\alpha)}{\alpha f'(\alpha)}\left(\frac{zf'(z)}{z-\alpha}+\frac{f(z)}{\alpha}\right) \\ &\quad +\bar{\varepsilon}\frac{\overline{f(\alpha)}}{\overline{\alpha f'(\alpha)}}\frac{z^2f'(z)}{1-\bar{\alpha}z}+o(1).\end{aligned} \tag{1.2}$$

In this paper we shall establish a new variational formula for the class of functions with positive real part. Some of the extremal problems taken up in the above papers by Robertson can be solved in more general forms and more directly by our new formula. This will be shown in examples of its applications.

§ 2. A variational formula for functions with positive real part.

LEMMA 1. *Let $f(z)$ be analytic in $|z|\leqq 1$ and let $|\alpha|<1$, then for $|z|<1$*

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{\xi+z}{\xi-z}\Re\frac{f(\xi)}{\xi-\alpha}d\varphi=\frac{f(z)-f(\alpha)}{z-\alpha}+\frac{z\overline{f(\alpha)}}{1-\bar{\alpha}z}-i\Im\frac{f(\alpha)-f(0)}{\alpha},\quad \xi=e^{i\varphi}. \tag{2.1}$$

PROOF. Since $[f(z)-f(\alpha)]/(z-\alpha)$ is analytic in $|z|\leqq 1$, we have

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{\xi+z}{\xi-z}\Re\frac{f(\xi)-f(\alpha)}{\xi-\alpha}d\varphi=\frac{f(z)-f(\alpha)}{z-\alpha}-i\Im\frac{f(\alpha)-f(0)}{\alpha}, \qquad |z|<1.$$

On the other hand

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{\xi+z}{\xi-z}\Re\frac{f(\alpha)}{\xi-\alpha}d\varphi=\frac{1}{2\pi}\int_0^{2\pi}\frac{\xi+z}{\xi-z}\Re\frac{\xi\overline{f(\alpha)}}{1-\bar{\alpha}\xi}d\varphi=\frac{z\overline{f(\alpha)}}{1-\bar{\alpha}z}, \qquad |z|<1.$$

Hence (2.1) follows.

We now denote by P the class of normalized functions $p(z)=1+p_1z+\cdots+p_kz^k+\cdots$ analytic and with positive real part in the unit circle. Our formula for this class is given by

THEOREM 1. *Let $p(z)\in P$. Then there exists a function $p^*(z)=p(z)+\delta p(z)$ belonging to P and with $\delta p(z)$ of the form*

$$\begin{aligned}\frac{2}{\rho}\delta p(z)&=\varepsilon\Big[p(z)\frac{1+\bar{\alpha}z}{1-\bar{\alpha}z}-\overline{p(\alpha)}\Big(p(z)-\frac{1+\bar{\alpha}z}{1-\bar{\alpha}z}\Big)-1\Big]\\ &\quad-\bar{\varepsilon}\Big[p(z)\frac{\alpha+z}{\alpha-z}+p(\alpha)\Big(p(z)-\frac{\alpha+z}{\alpha-z}\Big)-1\Big]+o(1),\end{aligned} \tag{2.2}$$

where α, ε are arbitrary complex numbers such that $|\alpha|<1$, $|\varepsilon|=1$, and ρ is a sufficiently small positive number.

PROOF. Setting $f(z)=1+\rho[\varepsilon/(1-\bar{\alpha}z)+\bar{\varepsilon}z/(z-\alpha)]$, $f(z)$ is continuous and positive on $|z|=1$ for α, ε, and ρ given above. Hence $\Re[p(rz)f(z)]$ is continuous and positive on $|z|=1$ for r in $0<r<1$. Therefore the function

$$g_r(z)=\frac{1}{2\pi}\int_0^{2\pi}\frac{\xi+z}{\xi-z}\Re[p(r\xi)f(\xi)]d\varphi, \qquad \xi=e^{i\varphi},$$

is analytic in $|z|<1$ and has a positive real part continuous in $|z|\leqq 1$. Making use of Lemma 1, we obtain

$$\begin{aligned}g_r(z)&=p(rz)+\rho\varepsilon\Big[p(rz)\frac{1}{1-\bar{\alpha}z}+\overline{p(r\alpha)}\frac{\bar{\alpha}z}{1-\bar{\alpha}z}\Big]\\ &\quad+\rho\bar{\varepsilon}\Big[p(rz)\frac{z}{z-\alpha}-p(r\alpha)\frac{\alpha}{z-\alpha}\Big]-i\rho\Im(\varepsilon+\bar{\varepsilon}p(r\alpha)).\end{aligned}$$

