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Semi-groups of operators in locally convex spaces

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The purpose of this paper is to extend some of the results in the theory of semi-groups of operators in Banach spaces to the case of locally convex topological vector spaces.

We consider a vector space $\mathfrak{X}$ with two locally convex topologies τ and σ which satisfy the conditions: (T 1) τ is stronger than σ; (T 2) τ has a base of neighborhoods of 0 composed of convex, circled and σ-closed sets; (T 3) $\mathfrak{X}_\tau$ is sequentially complete; (T 4) every continuous function $f(t)$ from $[0, 1]$ to $\mathfrak{X}_\sigma$ is Riemann integrable in σ. We shall call a family of linear operators $\{T_t\}_{t\geqq 0}$ in $\mathfrak{X}$ a (τ, σ) semi-group if it satisfies the conditions: (S 1) $T_0=I$ (identity); (S 2) $T_{t+s}=T_tT_s$; (S 3) $\{T_t\}$ is equicontinuous in $\mathfrak{L}(\mathfrak{X}_\tau, \mathfrak{X}_\tau)$; (S 4) T_t is, for every t, a σ-sequentially closed operator; (S 5) T_tx is a σ-continuous function for every x.

The well-known Hille-Yosida theory deals with the case $\tau=\sigma$ and when $\mathfrak{X}_\tau$ is a Banach space. The results have been generalized by Schwartz [8] when $\mathfrak{X}_\tau$ is a quasi-complete locally convex space. The theory in the case when $\mathfrak{X}$ is an adjoint space of a Banach space, τ is the strong topology and σ is the weak* topology is known as the theory of adjoint semi-groups by Feller [2] and Phillips [7].

In § 1 we give several sufficient conditions to assure the above assumptions. Especially it is shown that if $\mathfrak{X}_\tau$ is quasi-complete and if $\{T_t\}$ satisfies (S 1)—(S 3) and the condition that T_tx converges weakly to x as $t\to 0$ for every x, then $\{T_t\}$ is a (τ, τ) semi-group. § 2 is of preliminary nature.

The infinitesimal generators A_τ and A_σ are defined as usual by

$$A_\tau x=\tau\text{-}\lim_{t\to 0}\frac{1}{t}(T_t-I)x \quad \text{and} \quad A_\sigma x=\sigma\text{-}\lim_{t\to 0}\frac{1}{t}(T_t-I)x\,.$$

Thanks to the above assumptions, we can show that the Laplace transform

$$R(\lambda)x=\int_0^\infty e^{-\lambda t}T_txdt\,, \qquad \mathrm{Re}\,\lambda>0$$

is convergent as an improper σ-Riemann integral. $\{(\lambda R(\lambda))^m\}$ is equicontinuous in $\mathfrak{L}(\mathfrak{X}_\tau, \mathfrak{X}_\tau)$ with $\{T_t\}$ and $R(\lambda)$ is the resolvent of a τ-closed linear operator A, which we call the generator of $\{T_t\}$. We are mainly concerned in § 3 with $R(\lambda)$, A and its relationship with A_τ and A_σ, and prove that the relation $A_\tau \subset A \subset A_\sigma$ holds and the last two coincide if T_t is a σ-weakly continuous operator for every t. We seek also for the condition to certify the σ-weak continuity of $R(\lambda)$, which is used in § 7 to define dual semi-groups.

The domain $\mathfrak{D}(A)$ of A is σ-dense, but it is not necessarily τ-dense. We prove in § 4 that $T_t x$ is τ-continuous for $t \geqq 0$ if and only if $x \in \mathfrak{D}(A)^\tau$, the τ-closure of $\mathfrak{D}(A)$, and that the restriction S_t of T_t to $\mathfrak{D}(A)^\tau$ becomes a (τ, τ) semi-group. It will be of some interest that the generator A_0 of S_t coincides with A_τ. § 5 is again of preliminary nature.

A complete characterization of the generators of (τ, τ) semi-groups has been given by Hille, Yosida and Schwartz. In § 6 we give a simple proof for it and a generalization to a special class of (τ, σ) semi-groups including the duals of strongly continuous semi-groups in Banach spaces.

The notion of adjoint semi-groups is generalized in § 7, and is used to prove the uniqueness of the solution of the equation

$$\frac{d}{dt}x(t) = Ax(t) \qquad \text{with} \quad x(0) = x_0 .$$

It is proved that a τ-continuously differentiable solution is unconditionally unique and that if the dual space $\mathfrak{X}'_\sigma$ is quasi-complete relative to the Mackey topology $\tau(\mathfrak{X}'_\sigma, \mathfrak{X})$ and if T_t is a σ-weakly continuous operator for every t, then even a σ-weakly continuously differentiable solution is unique. (We note that this is the case for the duals of strongly continuous semi-groups in Banach spaces.) Previously we know that $x(t) = T_t x_0$ gives a τ-continuously differentiable solution if $x_0 \in \mathfrak{D}(A_0)$, and a σ-continuously differentiable solution if $x_0 \in \mathfrak{D}(A)$. The above theorem proves their uniqueness.

We develop in § 8 the general theory of analytic functions with values in locally convex spaces, and finally in § 9 we investigate the analytic semi-group T_t such that T_t is analytically extendable to the sector $|\arg t| < \theta$ and give various characterizations. This part considerably overlaps Yosida [**10**].

1. (τ, σ) semi-group of operators.

Let $\mathfrak{X}$ be a vector space over the complex number field. We assume that $\mathfrak{X}$ admits two locally convex topological structures τ and σ which satisfy the following conditions:

(T 1) τ is stronger than σ.

(T 2) There is a fundamental system of τ-neighborhoods of 0 composed of convex, circled and σ-closed sets.

(T 3) $\mathfrak{X}$ is sequentially complete relative to the topology τ.

(T 4) Every σ-continuous $\mathfrak{X}$-valued function $f(t)$ on the interval [0, 1] is

Riemann integrable in σ, i.e., the Riemann sum

$$S(f, \Delta)=\sum_{i=1}^{n} f(t_i')(t_i-t_{i-1})$$

$$\Delta: \quad 0=t_0 \leqq t_1' \leqq t_1 \leqq \cdots \leqq t_{n-1} \leqq t_n' \leqq t_n=1$$

converges to an element $x \in \mathfrak{X}$ relative to σ when Δ becomes finer.

(T 3) may be replaced by a weaker condition that for every convex circled σ-closed and τ-bounded set B the normed space $\mathfrak{X}_B$ generated by B is complete. (T 3) will be used only in this form.

We shall give several sufficient conditions to assure (T 1)—(T 4) in the following three propositions.

PROPOSITION 1.1. *If* (T 1) *and* (T 2) *are satisfied and if* $\mathfrak{X}$ *is sequentially complete relative to* σ, *then* (T 3) *and* (T 4) *hold.*

PROOF. Let x_n be a τ-Cauchy sequence. By (T 1) x_n is also a σ-Cauchy sequence. Thus it converges to an element x relative to σ. Take an arbitrary σ-closed τ-neighborhood V of 0. By assumption $x_n-x_m \in V$ holds for sufficiently large m and n. Letting m tend to infinity, we obtain $x_n-x \in V$ by (T 2). This proves that $x_n \rightarrow x$ relative to τ.

To prove (T 4) let $f(t)$ be a σ-continuous function on $[0,1]$. Since $f(t)$ is uniformly continuous, the totality of the Riemann sums $S(f, \Delta)$ forms a σ-Cauchy net, and $S(f, \Delta_n)=1/n \sum f(i/n)$, $n=1,2, \cdots$, an equivalent Cauchy sequence. Since $\mathfrak{X}$ is σ-sequentially complete, $S(f, \Delta_n)$ converges. Hence $f(t)$ is Riemann integrable.

PROPOSITION 1.2. *If* $\mathfrak{X}$ *is quasi-complete* (*relative to the Mackey topology*), *then the original topology* τ *and the weak topology* σ *satisfy* (T 1)—(T 4).

PROOF. (T 1), (T 2) and (T 3) are evident.

As for (T 4) a general result is known ([**3**], Chap. V, § 3. 3, Proposition 4), but we give here a direct proof for convenience sake.

If $f(t)$ is a weakly continuous $\mathfrak{X}$-valued function on $[0,1]$, then

$$F(x')=\int_0^1 \langle f(t), x' \rangle \, dt$$

converges for every $x' \in \mathfrak{X}'$ (the dual space of $\mathfrak{X}$) and represents a linear form on $\mathfrak{X}'$. We shall show that $F(x')=\langle x, x' \rangle$ for an $x \in \mathfrak{X}$. Then it will easily follow that x is the weak limit of the Riemann sums.

The range $\{f(t)\}$ forms a weakly compact and separable subset in $\mathfrak{X}$. Therefore it generates a separable closed subspace $\mathfrak{X}_f$ of $\mathfrak{X}$. $\mathfrak{X}_f$ is quasi-complete with $\mathfrak{X}$. And the weak topology of a closed subspace $\mathfrak{X}_f$ coincides with the induced topology from the weak topology of $\mathfrak{X}$. Thus we may assume that $\mathfrak{X}$ itself is separable and quasi-complete without loss of generality.

For a moment let $\mathfrak{X}$ be complete. Then by a theorem of Banach (cf. [**1**], Chap. IV, § 2 Théorème 4 and Remarque after the theorem.) a linear form $F(x')$ on the dual space $\mathfrak{X}'$ is identified with an element $x \in \mathfrak{X}$ if its restriction to every equicontinuous set in $\mathfrak{X}'$ is weakly* continuous, or since $\mathfrak{X}$ is separable, if for every sequence $x'_n \in \mathfrak{X}'$ which converges weakly* to an element x' we have $F(x'_n) \to F(x')$. But this immediately follows from Lebesgue's theorem.

When $\mathfrak{X}$ is not complete, the Riemann sum converges weakly to an element x in the completion $\hat{\mathfrak{X}}$ of $\mathfrak{X}$. x lies in the closed convex hull in $\hat{\mathfrak{X}}$ of the range $\{f(t)\}$. However, $\mathfrak{X}$ being quasi-complete, the closed convex hull in $\mathfrak{X}$ of a bounded set is complete, and hence closed in $\hat{\mathfrak{X}}$, i. e., it is also the closed convex hull in $\hat{\mathfrak{X}}$. This proves $x \in \mathfrak{X}$.

PROPOSITION 1.3. *Let $\mathfrak{X} = \mathfrak{Y}'$ be the dual space of a tonnelé space $\mathfrak{Y}$. Then the strong topology τ and the weak* topology σ satisfy the conditions* (T 1)—(T 4).

PROOF. (T 1) is clear. The set of the polar sets B° of bounded sets B in $\mathfrak{Y}$ forms a fundamental system of τ-neighborhoods of 0. B° is clearly weakly* closed. Hence (T 2) holds. By the Banach-Steinhaus theorem $\mathfrak{X}$ is quasi-complete, hence sequentially complete relative to the weak* topology σ. This implies (T 3), and (T 4) by Proposition 1.1.

The assumption in the proposition may be a little relaxed. Let us call a locally convex space $\mathfrak{Y}$ *countably tonnelé* if every barrel (closed convex and circled subset which absorbs every point in $\mathfrak{Y}$) which is the intersection of a countable family of neighborhoods of 0 is a neighborhood of 0. Then it is shown by a standard argument that every weak* Cauchy sequence x_n in $\mathfrak{Y}'$ is equicontinuous and thus converges to an element in $\mathfrak{Y}'$. (cf. Proposition 8.9.)

DEFINITION. We shall call a system of linear operators T_t, $t \geqq 0$, in $\mathfrak{X}$ a *σ-continuous semi-group of τ-equicontinuous operators*, or a *(τ, σ) semi-group* for short, if it satisfies the following conditions:

(S 1) $T_0 = I =$ the identity operator.

(S 2) $T_{t+s} = T_t T_s$ for every t, $s \geqq 0$.

(S 3) $\{T_t\}_{t \geqq 0}$ is equicontinuous in $\mathfrak{L}(\mathfrak{X}_\tau, \mathfrak{X}_\tau)$.

(S 4) T_t is for every $t \geqq 0$ a sequentially closed operator relative to σ, i. e., $x_n \to x(\sigma)$ and $y_n = T_t x_n \to y(\sigma)$ imply $y = T_t x$.

(S 5) $\sigma\text{-}\lim_{s \to t} T_s x = T_t x$ for every $x \in \mathfrak{X}$ and every $t \geqq 0$.

(S 3) means that for any τ-neighborhood V of 0 there is another τ-neighborhood U of 0 such that $T_t(U) \subset V$ for all t. Since we can take for U and V convex circled and σ-closed subsets, (S 3) has an alternative expression that for any τ-continuous and σ-lower semi-continuous semi-norm $p(x)$ on $\mathfrak{X}$ (we shall call such a semi-norm a *(τ, σ) semi-norm.)*, there is another (τ, σ) semi-norm $q(x)$ such that

$$p(T_t x) \leqq q(x) \quad \text{for all} \quad t \geqq 0 \quad \text{and all} \quad x \in \mathfrak{X}.$$

(S 4) is certainly satisfied if T_t is σ-continuous, or if σ is stronger than the weak topology associated with τ.

In certain circumstances the condition (S 5) may be replaced by a weaker one. The following three propositions give such conditions.