Let $g(z)=\lim_{r\to 1}g_r(z)$, then $g(z)$ is analytic and has a positive real part in $|z|<1$. Since $g(0)=1+\rho\Re(\varepsilon+\bar{\varepsilon}p(\alpha))>0$, the function

$$p^*(z)=\frac{g(z)}{1+\rho\Re(\varepsilon+\bar{\varepsilon}p(\alpha))}=g(z)-\rho p(z)\Re(\varepsilon+\bar{\varepsilon}p(\alpha))+o(\rho)$$

is a member of P. After a brief calculation we have $p^*(z)=p(z)+\delta p(z)$, where $\delta p(z)$ is a function of the form (2.2). Thus the theorem is proved.

From Theorem 1 we have at once

COROLLARY 1. *Let* $p(z)=1+\sum_{k=1}^{\infty}p_kz^k\in P$. *Then there exists a function* $p^*(z)=1+\sum_{k=1}^{\infty}(p_k+\delta p_k)z^k$ *belonging to* P *and with* δp_k *of the form*

$$\begin{aligned}\frac{2}{\rho}\delta p_k&=\varepsilon\Big[p_k+2\sum_{\nu=1}^{k}\bar{\alpha}^{\nu}p_{k-\nu}-\overline{p(\alpha)}(p_k-2\bar{\alpha}^k)\Big]\\&\quad-\bar{\varepsilon}\Big[p_k+2\sum_{\nu=1}^{k}\frac{1}{\alpha^{\nu}}p_{k-\nu}+p(\alpha)\Big(p_k-2\frac{1}{\alpha^k}\Big)\Big]+o(1),\qquad p_0=1,\end{aligned}\tag{2.3}$$

where α, ε *and* ρ *are given in the theorem.*

§ 3. An extremal problem connected with the coefficients of $p(z)$.

Making use of (1.1), Robertson [1] has proved without assuming Carathéodory's theorem the following.

THEOREM A. *Let* $p(z)=1+p_1z+\cdots+p_kz^k+\cdots\in P$. *Then* $|p_k|\leqq 2$ *for all* k, *and* $p_n=2$ *for a given* n *when and only when* $p(z)$ *is of the form*

$$p(z)=\frac{1+p_1z+\cdots+p_{n-1}z^{n-1}+z^n}{1-z^n},\qquad p_{n-k}=\bar{p}_k,\qquad 0<k<n.\tag{3.1}$$

On the other hand Hummel [3] has proved, making use of (1.2), the following.

THEOREM B[1]. *Let* S *be the class of normalized analytic functions starlike with respect to the origin in the unit circle, and let* $F(w_2,\cdots,w_n)$ *be analytic with respect to* $w_2,\cdots,w_n$ *in* $|w_k|\leqq k$, $k=2,\cdots,n$. *Then any function* $f(z)=z+a_2z^2+\cdots+a_kz^k+\cdots$ *in* S *which maximizes* $\Re F(a_2,\cdots,a_n)$ *must be of the form*

$$f(z)=z/\prod_{k=1}^{m}(1-\xi_kz)^{\mu_k},\qquad |\xi_k|=1,\quad \mu_k\geqq 0,\qquad \sum_{k=1}^{m}\mu_k=2,\qquad m\leqq n-1.$$

As is easily seen, Theorem B is equivalent to

THEOREM C. *Let* $F(w_1,\cdots,w_n)$ *be analytic with respect to* $w_1,\cdots,w_n$ *in* $|w_k|\leqq 2$, $k=1,\cdots,n$. *Then any function* $p(z)=1+p_1z+\cdots\cdots p_kz^k+\cdots$ *in* P *which maximizes* $\Re F(p_1,\cdots,p_n)$ *must be of the form*

$$p(z)=\sum_{k=1}^{m}\mu_k\frac{1+\xi_kz}{1-\xi_kz},\quad |\xi_k|=1,\quad \mu_k\geqq 0,\quad \sum_{k=1}^{m}\mu_k=1,\quad m\leqq n.\tag{3.2}$$

In this section we shall study the same kind of problem more in detail by applying our formula.

The above facts A, B and C, of course, can be derived immediately from Carathéodory's theorem [5] concerning the coefficient region of $p(z)$. We however do not assume the theorem of Carathéodory in the present discussion, since our purpose is to show a variational method.

1) Cf. Komatu [6].

LEMMA 2. *Let a rational function $p(z)$ be a member of P, and let $\Re p(z)=0$ for $|z|=1$. Then $p(z)$ has no poles in $|z|\neq 1$ but has at least one pole on $|z|=1$. Moreover all poles on $|z|=1$ of $p(z)$ are simple.*

PROOF. By virtue of the reflection principle $p(z)$ has no poles also in $|z|>1$. Therefore if $p(z)$ has not any pole on $|z|=1$, then $p(z)$ must be a constant. This however contradicts the assumption that $p(0)=1$ and $\Re p(e^{i\theta})=0$. Hence $p(z)$ has inevitably at least one pole on $|z|=1$. Next, let $e^{i\beta}$ be a pole of $p(z)$. If this pole is of order larger than one, then for r less than and sufficiently close to one the image curve of the arc $C: z=re^{i\theta}$, $\beta-\delta\leqq\theta\leqq\beta+\delta$, $\delta>0$, under $p(z)$ cuts the imaginary axis in at least one point. This also contradicts the assumption that $\Re p(z)>0$ for $|z|<1$. Thus the lemma is proved.

LEMMA 3. *Let $A(z)$ be a polynomial of z, and let $p(z)=A(z)/\prod_{k=1}^{m}(z-\bar{\xi}_k)$, $|\xi_k|=1$, $k=1,\cdots,m$, be a member of P. If $\Re p(z)=0$ for $|z|=1$, then $p(z)$ has a representation of the form* (3.2).

PROOF. Let the set of all poles of $p(z)$ be $\{\bar{\varepsilon}_1,\cdots,\bar{\varepsilon}_l\}$. Then from the facts stated in the preceding lemma $p(z)$ must be written in the form

$$p(z)=c+\sum_{k=1}^{l}\frac{c_k}{1-\varepsilon_k z},\qquad |\varepsilon_k|=1,$$

where each $-\bar{\varepsilon}_k c_k$ is the residue of $p(z)$ at $\bar{\varepsilon}_k$.

Since $\Re p(z)>0$ for $|z|<1$, the sign of $\Im p(z)$ on $|z|=1$ must change from negative to positive as z passes each $\bar{\varepsilon}_k$ in the positive direction. From this it follows that each c_k is real and positive, for we have $ic_k=\theta p(\bar{\varepsilon}_k e^{i\theta})+O(\theta)$ when $|\theta|$ is small. Next, because of $p(0)=1$ we have $c=1-\sum_{k=1}^{l}c_k$. Hence

$$p(z)=1+\sum_{k=1}^{l}\frac{c_k\varepsilon_k z}{1-\varepsilon_k z},\qquad c_k>0.$$

Moreover from the fact that $\Re p(z)=0$ for $|z|=1$ and $\Re[\varepsilon_k z/(1-\varepsilon_k z)]=-1/2$ for $|z|=1$, we have $\sum_{k=1}^{l}c_k=2$. Hence

$$p(z)=\sum_{k=1}^{l}\mu_k^*\frac{1+\varepsilon_k z}{1-\varepsilon_k z},\qquad \mu_k^*=\frac{c_k}{2}>0,\qquad \sum_{k=1}^{l}\mu_k^*=1.$$

On the other hand from the assumptions $\{\varepsilon_1,\cdots,\varepsilon_l\}$ is clearly a subset of $\{\xi_1,\cdots,\xi_m\}$. Consequently $p(z)$ is representable in the form (3.2).

THEOREM 2. *Let $F(w_1,\cdots,w_n)$ be analytic with respect to $w_1,\cdots,w_n$ in $|w_k|\leqq\max_{p\in P}|p_k|$, $k=1,\cdots,n$, and let $\lambda_k=F_{w_k}(p_1,\cdots,p_n)$, $k=1,\cdots,n$. If $p(z)=1+p_1z+\cdots+p_kz^k+\cdots$ is a function in P which maximizes (or minimizes) $\Re F(p_1,\cdots,p_n)$, then the coefficients $p_1,\cdots,p_n$ of $p(z)$ satisfy the equality*

$$\Re\sum_{k=1}^{n}\lambda_k p_k=\max_{|z|=1}(or\ \min)2\Re\sum_{k=1}^{n}\lambda_k z^k. \tag{3.3}$$