PROPOSITION 1.4. *Let a semi-group of operators* $\{T_t\}$ *in* $\mathfrak{X}$ *satisfy* (S 1)—(S 3) *and*

(S 5)$'_\tau$ $\tau\text{-}\lim_{t\to 0} T_t x = x$ *for every* x *in a dense set* $\mathfrak{D}$ *of* $\mathfrak{X}$.

Then (S 5) *holds with* σ *replaced by* τ.

PROOF. Let $x \in \mathfrak{D}$. We have

$$T_t x - T_s x = T_s(T_{t-s}x - x) \quad \text{or} \quad = T_t(x - T_{s-t}x)$$

according as $t \geqq s$ or $t < s$. When s tends to t, the term in the bracket tends to 0 relative to τ. Thus from the equi-continuity of T_t it follows that $T_t x - T_s x \to 0$. Next let x be an arbitrary element of $\mathfrak{X}$. For any τ-continuous semi-norm $p(x)$ and $\varepsilon > 0$, we can find a $y \in \mathfrak{D}$ such that $p(T_t x - T_t y) \leqq \varepsilon$ for all t. We have

$$\begin{aligned} p(T_t x - T_s x) &\leqq p(T_t x - T_t y) + p(T_t y - T_s y) + p(T_s y - T_s x) \\ &\leqq 2\varepsilon + p(T_t y - T_s y). \end{aligned}$$

This proves that $T_t x$ is τ-continuous in t.

PROPOSITION 1.5. *If a semi-group of operators* $\{T_t\}$ *satisfies* (S 1)—(S 3) *and*

(S 5)$''$ $\sigma\text{-}\lim_{t\to 0} T_t x = x$ *uniformly on every* τ*-bounded set in* $\mathfrak{X}$,

then (S 5) *holds.*

PROOF. This easily follows from the identity

$$T_t x - T_s x = (T_{t-s} - I)T_s x \quad \text{or} \quad = (I - T_{s-t})T_t x$$

and the boundedness of $\{T_t x\}$.

PROPOSITION 1.6. *Let* $\mathfrak{X}$ *be quasi-complete relative to* (*the Mackey topology asscociated with*) *a locally convex topology* τ, *and let* σ *be the weak topology associated with* τ. *If a semi-group of operators* T_t, $t \geqq 0$, *satisfies* (S 1)—(S 3) *and*

(S 5)$'$ $\sigma\text{-}\lim T_t x = x$ *for every* x *in a* T_t*-invariant dense set* $\mathfrak{D}$ *of* $\mathfrak{X}$,

then it satisfies (S 5) *with* σ *replaced by* τ.

PROOF. It follows from the semi-group property (S 2) and (S 5)$'$ that $T_t x$ is weakly right continuous for $t \geqq 0$, and from (S 3) that it is uniformly bounded.

We define the integral

$$I(\alpha, x) = \frac{1}{\alpha}\int_0^\alpha T_t x dt\,, \qquad 0 < \alpha < \infty\,,$$

as the Dunford-Pettis integral

$$\langle I(\alpha, x), x' \rangle = \frac{1}{\alpha}\int_0^\alpha \langle T_t x, x' \rangle dt \qquad \text{for} \quad x' \in \mathfrak{X}'\,.$$

It is clear that the function $\langle T_t x, x' \rangle$, which is bounded and right continuous, is integrable in the sense of Lebesgue. Since $\{T_t x\}$ is bounded, it easily follows that $I(\alpha, x)$ belongs to the bidual space $\mathfrak{X}''$. We shall show that $I(\alpha, x) \in \mathfrak{X}$. We have for any $t \geqq 0$

$$\langle T_t x, x' \rangle = \lim_{n\to\infty} \langle g_n(t), x' \rangle\,,$$

where $g_n(t)$ is the step function:

$$g_n(t) = T_{m\alpha/2^n} x \quad \text{for} \quad (m-1)\alpha/2^n < t \leqq m\alpha/2^n\,.$$

This shows that the range $\{T_t x\}$ is separable and by Lebesgue's theorem

$$\begin{aligned}\frac{1}{\alpha}\int_0^\alpha \langle T_t x, x' \rangle dt &= \lim_{n\to\infty} \frac{1}{\alpha}\int_0^\alpha \langle g_n(t), x' \rangle dt \\ &= \lim_{n\to\infty} \frac{1}{2^n \alpha} \sum_{m=1}^{2^n} \langle T_{m\alpha/2^n} x, x' \rangle\,.\end{aligned}$$

Thus $I(\alpha, x)$ is the weak limit in $\mathfrak{X}''$ of a sequence from a bounded set in $\mathfrak{X}$, the convex hull of $\{T_t x\}$. Now following the same argument as in the proof of Proposition 1.2, we obtain $I(\alpha, x) \in \mathfrak{X}$.

We shall show that (S 5) holds for any $y = I(\alpha, x)$ with $x \in \mathfrak{D}$. We have for any $x' \in \mathfrak{X}'$

$$\langle T_t y, x' \rangle = \langle y, T_t' x' \rangle = \frac{1}{\alpha}\int_0^\alpha \langle T_s x, T_t' x' \rangle ds = \frac{1}{\alpha}\int_0^\alpha \langle T_{t+s} x, x' \rangle ds\,,$$

thus

$$T_t y - T_{t+h} y = \frac{1}{\alpha}\Big(\int_t^{t+h} - \int_{t+\alpha}^{t+\alpha+h}\Big) T_s x\, ds\,.$$

So noticing that for any τ-continuous semi-norm $p(x)$ on $\mathfrak{X}$ there is a τ-continuous semi-norm $q(x)$ such that $p(T_t x) \leqq q(x)$, we have

$$p(T_t y - T_{t+h} y) \leqq 2|h| q(x)/\alpha\,.$$

On the other hand it follows from (S 5)$'$ that

$$x = \sigma\text{-}\lim_{\alpha\to 0} I(\alpha, x) \qquad \text{for any} \quad x \in \mathfrak{D}\,.$$

Therefore the set $\{I(\alpha, x)\}$ is dense in $\mathfrak{X}$. This completes the proof.

Hitherto we have concerned ourselves more with the topology τ. However, it is possible to start only with the topology σ.

For a vector space $\mathfrak{X}$ with a locally convex topology σ, the set of all barrels of $\mathfrak{X}$ forms a fundamental system of neighborhoods of 0 for a locally convex topology τ. We shall call this topology τ the *strong topology* associated with σ. τ is always stronger than σ, and is equal to σ if and only if $\mathfrak{X}$ is tonnelé relative to σ.

PROPOSITION 1.7. *Let $\mathfrak{X}$ be a locally convex topological vector space with the topology σ. We assume that $\mathfrak{X}$ is sequentially complete. Let a semi-group of operators T_t, $t \geqq 0$, in $\mathfrak{X}$ satisfy* (S 1), (S 2), (S 5) *and the following two conditions:*
(S 3)′ $\{T_t x\}_{t \geqq 0}$ *is bounded for every $x \in \mathfrak{X}$.*
(S 4)′ T_t *is a σ-weakly continuous operator in $\mathfrak{X}$ for every $t \geqq 0$.*
Then T_t is a (τ, σ) semi-group with the σ-strong topology τ.

PROOF. In view of the definition of τ and Proposition 1.1 the conditions (T 1)—(T 4) are valid for the topologies τ and σ. Clearly (S 4)′ implies (S 4). To prove (S 3) it is sufficient to show that $U = \cap T_t^{-1}(V)$ is a barrel for any barrel V in $\mathfrak{X}$. Obviously U satisfies the conditions for the barrels except that it absorbs every point x in $\mathfrak{X}$. However, it follows from the sequential completeness of $\mathfrak{X}$ that a barrel absorbs every bounded set (cf. [**1**], Chap. III, § 3, Lemme 1). This proves that $\{T_t x\} \subset \lambda V$, hence $x \in \lambda U$ for a $\lambda > 0$, completing the proof.

COROLLARY. *If, in addition to the assumptions of Proposition* 1.7, $\mathfrak{X}$ *is tonnelé, then a semi-group of operators T_t satisfying the conditions of Proposition* 1.7 *is equicontinuous.*

2. Operators and resolvents.

Before getting into the discussions we shall review a few results about operators and resolvents in locally convex spaces.

Let A be a linear operator with the domain $\mathfrak{D}(A)$ and the range $\mathfrak{R}(A)$ in a locally convex space $\mathfrak{X}$. A is said to be *closed* if the graph $\mathfrak{G}(A) = \{(x, Ax);\ x \in \mathfrak{D}(A)\}$ is closed in $\mathfrak{X} \times \mathfrak{X}$.

A complex number λ is said to belong to the *resolvent set* $\rho(A)$ if $(\lambda I - A)$ has an inverse $(\lambda I - A)^{-1}$ which belongs to $\mathfrak{L}(\mathfrak{X}, \mathfrak{X})$ (i. e. $(\lambda I - A)^{-1}$ is defined on $\mathfrak{X}$ and continuous). We denote $(\lambda I - A)^{-1}$ by $R(\lambda, A)$ or $R(\lambda)$ and call it the *resolvent* of A.

PROPOSITION 2.1. *Let A be a closed linear operator with a non-empty resolvent set. If λ and $\mu \in \rho(A)$, then we have*
(R 1) $(\lambda - \mu) R(\lambda) R(\mu) = (\lambda - \mu) R(\mu) R(\lambda) = R(\mu) - R(\lambda)$,
(R 2) $R(\lambda) x = 0$ *implies* $x = 0$.

Conversely if a family of operators $R(\lambda) \in \mathfrak{L}(\mathfrak{X}, \mathfrak{X})$ defined for λ in a non-empty set Λ of complex numbers satisfies (R 1) *and* (R 2) *for a $\lambda \in \Lambda$ and any $\mu \in \Lambda$, then $R(\lambda)$ is identical with the resolvent $R(\lambda, A)$ of a closed linear* *operator A with the resolvent set $\rho(A)\supset\Lambda$.*

PROPOSITION 2.2. *Let $\mathfrak{X}$ be sequentially complete. If $\rho(A)\ni\lambda$ and $\{(cR(\lambda))^n\}$ is equicontinuous for a $c>0$, then any number μ with $|\mu-\lambda|<c$ belongs to the resolvent set, and we have*

$$R(\mu)x=\sum_{n=0}^{\infty}(R(\lambda))^{n+1}(\lambda-\mu)^n x\,. \tag{2.1}$$

DEFINITION. A family of operators $R(\lambda)\in\mathfrak{L}(\mathfrak{X},\mathfrak{X})$ defined for $\lambda\in\Lambda$ and satisfying (R 1) for any λ, $\mu\in\Lambda$ is called a *pseudo-resolvent.*

PROPOSITION 2.3. *Let $\{R(\lambda)\}_{\lambda\in\Lambda}$ be a pseudo-resolvent. Then the kernel $\mathfrak{N}(R(\lambda))=\{x\,;\,R(\lambda)x=0\}$ and the image $\mathfrak{R}(R(\lambda))=\{R(\lambda)x\,;\,x\in\mathfrak{X}\}$ do not depend on $\lambda\in\Lambda$. $\{R(\lambda)\}$ is the resolvent of an operator A if and only if $\mathfrak{N}(R)=\mathfrak{N}(R(\lambda))=\{0\}$. In this case we have $\mathfrak{R}(R)=\mathfrak{R}(R(\lambda))=\mathfrak{D}(A)$.*

We omit the proofs of the above three propositions. They are the same as in the case of Banach spaces. The following proposition is due to Kato [**5**] and Yosida [**9**].

PROPOSITION 2.4. *Let $\{R(\lambda)\}_{\lambda\in\Lambda}$ be a pseudo-resolvent such that $\{\lambda_n R(\lambda_n)\}$ is equicontinuous for a sequence $\lambda_n\in\Lambda$ with $|\lambda_n|\to\infty$. Then we have*

$$\mathfrak{R}(R)^r=\{x\in\mathfrak{X}\,;\,\lim_{n\to\infty}\lambda_n R(\lambda_n)x=x\} \tag{2.2}$$

and

$$\mathfrak{N}(R)\cap\mathfrak{R}(R)^r=\{0\}\,, \tag{2.3}$$

where $\mathfrak{R}(R)^r$ is the closure of $\mathfrak{R}(R)$. Thus the restriction of $R(\lambda)$ to $\mathfrak{R}(R)^r$ is a resolvent of a closed linear operator in $\mathfrak{R}(R)^r$.

Moreover if $\{\lambda_n R(\lambda_n)x\}$ has a weakly relatively compact subsequence for every $x\in\mathfrak{X}$, then we have

$$\mathfrak{X}=\mathfrak{N}(R)+\mathfrak{R}(R)^r \qquad (\textit{direct sum}). \tag{2.4}$$

Consequently $R(\lambda)$ is a resolvent if and only if $\mathfrak{R}(R)$ is dense in $\mathfrak{X}$.

PROOF. On account of (R 1) we have, for any $R(\mu)y\in\mathfrak{R}(R)$,

$$\lambda_n R(\lambda_n)R(\mu)y-R(\mu)y=\frac{\mu}{\lambda_n-\mu}R(\mu)y-\frac{\lambda_n}{\lambda_n-\mu}R(\lambda_n)y\to 0 \tag{2.5}$$

as $n\to\infty$. Let $x\in\mathfrak{R}(R)^r$. Then, for any continuous semi-norm $p(x)$ and $\varepsilon>0$ there is a $y\in\mathfrak{R}(R)$ such that $p(x-y)\leqq\varepsilon$ and $p(\lambda_n R(\lambda_n)(x-y))\leqq\varepsilon$. Hence if n becomes sufficiently large, we have $p(\lambda_n R(\lambda_n)x-x)\leqq 3\varepsilon$. This proves (2.2), and hence (2.3).