If further λ_k, $k=1, \cdots, n$, *are not all zero for this* $p(z)$, *then the number of points* ξ *on* $|z|=1$ *at which*

$$\Re \sum_{k=1}^{n} \lambda_k \xi^k = \max_{|z|=1} (or \min) \Re \sum_{k=1}^{n} \lambda_k z^k \tag{3.4}$$

holds for $p(z)$ *is not larger than* n, *and if we denote all of such* ξ *by* ξ_k, $k=1, \cdots, m(\leqq n)$, *then* $p(z)$ *must be of the form*

$$p(z)=\sum_{k=1}^{m} \mu_k \frac{1+\xi_k z}{1-\xi_k z}, \qquad \mu_k \geqq 0, \qquad \sum_{k=1}^{m} \mu_k = 1. \tag{3.5}$$

PROOF. It suffices to prove the theorem only for the maximum problem. Since P is compact, there exists the function $p(z)$ which maximizes $\Re F(p_1, \cdots, p_n)$ in P. From (2.3) we have

$$\begin{aligned}\frac{2}{\rho}\delta\Re F(p_1, \cdots, p_n) &= \frac{2}{\rho}\Re \sum_{k=1}^{n} \lambda_k \delta p_k + o(1) \\ &= \Re\Big[\bar{\varepsilon}\Big\{\sum_{k=1}^{n} \bar{\lambda}_k\Big(\bar{p}_k + 2\sum_{\nu=1}^{k} \alpha^\nu \bar{p}_{k-\nu} - p(\alpha)(\bar{p}_k - 2\alpha^k)\Big) \\ &\quad - \sum_{k=1}^{n} \lambda_k\Big(p_k + 2\sum_{\nu=1}^{k}\frac{1}{\alpha^\nu} p_{k-\nu} + p(\alpha)\Big(p_k - 2\frac{1}{\alpha^k}\Big)\Big)\Big\}\Big] + o(1).\end{aligned}$$

Since $\delta\Re F(p_1, \cdots, p_n) \leqq 0$ for arbitrary ε and α such that $|\varepsilon|=1$, $|\alpha|<1$, we have on replacing α by z

$$p(z)=\frac{A(z)}{B(z)}, \qquad |z|<1,$$

if $B(z) \not\equiv 0$, where

$$A(z)=\sum_{k=1}^{n}\Big\{-i\Im(\lambda_k p_k) + \sum_{\nu=1}^{k}\Big(\bar{\lambda}_k \bar{p}_{k-\nu} z^\nu - \lambda_k p_{k-\nu}\frac{1}{z^\nu}\Big)\Big\},$$

$$B(z)=\sum_{k=1}^{n}\Big\{\Re(\lambda_k p_k) - \Big(\bar{\lambda}_k z^k + \lambda_k \frac{1}{z^k}\Big)\Big\}.$$

We here remark that $B(z)\equiv 0$ occurs only when $\lambda_1 = \cdots = \lambda_n = 0$. Since on $|z|=1$ $A(z)$ is pure imaginary and $B(z)$ is real, $p(z)$ is a rational function satisfying $\Re p(z)=0$ for $|z|=1$. Therefore by Lemma 2 $B(z)$ has at least one zero on $|z|=1$ in view of the fact that $A(z)$ has no poles on $|z|=1$. Hence

$$\min_{|z|=1} B(z) \leqq 0. \tag{3.6}$$

Next since the function $\phi(z)=(1+\bar{\alpha}z)/(1-\bar{\alpha}z)$, $|\alpha|<1$, is a member of P, if we set

$$p^*(z)=\frac{p(z)+\rho\phi(z)}{1+\rho}=p(z)+\rho(\phi(z)-p(z))+o(\rho),$$

then $p^*(z)$ is also a member of P. For this variation we have $\delta p_k = \rho(2\bar{\alpha}^k - p_k) + o(\rho)$, and $p(z)$ must satisfy also

$$\delta\Re F(p_1, \cdots, p_n) = \rho\Re\sum_{k=1}^{n}\lambda_k(2\bar{\alpha}^k - p_k) + o(\rho) \leqq 0\,,$$

so that

$$\Re\sum_{k=1}^{n}(\lambda_k p_k - 2\lambda_k\bar{\alpha}^k) \geqq 0\,, \qquad |\alpha| < 1\,.$$

Letting $|\alpha| \to 1$, we have $B(\alpha) \geqq 0$ for $|\alpha| = 1$, namely

(3.7) $$B(z) \geqq 0 \quad \text{for} \quad |z| = 1\,,$$

which combined with (3.6) yields $\min_{|z|=1} B(z) = 0$, i. e.,

$$\sum_{k=1}^{n}\Re(\lambda_k p_k) - \max_{|z|=1}\sum_{k=1}^{n} 2\Re(\lambda_k \bar{z}^k) = 0\,.$$

Hence (3.3) follows. Therefore we see further that if we denote by $\{\xi_1, \cdots, \xi_m\}$ the set of all points ξ on $|z| = 1$ at which (3.4) holds, then the set of their conjugates $\{\bar{\xi}_1, \cdots, \bar{\xi}_m\}$ coincides with that of all zeros on $|z| = 1$ of $B(z)$, where each multiple zero is taken into consideration only once. On the other hand $B(z)$ is of the form $B_1(z)/z^n$, where $B_1(z)$ is a polynomial of degree at most $2n$, and furthermore each zero on $|z| = 1$ of $B(z)$, namely that of $B_1(z)$, is of even order on account of (3.7). Hence $m \leqq n$.

Finally from Lemma 2 $p(z)$ has poles only on $|z| = 1$ in the whole plane and these poles are all simple. Therefore $p(z)$ must reduce to the form

$$p(z) = A_0(z) / \sum_{k=1}^{m}(z - \bar{\xi}_k)\,,$$

where $A_0(z)$ is a polynomial which may have some $z - \bar{\xi}_k$ as its factors. Thus by Lemma 3 $p(z)$ is representable in the form (3.5). This completes the proof.

COROLLARY 2. *Let λ_k, $k = 1, \cdots, n$, be constants. Then we have*

(3.8) $$\max_{p \in P}\,(or\ \min)\,\Re\sum_{k=1}^{n}\lambda_k p_k = \max_{|z|=1}\,(or\ \min)\,2\Re\sum_{k=1}^{n}\lambda_k z^k\,.$$

If further λ_k, $k = 1, \cdots, n$, are not all zero, then a necessary and sufficient condition that $p(z)$ be a function in P which maximizes (or minimizes) $\Re\sum_{k=1}^{n}\lambda_k p_k$ is that $p(z)$ is of the form (3.5) where $\{\xi_1, \cdots, \xi_m\}$, $m \leqq n$, is the set of all points ξ on $|z| = 1$ at which (3.4) holds.

PROOF. From the theorem the coefficients $p_1, \cdots, p_n$ of $p(z)$ which maximizes (or minimizes) $\Re\sum_{k=1}^{n}\lambda_k p_k$ in P satisfy (3.3). Hence (3.8) holds. Next it is clear from the theorem that the extremal function $p(z)$ must be of the form (3.5) with ξ_k defined in this corollary. Therefore it suffices to prove the converse. Let $p(z) = 1 + p_1 z + \cdots + p_k z^k + \cdots$ be a function of the form (3.5) with ξ_k defined in this corollary, then clearly $p(z) \in P$, and

$$\Re\sum_{k=1}^{n}\lambda_k p_k=\sum_{\nu=1}^{m}2\mu_\nu\Re\sum_{k=1}^{n}\lambda_k\xi_\nu^k=\sum_{\nu=1}^{m}\mu_\nu\max_{|z|=1}(or\ \min)\,2\Re\sum_{k=1}^{n}\lambda_k z^k$$

$$=\max_{p\in P}(\text{or }\min)\,\Re\sum_{k=1}^{n}\lambda_k p_k$$

on account of (3.4) and (3.8). Hence the converse holds, and so the proof is completed.

From this corollary we have at once

COROLLARY 3. *We have* $\max_{p\in P}(or\ \min)p_n=2(or\ -2)$. *Moreover a necessary and sufficient condition that* $p(z)$ *be a function in* P *which satisfies* $p_n=2(or\ -2)$ *for a given* n *is that* $p(z)$ *is of the form*

$$p(z)=\sum_{k=1}^{n}\mu_k\frac{1+\xi_k z}{1-\xi_k z},\qquad \xi_k^n=1(or\ -1),\qquad \mu_k\geqq 0,\qquad \sum_{k=1}^{n}\mu_k=1, \tag{3.9}$$

where ξ_k, $k=1,\cdots,n$, *are* n *roots of the equation* $\xi^n=1(or\ -1)$.

Corollary 3 is equivalent to Theorem A, for any function $p(z)$ given by (3.1) can be written in the form (3.9) if $p(z)\in P$, and conversely any function $p(z)$ given by (3.9) can be written in the form (3.1).

§ 4. Another extremal problem connected with the values of $p(z)$ and its derivatives.

Robertson [2] has proved, making use of (1.1), the following.