Next let $\{\lambda_n R(\lambda_n)x\}$ be weakly relatively compact (without loss of generality). Then there is a directed set $\{\nu\}$ and a function $n(\nu)$ with $n(\nu)\to\infty$ such that $\lambda_{n(\nu)}R(\lambda_{n(\nu)})x$ converges weakly to an element $x_1\in\mathfrak{X}$. x_1 belongs to $\mathfrak{R}(R)^r$, so that we have $\lambda_{n(\nu)}R(\lambda_{n(\nu)})(x-x_1)\to 0$ weakly. On the other hand from (2.5) with $y=x-x_1$ we have

$$R(\mu)(x-x_1)-R(\mu)\lambda_{n(\nu)}R(\lambda_{n(\nu)})(x-x_1)\to 0$$

strongly for any fixed μ. Since $R(\mu)$ is weakly continuous, we have thus $x-x_1\in\mathfrak{N}(R)$.

3. Infinitesimal generators and Laplace transform.

Let T_t, $t\geqq 0$, be a (τ,σ)-semi-group of operators in $\mathfrak{X}$. We define its τ-(resp. σ-) *infinitesimal generator* A_τ (resp. A_σ) by

$$A_\tau x=\tau\text{-}\lim_{t\to 0}\frac{1}{t}(T_tx-x)\qquad\left(\text{resp. } A_\sigma x=\sigma\text{-}\lim_{t\to 0}\frac{1}{t}(T_tx-x)\right)$$

when the right hand side exists.

It was crucial in the theory of semi-groups of operators in Banach spaces that the infinitesimal generator A has its resolvent $R(\lambda, A)=(\lambda I-A)^{-1}$ which is representable as the Laplace transform of T_t.

The *Laplace transform* of T_tx

$$R(\lambda)x=\int_0^\infty e^{-\lambda t}T_tx\,dt\,,\qquad\text{for}\quad \operatorname{Re}\lambda>0\,,\tag{3.1}$$

is defined in our case as an improper σ-Riemann integral. In fact, it follows from (T 4) and (S 5) that

$$R_N(\lambda)x=\int_0^N e^{-\lambda t}T_txdt$$

exists as a σ-Riemann integral and from (S 3) and (T 2) that for any (τ,σ) semi-norm $p(x)$ there is another $q(x)$ such that

$$\begin{aligned}p(R_N(\lambda)x-R_M(\lambda)x)&=p\Big(\int_M^N e^{-\lambda t}T_tx\,dt\Big)\\&\leqq(\operatorname{Re}\lambda)^{-1}|e^{-\operatorname{Re}\lambda N}-e^{-\operatorname{Re}\lambda M}|\,q(x)\,.\end{aligned}$$

This shows that $R_N(\lambda)x$ is a τ-Cauchy sequence. Thus $R(\lambda)x=\tau\text{-}\lim_{N\to\infty}R_N(\lambda)x$ exists.

We shall examine properties of $R(\lambda)x$. We see that $R(\lambda)$ is a linear operator from $\mathfrak{X}$ to $\mathfrak{X}$ and continuous relative to the topolygy τ. Moreover we have

PROPOSITION 3.1. *Let* $J(\lambda)=(\operatorname{Re}\lambda)^{-1}R(\lambda)$. *Then the family of operators* $\{J(\lambda)^m\}_{\operatorname{Re}\lambda>0,\ m=0,1,2,\cdots}$ *is equicontinuous in* $\mathfrak{L}(\mathfrak{X}_\tau,\mathfrak{X}_\tau)$ *with* $\{T_t\}_{t\geqq 0}$.

PROOF. By considering the Riemann sum, we can deduce from (S 2) and (S 4) that

$$T_tR(\lambda)x=R(\lambda)T_tx\,.\tag{3.2}$$

Hence we have

(3.3)
$$R(\mu)R(\lambda)x=\int_0^\infty e^{-\mu s}ds\int_0^\infty e^{-\lambda t}T_{t+s}xdt\,,$$

in the sense of an improper σ-Riemann integral.

Now let y' be an element of the dual space of $\mathfrak{X}$ relative to σ. We have

$$\langle R(\lambda)^2x,y'\rangle=\int_0^\infty e^{-\lambda s}ds\int_0^\infty e^{-\lambda t}\langle T_{t+s}x,y'\rangle dt$$
$$=\int_0^\infty te^{-\lambda t}\langle T_tx,y'\rangle dt\,.$$

This shows

$$R(\lambda)^2x=\int_0^\infty te^{-\lambda t}T_txdt\,.$$

The existence of the right hand side as an improper σ-Riemann integral is easily shown. Repeating the same arguments $m-1$ times we have

(3.4)
$$R(\lambda)^mx=\frac{1}{(m-1)!}\int_0^\infty t^{m-1}e^{-\lambda t}T_txdt\,.$$

Thus for any (τ,σ) semi-norm $p(x)$ there is a (τ,σ) semi-norm $q(x)$ such that

$$p(J(\lambda)^mx)\leqq\frac{(\mathrm{Re}\,\lambda)^m}{(m-1)!}\int_0^\infty t^{m-1}e^{-\mathrm{Re}\,\lambda t}dt\cdot\sup_{t\geqq0}p(T_tx)\leqq q(x)\,.$$

Corresponding problem whether $R(\lambda)$ is σ-continuous is rather difficult. We can not except that $\{R(\lambda)\}$ is σ-equicontinuous. For, τ-equicontinuous family of operators is, even relative to the τ-weak topology, not necessarily equicontinuous. Nevertheless we can state the following propositions.

PROPOSITION 3.2. *Let $\mathfrak{B}$ be a family of convex circled σ-closed and σ-bounded subsets of $\mathfrak{X}$. If a (τ,σ) semi-group T_t satisfies the following $\mathfrak{B}$-locally equibounded property:*

(S 3)$''_{\mathfrak{B}}$ *For every $B\in\mathfrak{B}$ there is a $D\in\mathfrak{B}$ such that $\cup T_t(B)\subset D$,*

then $\{J(\lambda)^m\}_{\mathrm{Re}\,\lambda>0,\ m=1,2,\dots}$ has the same property.

PROOF. The proof is similar to the preceding one.

PROPOSITION 3.3. *If we denote by $\mathfrak{B}_\tau$ the set of σ-closed convex circled τ-bounded sets, a (τ,σ) semi-group T_t satisfies* (S 3)$''_{\mathfrak{B}_\tau}$.

PROOF. From (T 2) we see that the σ-closed convex circled hull of a τ-bounded set is τ-bounded. Let B be a τ-bounded set and let V be an arbitrary τ-neighborhood of 0. Then there is a τ-neighborhood U of 0 such that $T_t(U)\subset V$. For a $\alpha>0$ we have $B\subset\alpha U$, therefore $T_t(B)\subset\alpha V$. This shows that $\cup T_t(B)$ is τ-bounded.

PROPOSITION 3.4. *If $\mathfrak{X}$ is sequentially complete relative to (the Mackey topology associated with) σ, or if the dual space $\mathfrak{X}'_\sigma$ relative to σ is sequentially complete relative to (the Mackey topology associated with) the weak* topology σ^*, then we have $\mathfrak{B}_\tau=\mathfrak{B}_\sigma$. Thus* (S 3)$''_{\mathfrak{B}_\sigma}$ *is satisfied with the set of all σ-*

bounded sets $\mathfrak{B}_\sigma$ for any (τ, σ) semi-group T_t.

PROOF. We see from (T 2) that a family of barrels relative to σ forms a fundamental system of τ-neighborhood of 0. If $\mathfrak{X}$ is sequentially complete, a barrel absorbs every bounded set. Let $\mathfrak{X}'_\sigma$ be sequentially complete relative to the Mackey topology. If V and B are a convex circled and σ-closed τ-neighborhood of 0 and a σ-bounded set, then the polar sets V° and B° are a bounded set and a barrel relative to σ^*. By assumption it follows that B° absorbs V°, hence V absorbs B.

PROPOSITION 3.5. *Let a (τ, σ) semi-group T_t satisfy* (S 4)$'$ *(cf. Proposition 1.7.). Then, for any λ with* $\mathrm{Re}\,\lambda > 0$, *$R(\lambda)$ is σ-weakly continuous on every τ-bounded and σ-weakly compact set in $\mathfrak{X}$. In particular it is sequentially σ-weakly continuous on every τ-bounded set.*

PROOF. Let B be a τ-bounded and σ-weakly compact set. We have to show that $x, x_\nu \in B$ and $\langle x_\nu, y'\rangle \to \langle x, y'\rangle$ for every $y' \in \mathfrak{X}'_\sigma$ imply that $\langle R(\lambda)x_\nu, y'\rangle \to \langle R(\lambda)x, y'\rangle$ for every $y' \in \mathfrak{X}'_\sigma$, where $\{\nu\}$ is a directed set. The family of continuous functions $\{\langle T_t x, y'\rangle\}_{x\in B}$ for $t \geqq 0$ is uniformly bounded for a fixed y'. We shall show that $\{\langle T_t x, y'\rangle\}_{x\in B}$ is a metrizable subset of the space of all functions $\mathbf{C}^{\mathbf{R}^+}$ with the pointwise topology. Then Lebesgue's theorem will show that the mapping which maps $\langle T_t x, y'\rangle$ to $\langle R(\lambda)x, y'\rangle$ is continuous. Since we know by (S 4)$'$ that the mapping which carries x to $\langle T_t x, y'\rangle$ is continuous, this will prove the proposition.

We note that $\{\langle T_t x, y'\rangle\}_{x\in B}$ is compact as the image of the compact set B by a continuous mapping. Consider the projection π from $\mathbf{C}^{\mathbf{R}^+}$ to $\mathbf{C}^{\mathbf{N}}$ which maps $f(t)$, $t \geqq 0$, to $f(t_i)$, $i = 1, 2, \cdots$, where t_i's are nonnegative rational numbers suitably indexed. Clearly π is continuous and maps the compact set $\{\langle T_t x, y'\rangle\}_{x\in B}$ in a one-to-one way into the metrizable space $\mathbf{C}^{\mathbf{N}}$. Thus $\{\langle T_t x, y'\rangle\}_{x\in B}$ is homeomorphic to a metrizable set. This completes the proof.

PROPOSITION 3.6. *Let T_t satisfy* (S 4)$'$. *If the dual space $\mathfrak{X}'_\sigma$ of $\mathfrak{X}$ with σ is quasi-complete relative to the Mackey topology $\tau(\mathfrak{X}'_\sigma, \mathfrak{X})$, then $R(\lambda)$ is σ-weakly continuous.*

PROOF. Let y be an arbitrary element in $\mathfrak{X}'_\sigma$. From Proposition 3.5 and Grothendieck's theorem ([3], Chap. II, Théorème 10) it follows that $\langle R(\lambda)x, y\rangle = \langle x, \hat{y}\rangle$ for a $\hat{y}$ in the completion $\hat{\mathfrak{X}}'$ of the space $\mathfrak{X}'_\sigma$ relative to the Mackey topology. It suffices to show that $\hat{y} \in \mathfrak{X}'_\sigma$. Since $R(\lambda)x$ is the σ-limit of the Riemann sums $S(e^{-\lambda t}T_t x, \Delta)$, $\hat{y}$ is the weak* limit of $S(e^{-\lambda t}T'_t y, \Delta)$. Note that $S(e^{-\lambda t}T'_t y, \Delta)$ is bounded (see Propositions 3.4 and 5.4). Then the same argument as in the proof of Proposition 1.2 leads to $y \in \mathfrak{X}'_\sigma$.

We now turn to the problem how $R(\lambda)x$ behaves when λ tends to $+\infty$.

PROPOSITION 3.7. *We have*

$$J(m\lambda)^m x \to x \qquad (\sigma) \quad \textit{for any } x \in \mathfrak{X} \tag{3.5}$$

and uniformly in $m=1, 2, \cdots$ *when* $\lambda>0$ *tends to* $+\infty$. *Moreover the convergence is uniform for* x *in a convex* σ*-closed and* τ*-bounded set* C *if* $T_t x$ *converges to* x *uniformly in* $x\in C$ *when* t *tends to* 0.

PROOF. Let $\pi(x)$ be a σ-continuous semi-norms. For any $\varepsilon>0$ there is a δ such that $\pi(T_t x-x)\leqq\varepsilon$ for $0\leqq t\leqq\delta$ and $x\in C$. Thus we have

$$\pi\Big((m-1)\,!^{-1}\int_0^\delta(\lambda mt)^{m-1}e^{-\lambda mt}(T_t x-x)\lambda m dt\Big)\leqq\varepsilon\,.$$

It is easily shown that $(m-1)\,!^{-1}\int_\delta^\infty(\lambda mt)^{m-1}e^{-\lambda mt}\lambda m dt$ tend to 0 uniformly in m, and $\{T_t x-x\}$ is uniformly bounded relative to τ. This proves (3.5).