THEOREM D. *Let* $F(w)$ *be analytic in* $\Re w>0$. *Then on* $|z|=r<1$

$$\min_{p\in P}\Re F(p(z))=\min_{|z|=r}\Re F\left(\frac{1+z}{1-z}\right).$$

THEOREM E. *Let* $F(w_0,w_1)$ *be analytic with respect to* w_0,w_1 *in* $\Re w_0>0$, $|w_1|<+\infty$. *Then on* $|z|=r<1$

$$\min_{p\in P}\Re F(p(z),zp'(z))=\min_{\alpha,\theta,\phi}\Re F(p_0(z),zp_0'(z)),$$

where

$$p_0(z)=\frac{1+\alpha}{2}\left(\frac{1+ze^{i\theta}}{1-ze^{i\theta}}\right)+\frac{1-\alpha}{2}\left(\frac{1+ze^{-i\theta}}{1-ze^{-i\theta}}\right),\qquad z=re^{i\phi},$$

$$-1\leqq\alpha\leqq 1,\qquad 0\leqq\theta\leqq 2\pi,\qquad 0\leqq\phi\leqq 2\pi.$$

Our formula permits us to solve this kind of problem in a more general form as follows.

THEOREM 3. *Let* $F(w_0,w_1,\cdots,w_n)$ *be analytic with respect to* $w_0,w_1,\cdots,w_n$ *in* $\Re w_0>0$, $|w_k|<+\infty$, $k=1,\cdots,n$, *and let* $\lambda_k=F_{w_k}(p(c),p'(c),\cdots,p^{(n)}(c))$, $k=0,\cdots,n$, $|c|<1$. *If* $p(z)$ *is a function in* P *which maximizes (or minimizes)* $\Re F(p(c),p'(c),\cdots,p^{(n)}(c))$ *for a given* c *in* $|z|<1$, *then* $p(z)$ *satisfies*

$$\Re\sum_{k=0}^{n}\lambda_k p^{(k)}(c)=\max_{|z|=1}(or\ \min)\,\Re\sum_{k=0}^{n}\lambda_k\phi^{(k)}(c,z)\,, \tag{4.1}$$

where

$$\phi(c,z)=\frac{1+cz}{1-cz}\,,\qquad \phi^{(k)}(c,z)=\Big[\frac{d^k}{d\zeta^k}\phi(\zeta,z)\Big]_{\zeta=c}=k\,!\frac{2z^k}{(1-cz)^{k+1}}\,,\qquad k\geqq 1\,.$$

If further λ_k, $k=0,\cdots,n$, are not all zero for this $p(z)$, then the number of points ξ on $|z|=1$ at which

$$\Re\sum_{k=0}^{n}\lambda_k\phi^{(k)}(c,\xi)=\max_{|z|=1}(or\ \min)\,\Re\sum_{k=0}^{n}\lambda_k\phi^{(k)}(c,z) \tag{4.2}$$

holds for $p(z)$ is not larger than $n+1$, and if we denote all of such ξ by ξ_k, $k=0,\cdots,m(\leqq n)$, then $p(z)$ must be of the form

$$p(z)=\sum_{k=0}^{m}\mu_k\frac{1+\xi_k z}{1-\xi_k z}\,,\qquad \mu_k\geqq 0\,,\qquad \sum_{k=0}^{m}\mu_k=1\,. \tag{4.3}$$

PROOF. Our proof will be carried forward in parallel with that of Theorem 2. It suffices to prove the theorem only for the maximum problem. .Since P is compact, there exists the function $p(z)$ which maximizes $\Re F(p(c),\cdots,p^{(n)}(c))$ in P. If we set

$$\phi(z)=\phi(z,\bar\alpha)=\frac{1+z\bar\alpha}{1-z\bar\alpha}\,,\qquad \psi(z)=\psi(z,\alpha)=\frac{\alpha+z}{\alpha-z}\,,$$

$$\phi_1(z)=\phi_1(z,\bar\alpha)=p(z)\phi(z,\bar\alpha),\qquad \psi_1(z)=\psi_1(z,\alpha)=p(z)\psi(z,\alpha)\,,$$

then from (2.2) we have

$$\frac{2}{\rho}\delta p(z)=\varepsilon[\phi_1(z)-\overline{p(\alpha)}(p(z)-\phi(z))-1]$$
$$-\bar\varepsilon[\psi_1(z)+p(\alpha)(p(z)-\psi(z))-1]+o(1)\,,$$

and hence

$$\frac{2}{\rho}\delta\Re F(p(c),\cdots,p^{(n)}(c))=\frac{2}{\rho}\Re\sum_{k=0}^{n}\lambda_k\delta p^{(k)}(c)+o(1)$$
$$=\Re[\bar\varepsilon\{\lambda_0-\bar\lambda_0+\sum_{k=0}^{n}(\bar\lambda_k\overline{\phi_1^{(k)}(c)}-\lambda_k\psi_1^{(k)}(c))$$
$$-p(\alpha)\sum_{k=0}^{n}(\bar\lambda_k\overline{p^{(k)}(c)}+\lambda_k p^{(k)}(c)-\bar\lambda_k\overline{\phi^{(k)}(c)}-\lambda_k\psi^{(k)}(c))\}]+o(1)\,.$$