PROPOSITION 3.8. *$R(\lambda)$ is the resolvent $(\lambda I-A)^{-1}$ of a τ-closed operator A with a σ-dense domain of definition.*

PROOF. From (3.3) we get for $\mu\neq\lambda$

$$\begin{aligned}R(\mu)R(\lambda)x&=\int_0^\infty T_r x dr\int_0^r e^{-\mu(r-t)}e^{-\lambda t}dt\\&=(\mu-\lambda)^{-1}(R(\lambda)x-R(\mu)x)\,.\end{aligned}$$

Thus $R(\lambda)$ satisfies the resolvent equation. In order to prove $\mathfrak{N}(R)=\{0\}$, it is sufficient to show that

$$A_\sigma R(\lambda)x=(\lambda R(\lambda)-I)x\qquad\text{for every } x\in\mathfrak{X}\,. \tag{3.6}$$

We have

$$\begin{aligned}\frac{T_h-I}{h}R(\lambda)x&=\frac{1}{h}\int_0^\infty e^{-\lambda s}T_{h+s}x ds-\frac{1}{h}\int_0^\infty e^{-\lambda s}T_s x ds\\&=\frac{e^{\lambda h}-1}{h}\int_0^\infty e^{-\lambda s}T_s x ds-e^{\lambda h}\frac{1}{h}\int_0^h e^{-\lambda s}T_s x ds\,.\end{aligned}$$

When h tends to zero, the first term converges to $\lambda R(\lambda)x$ in any topology and as in Proposition 3.7 we have

$$\frac{1}{h}\int_0^h e^{-\lambda s}T_s x ds\to x\qquad(\sigma)\,.$$

Thus (3.6) follows. (3.5) shows that $\mathfrak{D}(A)=\mathfrak{R}(R(\lambda))$ is σ-dense.

We shall call A the *infinitesimal generator* of $\{T_t\}$. From the uniqueness of the Laplace transform follows the

PROPOSITION 3.9. *The infinitesimal generator A determines the semi-group $\{T_t\}$ uniquely.*

Next we shall examine the relationship between A, A_τ and A_σ. First of all we have $A_\sigma\supset A$ from the above proof. The following proposition characterize the domain $\mathfrak{D}(A)$ of A.

PROPOSITION 3.10. *We have $x\in\mathfrak{D}(A)$ and $Ax=y$ if and only if*

$$T_t x-x=\int_0^t T_s y ds \tag{3.7}$$

for every t in the sense of a σ-Riemann integral.

PROOF. Let $x \in \mathfrak{D}(A)$ and $y = Ax$. Then for an element z we have $x = R(\lambda)z$ and $y = \lambda R(\lambda)z - z$. Since the both sides of (3.7) are σ-continuous function, the equality follows from the identity of their Laplace transforms. The Laplace transform of the left hand side is

$$R(\mu)R(\lambda)z - \mu^{-1}R(\lambda)z ,$$

and that of the right hand side is equal to

$$\int_0^\infty e^{-\mu t}dt \int_0^t T_s\{\lambda R(\lambda)z - z\}ds = \mu^{-1}R(\mu)(\lambda R(\lambda)z - z) .$$

Their equality follows from the resolvent equation.

Conversely (3.7) implies $R(\lambda)(\lambda x - y) = x$. For,

$$\begin{aligned}\int_0^\infty e^{-\lambda t}T_t(\lambda x - y)dt &= \int_0^\infty e^{-\lambda t}\Big\{\lambda x + \lambda \int_0^t T_s y ds - T_t y\Big\}dt \\ &= x + \int_0^\infty T_s y ds \int_s^\infty \lambda e^{-\lambda t}dt - \int_0^\infty e^{-\lambda t}T_t y dt \\ &= x .\end{aligned}$$

This proves $x \in \mathfrak{D}(A)$ and $Ax = y$.

COROLLARY. *We have*

$$x = \int_\alpha^\beta T_t z dt \in \mathfrak{D}(A) \quad \textit{and} \quad Ax = T_\beta z - T_\alpha z$$

for any $z \in \mathfrak{X}$ and $0 \leqq \alpha < \beta < \infty$.

PROOF. (3.7) is easily proved with $y = T_\beta z - T_\alpha z$.

PROPOSITION 3.11. *If T_t is a σ-weakly continuous operator for every $t \geqq 0$, then we have $A_\sigma = A$.*

PROOF. We have only to show that $A_\sigma \subset A$. Let x be an element in $\mathfrak{D}(A_\sigma)$. For every $y' \in \mathfrak{X}'_\sigma$ we have

$$\begin{aligned}\langle T_t A_\sigma x, y' \rangle = \langle A_\sigma x, T'_t y' \rangle &= \lim_{h \to 0} h^{-1} \langle (T_h - I)x, T'_t y' \rangle \\ &= \lim_{h \to 0} h^{-1} \langle (T_{t+h} - T_t)x, y' \rangle .\end{aligned}$$

Since $\langle T_t A_\sigma x, y' \rangle$ is continuous, this shows that $\langle T_t x, y' \rangle$ is differentiable. Thus we have

$$\langle T_t x, y' \rangle - \langle x, y' \rangle = \int_0^t \langle T_t A_\sigma x, y' \rangle dt .$$

From this we have (3.7), hence $x \in \mathfrak{D}(A)$ and $Ax = A_\sigma x$.

PROPOSITION 3.12. $A_\tau \subset A$ *for every (τ, σ)-semi-group of operators.*

PROOF. The same proof as in the previous proposition is applicable. $T'_t y'$ may not be in $\mathfrak{X}'_\sigma$, but it always belongs to $\mathfrak{X}'_\tau$.

PROPOSITION 3.13. *If $x \in \mathfrak{D}(A)$, then $T_t x \in \mathfrak{D}(A)$ for every t and we have $AT_t x = T_t Ax$. Moreover the $\mathfrak{X}$-valued function $x(t) = T_t x$ is continuously differentiable in t relative to σ and satisfies the differential equation*

$$\frac{d}{dx} x(t) = Ax(t) . \tag{3.8}$$

PROOF. The commutativity $AT_t = T_t A$ follows from (3.2) easily. (3.7) proves that $T_t x$ is continuously differentiable and $d/dt\, T_t x = T_t Ax$.

4. τ-center.

For a (τ, σ) semi-group T_t, $T_t x$ is not necessarily τ-continuous. The following proposition characterizes such an x that $T_t x$ is τ-continuous.

PROPOSITION 4.1. *Let T_t be a (τ, σ) semi-group and A be its generator. For an $x \in \mathfrak{X}$, $T_t x$ is τ-continuous for $t \geqq 0$ if and only if $x \in \mathfrak{D}(A)^\tau$, the τ-closure of the domain of A. And then $T_t x$ belongs to $\mathfrak{D}(A)^\tau$ for any t.*

PROOF. If $T_t x$ is τ-continuous at 0, we obtain $J(\lambda)x \to x$ (τ) as $\lambda \to \infty$ by the same way as in Proposition 3.7. Since $J(\lambda)x \in \mathfrak{D}(A)$, we have $x \in \mathfrak{D}(A)^\tau$.

If $x \in \mathfrak{D}(A)$, we have (3.7). Hence for any (τ, σ) semi-norm $p(x)$

$$p(T_t x - T_s x) \leqq |t-s| \cdot \sup_{r \geqq 0} p(T_r Ax)$$

holds. This shows that $T_t x$ is τ-(Lipschitz) continuous. If $x \in \mathfrak{D}(A)^\tau$, then for any (τ, σ) semi-norm $p(x)$ and $\varepsilon > 0$ there is a $y \in \mathfrak{D}(A)$ such that $p(T_t(x-y)) \leqq \varepsilon$ for any t. Thus we have

$$\begin{aligned} p(T_t x - T_s x) &\leqq p(T_t(x-y)) + p(T_t y - T_s y) + p(T_s(x-y)) \\ &\leqq 2\varepsilon + p(T_t y - T_s y) . \end{aligned}$$

When $|t-s|$ is sufficiently small, the right hand side becomes less than 3ε. $T_t x \in \mathfrak{D}(A)^\tau$ is proved similarly from the equicontinuity of T_t and the fact that $x \in \mathfrak{D}(A)$ implies $T_t x \in \mathfrak{D}(A)$.

COROLLARY 1. *If σ is stronger than the weak topology associated with τ, then $T_t x$ is τ-continuous for any $x \in \mathfrak{X}$.*

PROOF. $\mathfrak{D}(A)^\tau = \mathfrak{D}(A)^\sigma = \mathfrak{X}$.

COROLLARY 2. *If every τ-bounded sequence has a τ-weakly relatively compact subsequence, or in particular if $\mathfrak{X}$ with the topology τ is semi-reflexive, then every (τ, σ) semi-group T_t is τ-continuous.*

PROOF. For every $x \in \mathfrak{X}$, $\{nR(n)x\}$ is bounded, hence has a weakly relatively compact subsequence. From Proposition 2.4 it follows that $\mathfrak{D}(A)^\tau = \mathfrak{R}(R)^\tau = \mathfrak{X}$.

We shall denote $\mathfrak{D}(A)^\tau$ by $\mathfrak{X}_0$. The above proposition shows that the restriction of T_t to $\mathfrak{X}_0$ gives a (τ, τ) semi-group in $\mathfrak{X}_0$. We shall call this restriction the *τ-center* of T_t after Feller [2] and denote it by T_{0t}.

PROPOSITION 4.2. *$\mathfrak{X}_0$ is the τ-closure of the set of elements $x_{\alpha\beta}$ of the form*

$$x_{\alpha\beta}=\int_\alpha^\beta T_t x dt \tag{4.1}$$

with $x\in\mathfrak{X}$ and $0\leqq\alpha<\beta<\infty$.

PROOF. By Corollary of Proposition 3.10 we have $x_{\alpha\beta}\in\mathfrak{D}(A)\subset\mathfrak{X}_0$. Conversely if $x\in\mathfrak{X}_0$, then $T_t x$ is τ-continuous. Thus we have

$$h^{-1}x_{0h}=\frac{1}{h}\int_0^h T_t x dt\rightarrow x$$

relative to τ as $h\rightarrow 0$.

PROPOSITION 4.3. *Let $R_0(\lambda)$ and A_0 be the Laplace transform and the generator of T_{0t} respectively. Then $R_0(\lambda)$ is simply the restriction of $R(\lambda)$ to $\mathfrak{X}_0$ and A_0 is the maximal restriction of A with the range in $\mathfrak{X}_0$.*

PROOF. If $x\in\mathfrak{X}_0$, the integral in (3.1) converges in the sense of a τ-Riemann integral. Thus we have $R(\lambda)x=R_0(\lambda)x$. Hence it follows that $A_0\subset A$. By definition we have $\mathfrak{R}(A_0)\subset\mathfrak{X}_0$. Conversely if $x\in\mathfrak{D}(A)$ and $Ax\in\mathfrak{X}_0$, then we have (3.7) in the sense of a τ-Riemann integral. Thus it follows from Proposition 3.10 applied to T_{0t} and $\mathfrak{X}_0$ that $x\in\mathfrak{D}(A_0)$ and $A_0x=Ax$.

PROPOSITION 4.4. *A_0 coincides with the τ-infinitesimal generator A_τ. If $x\in\mathfrak{D}(A_0)$, then $x(t)=T_t x$ is continuously differentiable in t relative to τ and satisfies the differential equation* (3.8) *and $x(0)=x$.*

PROOF. The latter part is a special case of Proposition 3.13. Thus we have especially $A_0\subset A_\tau$. Conversely let $x\in\mathfrak{D}(A_\tau)$. Then Ax is the τ-limit of $h^{-1}(T_h x-x)$ where $T_h x$ and x are elements in $\mathfrak{D}(A)\subset\mathfrak{X}_0$. Thus we have $Ax\in\mathfrak{X}_0$. The uniqueness of the solution will be proved in § 7.

5. Dual operators.

Let $\mathfrak{X}$ and $\mathfrak{Y}$ be locally convex spaces and $\mathfrak{X}'$ and $\mathfrak{Y}'$ be their dual spaces. If A is a linear operator with the dense domain $\mathfrak{D}(A)\subset\mathfrak{X}$ and the range $\mathfrak{R}(A)\subset\mathfrak{Y}$, then the dual operator A' is defined by the following equation

$$\langle Ax, y'\rangle=\langle x, A'y'\rangle, \tag{5.1}$$

which should be satisfied for all $x\in\mathfrak{D}(A)$. A' is a linear operator with the domain $\mathfrak{D}(A')\subset\mathfrak{Y}'$ and the range $\mathfrak{R}(A')\subset\mathfrak{X}'$.

The results of this section are mostly due to Phillips [7].

PROPOSITION 5.1. *Every dual operator A' is closed relative to the weak* topologies of $\mathfrak{Y}'$ and $\mathfrak{X}'$.*

PROOF. (5.1) is equivalent to the equation

$$\langle (x, Ax), (A'y', -y')\rangle=0$$

in $(\mathfrak{X}\times\mathfrak{Y})'=\mathfrak{X}'\times\mathfrak{Y}'$. Thus we have $\{(A'y', -y')\}=\mathfrak{G}(A)^{\circ}$. This proves that $\mathfrak{G}(A)'=\{(y', A'y'); y'\in\mathfrak{D}(A')\}$ is closed.

PROPOSITION 5.2. *$\mathfrak{D}(A')$ is weakly* dense in $\mathfrak{Y}'$ if and only if A has a closed extension. And then its smallest closed extension is equal to the bidual A''.*

PROOF. If A has a closed extension, the closure $\mathfrak{G}(A)^{\circ\circ}$ of the graph $\mathfrak{G}(A)$ in $\mathfrak{X}\times\mathfrak{Y}$ turns out to be the graph of the smallest closed extension of A. This shows that the set $\mathfrak{D}(A')$ of the second components of $\mathfrak{G}(A)^{\circ}$ is weakly* dense in $\mathfrak{Y}'$.

Conversely let $\mathfrak{D}(A')$ be dense. The dual space of $\mathfrak{Y}'$ (resp. $\mathfrak{X}'$) with the weak* topology is $\mathfrak{Y}$ (resp. $\mathfrak{X}$) so that the bidual A'' of A is a linear operator from $\mathfrak{X}$ to $\mathfrak{Y}$. It is easy to see that $\mathfrak{G}(A'')=\mathfrak{G}(A)^{\circ\circ}$, hence A'' is a closed extension of A.

PROPOSITION 5.3. *We have $\mathfrak{D}(A')=\mathfrak{Y}'$ if and only if A is continuous relative to the weak topologies of $\mathfrak{X}$ and $\mathfrak{Y}$. And then A' is continuous relative to the weak* topologies of $\mathfrak{Y}'$ and $\mathfrak{X}'$.*

PROOF. If $\mathfrak{D}(A')=\mathfrak{Y}'$, the equation (5.1) shows that A is weakly continuous. Conversely if A is weakly continuous, then the form $\langle Ax, y'\rangle$ is a (densely defined) weakly continuous form for any $y'\in\mathfrak{Y}'$. Thus there is an $x'\in\mathfrak{X}'$ such that $\langle Ax, y'\rangle=\langle x, x'\rangle$ for all $x\in\mathfrak{D}(A)$.

Let this be the case. Then the smallest closed extension A'' of A is an everywhere defined weakly continuous operator. From $\mathfrak{D}(A'')=\mathfrak{X}$ it follows that A' is weakly* continuous.

PROPOSITION 5.4. *Let $\mathfrak{S}$ (resp. $\mathfrak{S}_1$) be a family of bounded sets in $\mathfrak{X}$ (resp. $\mathfrak{Y}$) which covers $\mathfrak{X}$ (resp. $\mathfrak{Y}$). If a family of weakly continuous operators $\{T_\alpha\}$ from $\mathfrak{X}$ to $\mathfrak{Y}$ has the $(\mathfrak{S}, \mathfrak{S}_1)$ equibounded property that for any $B\in\mathfrak{S}$ there is a $C\in\mathfrak{S}_1$ such that $T_\alpha(B)\subset C$ for all α, then the family of dual operators $\{T'_\alpha\}$ is equicontinuous relative to the topologies of uniform convergence on $\mathfrak{S}_1$ and $\mathfrak{S}$. Especially if $\{T_\alpha\}$ is locally equibounded, thus in particular if it is equicontinuous relative to the original topologies of $\mathfrak{X}$ and $\mathfrak{Y}$, then $\{T'_\alpha\}$ is strongly equicontinuous.*

PROOF. The set of polar sets B° of $B\in\mathfrak{S}_1$ (resp. $\mathfrak{S}$) forms a base of neighborhoods of 0 relative to the topology of uniform convergence on $\mathfrak{S}_1$ (resp. $\mathfrak{S}$). Let $B\in\mathfrak{S}$ and $C\in\mathfrak{S}_1$ be such that $T_\alpha(B)\subset C$. Then

$$\sup|\langle x, T'_\alpha y'\rangle|=\sup|\langle T_\alpha x, y'\rangle|\leqq 1$$

for $x\in B$ and $y'\in C^{\circ}$ proves that $T'_\alpha(C^{\circ})\subset B^{\circ}$ and hence the proposition.

PROPOSITION 5.5. *A' is one to one if and only if the range $\mathfrak{R}(A)$ is dense in $\mathfrak{Y}$.*

PROPOSITION 5.6. *If $\mathfrak{R}(A')$ is weakly* dense in $\mathfrak{X}$ then A has an inverse. If A is closed, the converse is also true.*

PROOF. The above two propositions follow easily from the equation (5.1) and the Hahn-Banach theorem. We note that if A is closed, then we have $A=A''$.

PROPOSITION 5.7. *If $\mathfrak{D}(A)$ and $\mathfrak{R}(A)$ are dense in $\mathfrak{X}$ and $\mathfrak{Y}$ respectively, and if A has an inverse A^{-1}, then $(A')^{-1}$ and $(A^{-1})'$ exist and we have $(A')^{-1}=(A^{-1})'$.*

PROOF. The existence of $(A')^{-1}$ follows from Proposition 5.5, and that of $(A^{-1})'$ is clear.

If $y\in\mathfrak{D}(A^{-1})$ and $x'\in\mathfrak{D}((A')^{-1})$, then we have

$$\langle A^{-1}y, x'\rangle=\langle A^{-1}y, A'(A')^{-1}x'\rangle=\langle y, (A')^{-1}x'\rangle .$$

This proves that $x'\in\mathfrak{D}((A^{-1})')$ and $(A^{-1})'x'=(A')^{-1}x'$.

Next, let $x\in\mathfrak{D}(A)$ and $x'\in\mathfrak{D}((A^{-1})')$. Then

$$\langle Ax, (A^{-1})'x'\rangle=\langle A^{-1}Ax, x'\rangle=\langle x, x'\rangle$$

proves that $(A^{-1})'x'\in\mathfrak{D}(A')$ and $A'(A^{-1})'x'=x$, and hence $(A^{-1})'\subset(A')^{-1}$.

PROPOSITION 5.8. *Let A be a densely defined operator from $\mathfrak{X}$ to $\mathfrak{X}$, and B be a weakly continuous operator from $\mathfrak{X}$ to $\mathfrak{X}$. Then we have*

$$(A+B)'=A'+B' \tag{5.2}$$

and

$$(BA)'=A'B' . \tag{5.3}$$

PROOF. If $x\in\mathfrak{D}(A)$ and $x'\in\mathfrak{D}(A')$, then we have

$$\langle (A+B)x, x'\rangle=\langle x, (A'+B')x'\rangle .$$

Thus it follows that $(A+B)'\supset A'+B'$. Replacing $A+B$ by A and B by $-B$, we have $A'\supset(A+B)'-B'$, and hence $(A+B)'\subset A'+B'$.

Similarly

$$\langle Ax, B'x'\rangle=\langle BAx, x'\rangle=\langle x, (BA)'x'\rangle$$

proves $(BA)'\subset A'B'$, and

$$\langle BAx, x'\rangle=\langle Ax, B'x'\rangle=\langle x, A'B'x'\rangle$$

proves $A'B'\subset(BA)'$.

PROPOSITION 5.9. *Let A be a densely defined operator with a nonempty resolvent set. If λ belongs to the resolvent set $\rho(A)$, then λ belongs to $\rho(A')$ and we have*

$$(\lambda I-A')^{-1}=((\lambda I-A)^{-1})' . \tag{5.4}$$

6. Generation of semi-groups.

The problem of this section is to constuct a semi-group T_t with a given infinitesimal generator A. First we consider the case when $\sigma=\tau$. The following generalization of the Hille-Yosida theorem is due to Schwartz [8].

PROPOSITION 6.1. *Let $\mathfrak{X}_\tau$ be a sequentially complete locally convex space. Then, an operator A is the infinitesimal generator of a (τ, τ) semi-group of operators if and only if it satisfies the following conditions:*

(G 1) *A is a closed linear operator with a dense domain $\mathfrak{D}(A)$.*

(G 2) *There is a sequence of positive numbers λ_n with $\lambda_n \to \infty$ such that $\lambda_n \in \rho(A)$ and the family of operators $\{(I-\lambda_n^{-1}A)^{-m}\}$ is equicontinuous in $n, m=1, 2, \cdots$.*

PROOF. Necessity follows from Propositions 3.1 and 3.7. We shall prove the sufficiency by the second method of Hille-Phillips [4].

As in Proposition 3.1 we set $J(\lambda)=\operatorname{Re}\lambda R(\lambda)$ for $\lambda \in \rho(A)$. When $\lambda>0$, we have $J(\lambda)=(I-\lambda^{-1}A)^{-1}$. First we shall prove

(G 2)$''$ Every λ with $\operatorname{Re}\lambda>0$ belongs to the resolvent set $\rho(A)$, and $\{J(\lambda)^m\}$ is equicontinuous in $\operatorname{Re}\lambda>0$ and $m=0, 1, 2, \cdots$.

It follows from Proposition 2.2 that $\lambda \in \rho(A)$ if $|\lambda-\lambda_n|<\lambda_n$. Letting n sufficiently large, we see that every λ with $\operatorname{Re}\lambda>0$ belongs to $\rho(A)$. By Proposition 2.2 we have also

$$R(\lambda)^m x=\sum_{k=0}^{\infty}\binom{m+k-1}{k}(\lambda_n-\lambda)^k R(\lambda_n)^{m+k}x. \tag{6.1}$$

Thus for any continuous semi-norm $p(x)$ we have

$$p(J(\lambda)^m x) \leqq (\operatorname{Re}\lambda/(|\lambda_n|-|\lambda_n-\lambda|))^m \sup p(J(\lambda_n)^m x).$$

Let n tend to ∞. Then the factor in the bracket tends to 1.

Next we shall show that

(G 1)$''$ $J(\lambda)^m x \to x$ for any $x \in \mathfrak{X}$ and m as $\lambda \to \infty$.

Since $\mathfrak{D}(A)$ is dense, Proposition 2.4 proves (G 1)$''$ for $m=1$. If $m>1$, we have $J(\lambda)^m x-x=J(\lambda)(J(\lambda)^{m-1}x-x)+(J(\lambda)x-x)$. $J(\lambda)$ being equicontinuous, both terms converge to 0.

We now define

$$T_t^{(n)}=J\left(\frac{n}{t}\right)^n=\left(I-\frac{tA}{n}\right)^{-n}. \tag{6.2}$$

It follows from (G 2)$''$ that $T_t^{(n)}$ is equicontinuous in $t>0$ and $n=1, 2, \cdots$, and from (G 1)$''$ that for every $x \in \mathfrak{X}$ and n, $T_t^{(n)}x \to x$ as $t \to 0$. We see easily that $T_t^{(n)}x$ is differentiable (even analytic) in $t>0$, and for $x \in \mathfrak{D}(A)$ we have

$$\frac{d}{dt}T_t^{(n)}x=\left(I-\frac{tA}{n}\right)^{-n-1}Ax=T_t^{(n)}J\left(\frac{n}{t}\right)Ax, \tag{6.3}$$

thus

$$T_t^{(n)}x-x=\int_0^t T_s^{(n)}J(n/s)Axds. \tag{6.4}$$

We shall prove that $T_t^{(n)}x$ has a limit. If $x \in \mathfrak{D}(A)$, we have

$$T_t^{(n)}x - T_t^{(m)}x = \int_0^t \frac{d}{ds}(T_{t-s}^{(m)}T_s^{(n)}x)ds$$
$$= \int_0^t T_{t-s}^{(m)}T_s^{(n)}(J(n/s) - J(m/(t-s)))Axds\,.$$

Since $T_{t-s}^{(m)}T_s^{(n)}$ is equicontinuous, for any continuous semi-norm $p(x)$ there is a continuous semi-norm $q(x)$ such that

$$p(T_t^{(n)}x - T_t^{(m)}x) \leqq t \sup_{0 \leqq s \leqq t} q((J(n/s) - J(m/(t-s)))Ax)\,. \tag{6.5}$$

Therefore by (G 1)″ $T_t^{(n)}x$ forms a Cauchy sequence. Since $\mathfrak{D}(A)$ is dense and $T_t^{(n)}$ is equicontinuous, it follows that $T_t^{(n)}x$ is a Cauchy sequence for every x. Let

$$T_t x = \lim_{n\to\infty} T_t^{(n)}x \tag{6.6}$$

be its limit. As is seen from (6.5) the convergence is uniform on every interval $0 \leqq t \leqq t_0$. Thus $T_t x$ is continuous for $t \geqq 0$.

The semi-group property (S 2) is proved as follows. We have

$$T_t^{(mn)}x = (T_{t/m}^{(n)})^m x\,.$$

Thus letting n tends to ∞, we obtain

$$T_t x = (T_{t/m})^m x\,. \tag{6.7}$$

for any $x \in \mathfrak{X}$ and m. The proof is similar to that of (G 1)″. (6. 7) with the continuity proves (S 2).

At last we have to show that A is the infinitesimal generator of T_t. Let $x \in \mathfrak{D}(A)$. We have

$$T_s^{(n)}J(n/s)Ax - T_s Ax = T_s^{(n)}(J(n/s) - I)Ax + (T_s^{(n)} - T_s)Ax \to 0$$

uniformly in $s \in [0, t]$ as $n \to \infty$. Therefore letting n tend to ∞ in (6.3), we obtain

$$T_t x - x = \int_0^t T_s Axds\,. \tag{6.8}$$

This shows that A is a restriction of the infinitesimal generator $\hat{A}$ of T_t by Proposition 3.10. However $(I-A)$ and $(I-\hat{A})$ both map $\mathfrak{D}(A)$ and $\mathfrak{D}(\hat{A})$ onto $\mathfrak{X}$ in a one-to-one way. Thus they must coincide with each other. This completes the proof.