Since $\delta\Re F(p(c),\cdots,p^{(n)}(c))\leqq 0$ for arbitrary ε and α such that $|\varepsilon|=1$, $|\alpha|<1$, we have on replacing α by z

$$p(z)=\frac{A(z)}{B(z)}\,,\qquad |z|<1\,,$$

if $B(z)\not\equiv 0$, where

$$A(z)=\lambda_0-\bar\lambda_0+\sum_{k=0}^{n}\{\bar\lambda_k\overline{\phi_1^{(k)}(c,\bar z)}-\lambda_k\psi_1^{(k)}(c,z)\}\,,$$

$$B(z)=\sum_{k=0}^{n}\{2\Re(\lambda_k p^{(k)}(c))-(\bar{\lambda}_k\overline{\phi^{(k)}(c,\bar{z})}+\lambda_k\psi^{(k)}(c,z))\}\,,$$

and (k) denotes the k-th derivative with respect to the first variable. We here remark that $B(z)\equiv 0$ occurs too only when $\lambda_0\cdots=\lambda_n=0$, and both $A(z)$ and $B(z)$ are rational fractions with denominator $[(1-\bar{c}z)(z-c)]^{n+1}$.

Since $\psi(z,\alpha)=\phi(z,1/\alpha)$ and $\psi_1(z,\alpha)=\phi_1(z,1/\alpha)$, we have for $|z|<1$, $|\alpha|=1$

$$\psi^{(k)}(z,\alpha)=\phi^{(k)}(z,1/\alpha)=\phi^{(k)}(z,\bar{\alpha})\,,$$
$$\psi_1^{(k)}(z,\alpha)=\phi_1^{(k)}(z,1/\alpha)=\phi_1^{(k)}(z,\bar{\alpha})\,.$$

From this we see that on $|z|=1$ $A(z)$ is pure imaginary while $B(z)$ is real. Hence $\Re p(z)=0$ for $|z|=1$. Therefore by Lemma 2 $B(z)$ has at least one zero on $|z|=1$ in view of the fact that $A(z)$ has no poles on $|z|=1$, so that

$$\min_{|z|=1} B(z)\leqq 0\,. \tag{4.4}$$

Next since $\phi(z)\in P$, the function

$$p^*(z)=\frac{p(z)+\rho\phi(z)}{1+\rho}=p(z)+\rho(\phi(z)-p(z))+o(\rho)$$

is also a member of P. For this variation, $p(z)$ must satisfy

$$\delta\Re F(p(c),\cdots,p^{(n)}(c))=\rho\Re\sum_{k=0}^{n}\lambda_k(\phi^{(k)}(c)-p^{(k)}(c))+o(\rho)\leqq 0$$

too, so that

$$\Re\sum_{k=0}^{n}\{\lambda_k p^{(k)}(c)-\lambda_k\phi^{(k)}(c,\bar{\alpha})\}\geqq 0\,,\qquad |\alpha|<1\,.$$

Letting $|\alpha|\to 1$, we have $B(\alpha)\geqq 0$ for $|\alpha|=1$, namely

$$B(z)\geqq 0 \qquad \text{for} \qquad |z|=1\,, \tag{4.5}$$

which combined with (4.4) yields $\min_{|z|=1} B(z)=0$, i. e.,

$$\Re\sum_{k=0}^{n}\lambda_k p^{(k)}(c)-\max_{|z|=1}\Re\sum_{k=0}^{n}\lambda_k\phi^{(k)}(c,\bar{z})=0\,.$$

Hence (4.1) follows. Therefore we see further that if we denote by $\{\xi_0,\cdots,\xi_m\}$ the set of all points ξ on $|z|=1$ at which (4.2) holds, then the set of their conjugates $\{\bar{\xi}_0,\cdots,\bar{\xi}_m\}$ coincides with that of all zeros on $|z|=1$ of $B(z)$, where each multiple zero is taken into consideration only once. On the other hand $B(z)$ is of the form $B_1(z)/[(1-\bar{c}z)(z-c)]^{n+1}$, where $B_1(z)$ is a polynomial of degree at most $2(n+1)$, and furthermore each zero on $|z|=1$ of $B(z)$, namely that of $B_1(z)$, is of even order on account of (4.5). Hence $m\leqq n$.