When σ is strictly weaker than (the weak topology associated with) τ, a complete characterization of the generators of (τ, σ) semi-groups seems very difficult. We deal only with a special case, which includes the duals of continuous semi-groups in Banach spaces.

PROPOSITION 6.2. *Let $\mathfrak{X}$ be a vector space with local convex topologies τ and σ satisfying* (T 1)—(T 4), *and let $\mathfrak{B}$ be a family of convex circled and*

σ-weakly compact sets which covers $\mathfrak{X}$. Assume that the dual space $\mathfrak{X}'_\sigma$ relative to σ is quasi-complete relative to the topology of uniform convergence on the sets of $\mathfrak{B}$. Then the infinitesimal generator A of a (τ, σ) semi-group with the $\mathfrak{B}$-locally equibounded property (S 3)$''_{\mathfrak{B}}$ *satisfies the conditions:*

(G 1)$'$ *A is a σ-closed linear operator with a σ-dense domain.*

(G 2)$'$ *$\{(I-\lambda_n^{-1}A)^{-m}\}_{n,m=1,2,\dots}$ is τ-equicontinuous and $\mathfrak{B}$-locally equibounded for a sequence $\lambda_n \to \infty$.*

(G 3)$'$ *$(I-\lambda^{-1}A)^{-1}$ is a σ-weakly continuous operator for every $\lambda > 0$.*

(G 4)$'$ *For every x we have*

$$(I-n^{-1}tA)^{-n}x \to x \qquad (\sigma)$$

as $t \to 0$ uniformly in $n = 1, 2, \cdots$.

Conversely if an operator A satisfies the conditions (G 1)$'$—(G 3)$'$, *then it generates a (τ, ω) semi-group T_t with the property* (S 3)$''_{\mathfrak{B}}$, *where ω is the topology in $\mathfrak{X}$ of uniform convergence on the compact sets in $\mathfrak{X}'_{\sigma\mathfrak{B}}$. Moreover if* (G 4)$'$ *is satisfied, then $T_t x$ is σ-right continuous for every x, and if the convergence in* (G 4)$'$ *is uniform on every τ-bounded set in $\mathfrak{B}$, then T_t is a (τ, σ) semi-group.*

PROOF. Necessity follows from Propositions 3.1, 3.2, 3.6 and 3.7. To prove sufficiency, let A be an operator satisfying (G 1)$'$—(G 3)$'$, and let $\mathfrak{Y}$ be the locally convex space $\mathfrak{X}'_\sigma$ with the topology of uniform convergence on $\mathfrak{B}$. It follows from Mackey's theorem ([**1**], Chap. IV, § 2, Theorem 2) that $\mathfrak{Y}' = \mathfrak{X}$. According to the propositions in § 5, we see from (G 1)$'$ that the dual operator A' of A is a closed linear operator with a dense domain in $\mathfrak{Y}$, and from (G 2)$'$ and (G 3)$'$ that $\{(I-\lambda_n^{-1}A')^{-m}\}$ is equicontinuous in $\mathfrak{Y}$. Thus A' satisfies the conditions (G 1) and (G 2) in Proposition 6.1. Let T'_t be the semi-group in $\mathfrak{Y}$ generated by A', and let T_t be its dual operator T''_t. We shall prove that T_t is the desired semi-group.

(S 1) and (S 2) follow from corresponding properties for T'_t. Here we note that

$$T_t x = \sigma\text{-weak}\lim_{n\to\infty}\left(I-\frac{tA}{n}\right)^{-n} x \tag{6.9}$$

for every $x \in \mathfrak{X}$. By the same way as in the preceding proof it is proved from (G 2)$'$ that $\{(I-n^{-1}tA)^{-n}\}_{t>0, n=1,2,\dots}$ is τ-equicontinuous. Thus for any (τ, σ) semi-norm $p(x)$ there is a (τ, σ) semi-norm $q(x)$ such that $p((I-n^{-1}tA)^{-n}x) \leqq q(x)$. Since the set $\{x ; p(x) \leqq c\}$ is σ-weakly closed, it follows that $p(T_t x) \leqq q(x)$. This proves that $\{T_t\}$ is τ-equicontinuous. The $\mathfrak{B}$-local equiboundedness (S 3)$''_{\mathfrak{B}}$ is proved by a similar way. (S 4) is clear. To prove (S 5) relative to ω, it is sufficient to show that when s tends to t, $T'_s y$ converges to $T'_t y$ uniformly on every compact set in $\mathfrak{Y}$. But this follows from (S 3) and (S 5) for T'_t by the Ascoli-Arzela theorem.

The topology ω may not satisfy the condition (T 4). However, since the range $T_t x$ is contained in a convex circled and weakly complete set for every x, it is easily shown that the integrals (3.1) and (3.7) converge in $\mathfrak{X}$ relative to ω. If $x \in \mathfrak{D}(A)$, then we have

$$\int_0^t \langle T_s^{(n)} Ax, y \rangle ds = \int_0^t \langle x, T_s'^{(n)} A'y \rangle ds$$

for every $y \in \mathfrak{D}(A')$. When n tends to ∞, the left hand side tends to $\int_0^t \langle T_s Ax, y \rangle ds$, and the right hand side tends to $\int_0^t \langle x, T_s' A'y \rangle ds = \langle x, T_t' y - y \rangle = \langle T_t x - x, y \rangle$. Thus we obtain (6.8.). By the same way as in the proof of Proposition 6.1, we can show that A is the infinitesimal generator of T_t.

When (G 4)$'$ is satisfied, (6.9) proves that $T_t x$ is σ-right continuous at 0, and hence at every t by the semi-group property. The last statement follows from Proposition 1.5.

7. Dual semi-groups.

We shall give a generalization of Phillips' theorem [7].

PROPOSITION 7.1. *Let T_t be a (τ, σ) semi-group of operators in $\mathfrak{X}$ satisfying the condition* (S 4)$'$. *In the dual space $\mathfrak{X}_\sigma'$, let τ' be the topology of uniform convergence on τ-bounded sets in $\mathfrak{X}$ and σ' be the weak* topology relative to σ. If $\mathfrak{X}_\sigma'$ is quasi-complete relative to the Mackey topology $\tau(\mathfrak{X}_\sigma', \mathfrak{X})$ (e.g. if $\mathfrak{X}_\sigma$ is tonnelé or if $\mathfrak{X}_\sigma$ is the weak dual space of a quasi-complete tonnelé space), then the family of dual operators $\{T_t'\}$ forms a (τ', σ') semi-group in $\mathfrak{X}_\sigma'$. Moreover the Laplace transform and the infinitesimal generator of T_t' are equal to the dual operators of the Laplace transform $R(\lambda)$ and the generator A of T_t respectively.*

PROOF. (T 1) and (T 2) are clear, and (T 4) follows from Proposition 1.2. In view of Proposition 3.4, we see that τ' is the strong topology associated with the duality of $\mathfrak{X}'$ and $\mathfrak{X}$. Thus it is stronger than the Mackey topology. Now (T 3) follows as in Proposition 1.1.

Further, (S 1), (S 2), (S 4) and (S 5) are clear, and by Proposition 5.4 we have (S 3).

Let $\tilde{R}(\lambda)$ and $\tilde{A}$ be the Laplace transform and the generator of T_t' respectively. We have

$$\begin{aligned}\langle R(\lambda)x, x' \rangle &= \int_0^\infty e^{-\lambda t} \langle T_t x, x' \rangle dt \\ &= \int_0^\infty e^{-\lambda t} \langle x, T_t' x' \rangle dt \\ &= \langle x, \tilde{R}(\lambda) x' \rangle\end{aligned}$$

for every $x \in \mathfrak{X}$ and $x' \in \mathfrak{X}'$. Hence it follows that $\tilde{R}(\lambda) = R(\lambda)'$, and this implies by Proposition 5.7 $\tilde{A} = A'$.

DEFINITION. Let $\mathfrak{X}$ and T_t satisfy the conditions in Proposition 7.1. We denote by $\mathfrak{X}^+$ the τ'-closure $\mathfrak{D}(A')^{\tau'}$ of $\mathfrak{D}(A')$, and by T_t^+ the restriction of T_t' to $\mathfrak{X}^+$. We shall call T_t^+ the *dual semi-group* of T_t.

PROPOSITION 7.2. *The dual semi-group T_t^+ is a (τ', τ') semi-group in $\mathfrak{X}^+$. The infinitesimal generator A^+ of T_t^+ is the largest restriction of A' with the range in $\mathfrak{X}^+$.*

PROOF. Although this is a consequence of Propositions 4.1 and 4.3, we shall give another proof based on Proposition 6.1.

It follows by Propositions 5.9 and 5.4 that A' has the resolvent $R(\lambda, A') = (\lambda I - A')^{-1}$ such that $\{(\mathrm{Re}\,\lambda R(\lambda, A'))^m\}_{\mathrm{Re}\,\lambda>0, m=1,2,\dots}$ is equicontinuous as a family of operators from $\mathfrak{X}'_{\tau'}$ to $\mathfrak{X}'_{\tau'}$.

Let $R(\lambda)$ be the restriction of $R(\lambda, A')$ to $\mathfrak{X}^+$. $R(\lambda)$ satisfies the resolvent equation and $\mathfrak{N}(R) = \{0\}$ by Proposition 2.4. Thus it is the resolvent of a closed linear operator A^+ in $\mathfrak{X}^+$. By Proposition 2.4 we have also $\tau'\text{-}\lim_{\lambda\to\infty} \lambda R(\lambda)x = x$ for every $x \in \mathfrak{X}^+$. Thus A^+ satisfies (G 1) and (G 2), and so it generates a (τ', τ') semi-group T_t^+ in $\mathfrak{X}^+$. We shall show that T_t^+ is the restriction of T_t' to $\mathfrak{X}^+$. For every $x \in \mathfrak{X}$ and $x' \in \mathfrak{X}^+$, we have

$$\langle (I - tA/m)^{-m}x, x' \rangle = \langle x, (I - tA^+/m)^{-m}x' \rangle .$$

Let m tends to ∞. Then Proposition 6.1 proves $\langle T_t x, x' \rangle = \langle x, T_t^+ x' \rangle$. Hence we have $T_t' x' = T_t^+ x'$.

It remains to show that A^+ is the maximal restriction of A' with the range in $\mathfrak{X}^+$. By the definition we have $A^+ \subset A'$. On the other hand if $x' \in \mathfrak{D}(A')$ and $A'x' \in \mathfrak{X}^+$, then we have $(\lambda I - A')x' \in \mathfrak{X}^+$. Hence we obtain

$$x' = (\lambda I - A^+)^{-1}(\lambda I - A')x \in \mathfrak{D}(A^+) .$$

As an application of the dual semi-group we shall show the uniqueness of the solution of the Cauchy problem for the differential equation

$$\frac{d}{dt}x(t) = Ax(t) . \tag{7.1}$$

PROPOSITION 7.3. *Let T_t be a (τ, σ) semi-group with a dual semi-group T_t^+, and let A be its infinitesimal generator. Let an $\mathfrak{X}$-valued function $x(t)$ satisfy the conditions:* i) *$x(t)$ is continuous relative to the σ-weak topology on $[0, \delta]$ and $x(0) = 0$;* ii) *$x(t)$ is differentiable relative to the σ-weak topology, belongs to $\mathfrak{D}(A)$ and satisfies the equation* (7.1) *on $(0, \delta)$. Then $x(t)$ is identically equal to* 0.

PROOF. For every $a \in (0, \delta)$ and every $x^+ \in \mathfrak{X}^+$ we have

$$\left\langle \frac{T_{a-s}x(s)-T_{a-t}x(t)}{s-t}, x^+ \right\rangle = \left\langle \frac{x(s)-x(t)}{s-t} - Ax(t), T_{a-t}^+x^+ \right\rangle$$
$$+\left\langle \frac{x(s)-x(t)}{s-t} - Ax(t), (T_{a-s}^+ - T_{a-t}^+)x^+ \right\rangle$$
$$+\left\langle T_{a-s}Ax(t), x^+ \right\rangle$$
$$+\left\langle \frac{T_{a-s}-T_{a-t}}{s-t} x(t), x^+ \right\rangle .$$

Let s tend to $t \in (0, a)$. Then the first term tends to 0 because of the differentiability of $x(t)$. By Proposition 3.4 and the τ'-continuity of $T_{a-t}^+x^+$ the second term also tends to 0. The third term converges to $\langle T_{a-t}Ax(t), x^+\rangle$, and finally from Proposition 3.13 we see that the fourth term converges to $\langle -T_{a-t}Ax(t), x^+\rangle$. Thus we have

$$\frac{d}{dt}\langle T_{a-t}x(t), x^+\rangle = 0$$

for any $t \in (0, a)$. Similarly we have

$$\langle T_{a-t}x(t), x^+\rangle = \langle x(t), T_a^+x^+\rangle + \langle x(t), (T_{a-t}^+ - T_a^+)x^+\rangle \to 0$$

as $t \to 0$, and

$$\langle T_{a-t}x(t), x\rangle = \langle x(t), x^+\rangle + \langle x(t), (T_{a-t}^+ - I)x^+\rangle \to \langle x(a), x^+\rangle$$

as $t \to a$. Hence we obtain $\langle x(a), x^+\rangle = 0$. Since $\mathfrak{X}^+$ is weakly* dense in $\mathfrak{X}'$, this proves $x(a)=0$.