The last part of the theorem also can be proved just as in the proof of Theorem 2.

COROLLARY 4. *Let λ_k, $k=0,\cdots,n$, be constants. Then for c given in $|z|<1$ we have*

$$\max_{p\in P}(or\ \min)\,\Re\sum_{k=0}^{n}\lambda_k p^{(k)}(c)=\max_{|z|=1}(or\ \min)\,\Re\sum_{k=0}^{n}\lambda_k\phi^{(k)}(c,z)\,. \tag{4.6}$$

If further λ_k, $k=0,\cdots,n$, *are not all zero, then a necessary and sufficient condition that* $p(z)$ *be a function in* P *which maximizes (or minimizes)* $\Re\sum_{k=0}^{n}\lambda_k p^{(k)}(c)$ *is that* $p(z)$ *is of the form* (4.3) *where* $\{\xi_0,\cdots,\xi_m\}$, $m\leqq n$, *is the set of all points* ξ *on* $|z|=1$ *at which* (4.2) *holds.*

This corollary can be proved in the same way as used in the proof of Corollary 2.

Theorem D is a special case of Theorem 3. We shall finally show another corollary which is a generalization of Theorem E.

COROLLARY 5. *Let* $F(w_0,w_1,\cdots,w_n)$ *be analytic with respect to* $w_0,w_1,\cdots,w_n$ *in* $\Re w_0>0$, $|w_k|<+\infty$, $k=1,\cdots,n$. *Then for* r *in* $0\leqq r<1$ *we have*

$$\begin{aligned}&\max_{|z|=r,p\in P}(or\ \min)\,\Re F(p(z),zp'(z),\cdots,z^n p^{(n)}(z))\\&\qquad=\max_{\mu_k,\xi_k}(or\ \min)\,\Re F(p_0(r),rp_0'(r),\cdots,r^n p_0^{(n)}(r))\,,\end{aligned} \tag{4.7}$$

where

$$p_0(z)=\sum_{k=0}^{n}\mu_k\frac{1+\xi_k z}{1-\xi_k z},\qquad |\xi_k|=1\,,\qquad \mu_k\geqq 0\,,\qquad \sum_{k=0}^{n}\mu_k=1\,. \tag{4.8}$$

Moreover any function $p(z)$ *in* P *which maximizes (or minimizes)* $\max_{|z|=r}(or\ \min)$ $\Re F(p(z),zp'(z),\cdots,z^n p^{(n)}(z))$ *must be of the form* (4.8).

PROOF. It suffices to prove the corollary for the maximum problem. Let $G(w_0,w_1,\cdots,w_n)=F(w_0,rw_1,\cdots,r^n w_n)$. Then we have

$$\begin{aligned}&\max_{p\in P}\Re G(p(r),p'(r),\cdots,p^{(n)}(r))\\&\qquad=\max_{|z|=r,p\in P}\Re F(p(z),zp'(z),\cdots,z^n p^{(n)}(z))\end{aligned} \tag{4.9}$$

from the fact that if $p(z)\in P$, then $p_1(z)=p(e^{i\theta}z)\in P$ and $r^k p_1^{(k)}(r)=(re^{i\theta})^k p^{(k)}(re^{i\theta})$ for an arbitrary real θ. On the other hand from Theorem 3 any function $p(z)$ which maximizes $\Re G(p(r),p'(r),\cdots,p^{(n)}(r))$ must be of the form (4.8). Moreover every function $p_0(z)$ given by (4.8) is clearly a member of P. Hence (4.7) holds. Next let $p(z)$ be a function in P which satisfies for $z_0=re^{i\beta}$

$$\Re F(p(z_0),\cdots,z_0^n p^{(n)}(z_0))=\max_{|z|=r,p\in P}\Re F(p(z),\cdots,z^n p^{(n)}(z))\,,$$

then because of (4.9) we have, for $p_2(z)=p(e^{i\beta}z)\in P$,

$$\Re G(p_2(r),\cdots,p_2^{(n)}(r))=\max_{p\in P}\Re G(p(r),\cdots,p^{(n)}(r))\,.$$

Therefore from Theorem 3 $p_2(z)$ must be of the form (4.8), so that the function $p(z)=p_2(e^{-i\beta}z)$ also must be of the same form. This completes the proof.

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