By a similar way we can prove a uniqueness theorem weaker but with no stringent condition on the topologies.

PROPOSITION 7.4. *Let T_t be a (τ, σ) semi-group, and A be its infinitesimal generator. A solution $x(t)$ of* (7.1) *which is τ-continuous on $[0, \delta]$ and τ-differentiable on $(0, \delta)$ and satisfies $x(0)=0$ is identically equal to* 0.

PROOF. We have

$$\frac{T_{a-s}x(s)-T_{a-t}x(t)}{s-t} = T_{a-s}\left(\frac{x(s)-x(t)}{s-t} - Ax(t)\right)$$
$$+T_{a-s}Ax(t) + \frac{T_{a-s}-T_{a-t}}{s-t}x(t).$$

When s tends to t, the first term tends to 0 relative to τ by the equicontinuity of $\{T_{a-s}\}$, and the second and the third terms tend to $T_{a-t}Ax(t)$ and $-T_{a-t}Ax(t)$ relative to σ respectively. Thus $T_{a-t}x(t)$ is continuously differentiable relative to σ and the derivative vanishes. Similarly it follows from the equicontinuity of $\{T_{a-t}\}$ that $T_{a-t}x(t)$ is τ-continuous on $[0, a]$. Consequently we have $x(a) = T_ax(0) = 0$.

Proposition 7.4 verifies the uniqueness of the solution given in Proposition 4.4, and Proposition 7.3 verifies the uniqueness of the solution given in Proposition 3.13 under the conditions that $\mathfrak{X}'_\sigma$ is quasi-complete relative to the Mackey topology $\tau(\mathfrak{X}'_\sigma, \mathfrak{X})$ and that T_t satisfies (S 4)$'$.

8. Analytic functions in locally convex spaces.

In this section we shall develop the theory of analytic functions with values in locally convex spaces as a preparation for the next section.

DEFINITION. Let $\mathfrak{X}$ be a locally convex space over the complex number field, and let $f(\zeta)=f(\xi+i\eta)$ be an $\mathfrak{X}$-valued function defined in a domain on the complex plane. We call $f(\zeta)$ *analytic* if it satisfies the conditions:

(A 1) $f(\zeta)$ is infinitely differentiable as an $\mathfrak{X}$-valued function of real variables ξ and η.

(A 2) It satisfies the Cauchy-Riemann differential equation

$$\frac{\partial}{\partial\bar{\zeta}}f(\zeta)=\frac{1}{2}\Big(\frac{\partial}{\partial\xi}+i\frac{\partial}{\partial\eta}\Big)f(\zeta)=0\,.$$

PROPOSITION 8.1. *If $f(\zeta)$ is an analytic function, then its derived functions $f^{(n)}(\zeta)$ are analytic and we have*

$$\oint f(\zeta)d\zeta=0\,, \tag{8.1}$$

$$\frac{n\,!}{2\pi i}\oint\frac{f(\zeta)}{(\zeta-\alpha)^{n+1}}d\zeta=f^{(n)}(\alpha) \tag{8.2}$$

for any closed curve encircling α, where the integrals are taken in the sense of Riemann and $f^{(n)}(\alpha)=\partial^n f(\xi+i\eta)/\partial\xi^n|_{\xi+i\eta=\alpha}$.

PROOF. We remark that for any $x'\in\mathfrak{X}'$ $\langle f(\zeta), x'\rangle$ is analytic as a numerical function and we have $\langle f(\zeta), x'\rangle^{(n)}=\langle f^{(n)}(\zeta), x'\rangle$. Thus $f^{(n)}(\zeta)$ satisfies (A 2). Since $f(\zeta)$ is continuous, the Riemann sums form a Cauchy net, and as the above remark shows they converge weakly to the right hand side. By the same argument as in the proof of Proposition 1.1, we see that the Riemann sums converge to the right hand side relative to the original topology of $\mathfrak{X}$.

PROPOSITION 8.2. *If $f(\zeta)$ is analytic in the circle $|\zeta-\alpha|<\rho$, then $r^n f^{(n)}(\alpha)/n\,!$ converges to 0 for any $|r|<\rho$, and we have*

$$f(\zeta)=\sum_{n=0}^{\infty}\frac{f^{(n)}(\alpha)}{n\,!}(\zeta-\alpha)^n \tag{8.3}$$

for any ζ with $|\zeta-\alpha|<\rho$, where the series converges uniformly on every compact set in the circle in the sense of Mackey, a fortiori relative to the original topology.

PROOF. The assertions easily follow from the integral formula (8.2). We have only to take for the path of integration the circle with the radius ρ', $r < \rho' < \rho$ and note that $\{f(\zeta)\}$ on the circle forms a compact, hence a bounded set in $\mathfrak{X}$.

PROPOSITION 8.3. *We assume that every continuous function defined on* $[0, 1]$ *and with values in* $\mathfrak{X}$ *is Riemann integrable. Then, every* $\mathfrak{X}$*-valued continuous function* $f(\zeta)$ *which satisfies* (8.1) *for any closed curve is analytic.*

PROOF. From the assumptions we easily obtain Cauchy's integral formula

$$\frac{1}{2\pi i}\oint\frac{f(\zeta)}{(\zeta-\alpha)}d\zeta=f(\alpha)\,. \tag{8.4}$$

Thus we have

$$\frac{f(\alpha+\xi)-f(\alpha)}{\xi}-\frac{1}{2\pi i}\oint\frac{1}{(\zeta-\alpha)^2}f(\zeta)d\zeta$$
$$=\frac{1}{2\pi i}\oint\Big(\frac{1}{(\zeta-\alpha-\xi)(\zeta-\alpha)}-\frac{1}{(\zeta-\alpha)^2}\Big)f(\zeta)d\zeta\,.$$

When ξ tends to 0, the right hand side converges to 0. This shows that $f(\zeta)$ is differentiable. Similarly, or noting that $f^{(n-1)}(\zeta)$ satisfies the assumptions in the proposition, we see that $f(\zeta)$ is infinitely differentiable. The Cauchy-Riemann equation is proved similarly.

REMARK. In order to assert (8.1), it is sufficient to show that $\langle f(\zeta), x'\rangle$ is analytic for x' in a total set of $\mathfrak{X}'$.

PROPOSITION 8.4. *Let* $\mathfrak{X}$ *be sequentially complete, and let* $\{a_n\}$ *be a sequence of elements in* $\mathfrak{X}$ *such that* $\{\rho^n a_n\}$ *is bounded for a* $\rho>0$. *Then*

$$f(\zeta)=\sum_{n=0}^{\infty}a_n\zeta^n \tag{8.5}$$

converges for $|\zeta|<\rho$ *and represents an analytic function such that* $a_n = f^{(n)}(0)/n\,!$.

PROOF. The convergence follows from the sequential completeness. We see easily that $f(\zeta)$ is continuous and satisfies (8.1). The last statement easily follows from (8.2).

PROPOSITION 8.5. *If* $\mathfrak{X}$ *is sequentially complete (relative to the Mackey topology), every scalarly analytic function (i.e. a function* $f(\zeta)$ *such that* $\langle f(\zeta), x'\rangle$ *is analytic for every* $x'\in\mathfrak{X}'$*) is analytic.*

PROOF. Since $\mathfrak{X}$ is sequentially complete, the scalarly infinitely differentiable function $f(\zeta)$ is infinitely differentiable (cf. [3], Chap. III, § 8, Proposition 15). From Proposition 8.3 it follows that a continuous and scalarly analytic function is analytic.

PROPOSITION 8.6. *Let two locally convex topologies* τ *and* σ *in* $\mathfrak{X}$ *satisfy the conditions* (T 1) *and* (T 2). *If an* $\mathfrak{X}$*-valued function* $f(\zeta)$ *is* σ*-analytic and* τ*-bounded on every compact subset of the domain of definition, then it is*

τ-analytic.

PROOF. We have

$$f^{(n)}(\zeta)=\int_{\alpha}^{\zeta} f^{(n+1)}(z)dz+f^{(n)}(\alpha) \tag{8.6}$$

in the sense of a σ-Riemann integral. The proof is similar to that of Proposition 8.1. If ρ is small, there is a convex circled and τ-bounded set B such that $f^{(n+1)}(z)\in B$ for any z with $|z-\alpha|<\rho$. Because of (T 2) we may assume that B is σ-closed. Thus we have, for ζ with $|\zeta-\alpha|<\rho$, $f^{(n)}(\zeta)-f^{(n)}(\alpha)\in|\zeta-\alpha|B$. This shows that $f^{(n)}(\zeta)$ is continuous relative to τ. Now the Riemann sums which define the integral of (8.6) form a Cauchy net relative to τ and converge relative to σ. Hence, as in the proof of Proposition 1.1, we see that it converges to the same limit relative to τ. We have thus (8.6) in the sense of a τ-Riemann integral. This shows that $f^{(n)}(\zeta)$ is differentiable relative to τ and its derivative is $f^{(n+1)}(\zeta)$. Thus $f(\zeta)$ satisfies (A 1). (A 2) is clear.

COROLLARY. *If $\mathfrak{X}_{\sigma}$ satisfies the condition in Proposition* 3.4, *then every analytic function $f(\zeta)$ is also analytic relative to the strong topology associated with the original topology.*

PROOF. The original topology and the strong topology have the same family of bounded sets. (T 2) is evident.

DEFINITION. An $\mathfrak{X}$-valued function $f(\xi)$ defined on a real interval is called a *real analytic function* if it is infinitely differentiable and if it is represented by the Taylor series (8.3) around every point α in the interval with a positive radius of convergence.

PROPOSITION 8.7. *Let $\mathfrak{X}$ be sequentially complete. Then, an infinitely differentiable function $f(\xi)$ is real analytic if and only if for every compact set K in the interval there is a $\rho_K>0$ such that $\{\rho_K^n f^{(n)}(\xi)/n!\}_{\xi\in K, n=0,1,2,\cdots}$ is bounded. In this case $f(\xi)$ can be extended to a complex analytic function $f(\zeta)$ defined at least for ζ with* $\mathrm{dis}(\zeta, K)<\rho_K$.

PROOF. If $f(\xi)$ is represented by (8.3) for $|\xi-\alpha|<\rho$, then it is easily proved that, with $\rho_1=\rho/3$, $\{\rho_1^n f^{(n)}(\xi)/n!\}_{|\xi-\alpha|<\rho_1, n=0,1,\cdots}$ is bounded. Thus only if part follows.

Conversely if $f(\xi)$ satisfies the conditions, then the Taylor series converges and expresses an analytic function $f_{\alpha}(\zeta)$ for ζ with $|\zeta-\alpha|<\rho_{\{\alpha\}}$. We have to show that $f_{\alpha}(\zeta)=f_{\beta}(\zeta)$ when ζ belongs to the intersection of the two disks. To do so, it is sufficient to show that $\langle f_{\alpha}(\xi), x'\rangle=\langle f(\xi), x'\rangle$ for real ξ with $|\xi-\alpha|<\rho_{\{\alpha\}}$ and $x'\in\mathfrak{X}'$. But this is easily proved by estimating the residual term in the Taylor expansion. This completes the proof.

Next we consider operator valued functions. Let $\mathfrak{X}$ and $\mathfrak{Y}$ be two locally convex spaces. We denote by $\mathfrak{L}_{\mathfrak{S}}(\mathfrak{X}, \mathfrak{Y})$ the space of continuous linear operators from $\mathfrak{X}$ to $\mathfrak{Y}$ with the topology of the uniform convergence on every set in $\mathfrak{S}$, where $\mathfrak{S}$ is a family of bounded sets in $\mathfrak{X}$ which covers $\mathfrak{X}$. $\mathfrak{L}_{\mathfrak{S}}(\mathfrak{X}, \mathfrak{Y})$ is a locally convex space.

As a corollary of Proposition 8.6 we have the

PROPOSITION 8.8. *Let $f(\zeta)$ be an $\mathfrak{L}_{\mathfrak{S}}(\mathfrak{X}, \mathfrak{Y})$ valued analytic function. If $\{f(\zeta)\}_{\zeta\in K}$ is equicontinuous for every compact set K in the domain, then $f(\zeta)$ is analytic in $\mathfrak{L}_{\mathfrak{B}}(\mathfrak{X}, \mathfrak{Y})$ with $\mathfrak{B}$ the set of all bounded sets in $\mathfrak{X}$.*

We shall call such an analytic function $f(\zeta)$ a *locally equicontinuous* analytic function.

PROPOSITION 8.9. *An analytic function $f(\zeta)$ with values in $\mathfrak{L}(\mathfrak{X}, \mathfrak{Y})$ is locally equicontinuous if and only if $\{\rho_1^n f^{(n)}(\alpha)/n!\}$ is equicontinuous for every α with a $\rho_1 > 0$.*

PROOF. It is sufficient to note that the closed convex circled hull of equicontinuous set is equicontinuous.

COROLLARY 1. *An analytic function $f(\zeta)$ in $\mathfrak{L}(\mathfrak{X}, \mathfrak{Y})$ is locally equicontinuous if and only if its derived function $f'(\zeta)$ is locally equicontinuous.*

COROLLARY 2. *Assume that every set in $\mathfrak{X}$ which is an intersection of a countable set of convex circled and closed neighborhoods of* 0 *and absorbs every set in $\mathfrak{S}$ is a neighborhood of* 0 *(e.g., $\mathfrak{X}$ is a tonnelé space or a DF space). Then every $\mathfrak{L}_{\mathfrak{S}}(\mathfrak{X}, \mathfrak{Y})$ analytic function is locally equicontinuous.*

PROOF. It is sufficient to show that every countable and $\mathfrak{L}_{\mathfrak{S}}(\mathfrak{X}, \mathfrak{Y})$ bounded set $\{u_n\}$ of continuous linear operators is equicontinuous. Let V be a convex circled and closed neighborhood of 0 in $\mathfrak{Y}$. The set $U = \cap u_n^{-1}(V)$ is the intersection of a countable set of convex circled and closed neighborhoods and, since $\{u_n\}$ is bounded, absorbs every set in $\mathfrak{S}$. Thus U is a neighborhood of 0.

PROPOSITION 8.10. *Let $\mathfrak{X}$ be countably tonnelé (cf. Proposition* 1.3*) and $\mathfrak{Y}$ be sequentially complete. Then, an $\mathfrak{L}_{\mathfrak{S}}(\mathfrak{X}, \mathfrak{Y})$ valued function $f(\zeta)$ is analytic if and only if $\langle f(\zeta)x, y'\rangle$ is analytic for every $x\in\mathfrak{X}$ and $y'\in\mathfrak{Y}'$. In this case every analytic function is locally equicontinuous.*

PROOF. Only if part is evident. To prove the converse first we consider the case when $\mathfrak{S}$ is the set s of all finite sets in $\mathfrak{X}$. We know that the dual space of $\mathfrak{L}_s(\mathfrak{X}, \mathfrak{Y})$ is $\mathfrak{X}\otimes\mathfrak{Y}'$ ([**1**], Chap. IV, § 2, Proposition 11), and it is easily shown that $\mathfrak{L}_s(\mathfrak{X}, \mathfrak{Y})$ is sequentially complete. Therefore it follows from Proposition 8.5 that $f(\zeta)$ is analytic in $\mathfrak{L}_s(\mathfrak{X}, \mathfrak{Y})$. We see by Corollary 2 of Proposition 8.9 that $f(\zeta)$ is locally equicontinuous. Thus it is analytic in $\mathfrak{L}_{\mathfrak{S}}(\mathfrak{X}, \mathfrak{Y})$.

PROPOSITION 8.11. *Let $\mathfrak{X}$, $\mathfrak{Y}$, and $\mathfrak{Z}$ be locally convex spaces. If $f(\zeta)$ is a locally equicontinuous analytic function in $\mathfrak{L}(\mathfrak{Y}, \mathfrak{Z})$ and $g(\zeta)$ is an analytic function in $\mathfrak{L}_{\mathfrak{S}}(\mathfrak{X}, \mathfrak{Y})$, then the product $f(\zeta)g(\zeta)$ is an analytic function in $\mathfrak{L}_{\mathfrak{S}}(\mathfrak{X}, \mathfrak{Z})$.*

And we have

$$\frac{d}{d\zeta}(f(\zeta)g(\zeta))=\frac{d}{d\zeta}f(\zeta)\cdot g(\zeta)+f(\zeta)\cdot\frac{d}{d\zeta}g(\zeta)\,. \tag{8.7}$$

PROOF. First we prove the continuity in ζ. Let the disk $|\zeta-\alpha|\leqq\rho$ be contained in the domain. For any continuous semi-norm $p(z)$ in $\mathfrak{Z}$, there is a continuous semi-norm $q(y)$ in $\mathfrak{Y}$ such that $p(f(\zeta)y)\leqq q(y)$ for any $y\in\mathfrak{Y}$ and ζ in the disk. We have for any $x\in\mathfrak{X}$ and any α, ζ in the disk

$$\begin{aligned}&p((f(\zeta)g(\zeta)-f(\alpha)g(\alpha))x)\\&\leqq p(f(\zeta)(g(\zeta)-g(\alpha))x)+p((f(\zeta)-f(\alpha))g(\alpha)x)\\&\leqq q((g(\zeta)-g(\alpha))x)+p((f(\zeta)-f(\alpha))g(\alpha)x)\,.\end{aligned}$$

Let x be in a set $B\in\mathfrak{S}$ and let ζ tend to α. Then the first term uniformly tends to 0, and since $g(\alpha)B$ is a bounded set in $\mathfrak{Y}$, the second term also tends to 0 uniformly in x. This proves the continuity.

Similarly we have

$$\begin{aligned}&p\Big(\frac{1}{\zeta-\alpha}(f(\zeta)g(\zeta)-f(\alpha)g(\alpha))x-(f'(\alpha)g(\alpha)+f(\alpha)g'(\alpha))x\Big)\\&\leqq q\Big(\Big(\frac{g(\zeta)-g(\alpha)}{\zeta-\alpha}-g'(\alpha)\Big)x\Big)+p((f(\zeta)-f(\alpha))g'(\zeta)x)\\&\quad+p\Big(\Big(\frac{f(\zeta)-f(\alpha)}{\zeta-\alpha}-f'(\alpha)\Big)g(\alpha)x\Big)\,.\end{aligned}$$

Let $x\in B$. When ζ tends to α, the first term uniformly tends to 0 and, since $\{g'(\zeta)B\}_{|\zeta-\alpha|\leqq\rho}$ is bounded in $\mathfrak{Y}$, the second term also tends to 0. Finally it follows from the differentiability of $f(\zeta)$ in $\mathfrak{L}_{\mathfrak{B}}(\mathfrak{Y},\mathfrak{Z})$ and the boundedness of $g(\alpha)B$ that the third term converges to 0. This proves that $f(\zeta)g(\zeta)$ is differentiable and that its derived function is given by (8.7).

Now the infinite differentiability is proved recursively. Replacing $\zeta-\alpha$ by $\bar\zeta-\bar\alpha$ we see that $f(\zeta)g(\zeta)$ satisfies the Cauchy-Riemann equation.

Lastly we give a special proposition which will be used in the next section.

PROPOSITION 8.12. *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be locally convex spaces, and let $f(\zeta)$ be an $\mathfrak{L}_{\mathfrak{S}}(\mathfrak{X},\mathfrak{Y})$-valued analytic function defined in the open sector $|\arg\zeta|<\pi/2+\theta$ with $\theta>0$. If $\{\zeta f(\zeta)\}$ is equicontinuous in every subsector $|\arg\zeta|\leqq\pi/2+\theta-\varepsilon$, $\varepsilon>0$, $\zeta\neq0$, then the family of operators $\{\zeta^{n+1}f^{(n)}(\zeta)/n\,!\}$ is equicontinuous in every sector $|\arg\zeta|\leqq\theta-\varepsilon$, $\varepsilon>0$, $\zeta\neq0$.*

PROOF. Using the integral formula

$$f'(\alpha)=\frac{1}{2\pi i}\int_{|\zeta-\alpha|=c|\alpha|}\frac{f(\zeta)}{(\zeta-\alpha)^2}d\zeta\,,$$

we see easily that $\{\zeta^2f'(\zeta)\}$ is equicontinuous in every subsector $|\arg\zeta|\leqq\pi/2$

$+\theta-\varepsilon$. The assertion now follows from the following

LEMMA. *Let $\mathfrak{E}$ be a convex circled closed and equicontinuous set in $\mathfrak{L}_{\mathfrak{S}}(\mathfrak{X}, \mathfrak{Y})$. If $g(\zeta)$ is an $\mathfrak{L}_{\mathfrak{S}}(\mathfrak{X}, \mathfrak{Y})$-valued analytic function defined in the half plane* $\operatorname{Re}\zeta>0$ *such that $\zeta^2 g(\zeta)\in\mathfrak{E}$, then we have $\xi^{n+2}g^{(n)}(\xi)/(n+1)!\in 2\mathfrak{E}$ for any $\xi>0$ and $n=0, 1, 2, \cdots$.*

PROOF. Deforming the path of integration in (8.2) we have

$$\frac{g^{(n-1)}(\xi)}{(n-1)!}=\frac{-1}{2\pi i}\int_{\xi/(n+1)-i\infty}^{\xi/(n+1)+i\infty}\frac{1}{(\zeta-\xi)^n}g(\zeta)d\zeta .$$

This yields

$$\begin{aligned}\frac{\xi^{n+1}g^{(n-1)}(\xi)}{n!}&\in\frac{\xi^{n+1}}{2\pi n}\left(\frac{n+1}{n\xi}\right)^n\int_{-\infty}^{\infty}\frac{d\tau}{\tau^2+(\xi/(n+1))^2}\mathfrak{E}\\&=\frac{1}{2}\left(\frac{n+1}{n}\right)^{n+1}\mathfrak{E}\subset 2\mathfrak{E} .\end{aligned}$$

9. Analytic semi-groups.

DEFINITION. A (τ, σ) semi-group T_t is called an *analytic semi-group of class* $\mathfrak{H}(\theta)$ if it is real analytic in $\mathfrak{L}_s(\mathfrak{X}_\tau, \mathfrak{X}_\sigma)$ for $t>0$ and has an analytic continuation T_t defined in the sector $|\arg t|<\theta$ with $\theta>0$ such that $\{T_t\}$ is equicontinuous in $\mathfrak{L}(\mathfrak{X}_\tau, \mathfrak{X}_\tau)$ on every subsector $|\arg t|\leqq\theta-\varepsilon$, $\varepsilon>0$.

PROPOSITION 9.1. *If T_t is a (τ, σ)-analytic semi-group of class $\mathfrak{H}(\theta)$, then the continuation T_t is analytic as an $\mathfrak{L}_{\mathfrak{B}_\tau}(\mathfrak{X}_\tau, \mathfrak{X}_\tau)$-valued function in the sector $|\arg t|<\theta$, and satisfies the semi-group property* (S 2) *for any t and s in the sector. Furthermore for any x in the τ-center $\mathfrak{D}(A)^\tau$ we have $T_t x\to x$ relative to τ as t tends to 0 through the Stolz region $|\arg t|\leqq\theta-\varepsilon$, $\varepsilon>0$.*

PROOF. The analyticity in $\mathfrak{L}_{\mathfrak{B}_\tau}(\mathfrak{X}_\tau, \mathfrak{X}_\tau)$ follows from Propositions 8.6 and 8.8. The unique continuation property of analytic functions proves the semi-group property.

If $x=T_\delta y$ with $\delta>0$, then clearly we have $T_t x\to x$ relative to τ as t tends to 0. Since we have $T_\delta y\to y$ (τ) for any $y\in\mathfrak{X}_0=\mathfrak{D}(A)^\tau$ as $\delta\to 0$, the set $\{T_\delta y; y\in\mathfrak{X}_0, \delta>0\}$ is dense in $\mathfrak{X}_0$. Now noticing that T_t is equicontinuous on the Stolz region $|\arg t|\leqq\theta-\varepsilon$, we can prove that for any $x\in\mathfrak{X}_0$ $T_t x$ tends to x relative to τ as $t\to 0$ in the region by the same way as in the proof of Proposition 1.4.

From now on we shall identify the original semi-group and its continuation.

PROPOSITION 9.2. *Let T_t be a (τ, σ) semi-group and $R(\lambda)$ be its Laplace transform. Then the following conditions are equivalent.*

(I) *T_t is an analytic semi-group of class $\mathfrak{H}(\theta)$.*

(II) *$\lambda R(\lambda)$ can be continued analytically to the open sector $|\arg\lambda|<\theta+\pi/2$ as an $\mathfrak{L}_{\mathfrak{S}}(\mathfrak{X}_\tau, \mathfrak{X}_\tau)$-valued function and equicontinuous on every subsector $|\arg\lambda|\leqq\theta+\pi/2-\varepsilon$, $\varepsilon>0$, $\lambda\neq 0$.*

(III) *The family of operators* $\{(\lambda R(\lambda))^n\}_{n=0,1,2,\cdots}$ *is equicontinuous in* $\mathfrak{L}(\mathfrak{X}_\tau, \mathfrak{X}_\tau)$ *on every sector* $|\arg\lambda|\leqq\theta-\varepsilon$, $\varepsilon>0$, $\lambda\neq 0$.

(IV) *If* $t>0$, *then* $T_t x\in\mathfrak{D}(A)$ *for any* x *and the family of operators* $\{(cnt\,AT_t)^n/n\,!\}_{n=0,1,2,\cdots,t>0}$ *is equicontinuous is* $\mathfrak{L}(\mathfrak{X}_\tau, \mathfrak{X}_\tau)$ *for any* c *with* $0<c<\sin\theta$.

PROOF. 1° (I) implies (II). If $|\arg\lambda|<\theta+\pi/2$, we can find an ω such that $|\arg\lambda+\omega|<\pi/2$ and $|\omega|<\theta$. We define $R(\lambda)$ for such λ by

$$R(\lambda)x=\int_{+0}^{e^{i\omega}\infty}e^{-\lambda t}T_t x dt$$

in the sense of an improper σ-integral. From (I) it easliy follows that the integral converges and represents a continuous linear operator $R(\lambda)$ independent of the choice of ω. If $|\arg\lambda|<\pi/2$, this operator coincides with that given in §3. Since the analyticity in λ is clear, we see that it gives an analytic extension. We have, for any (τ,σ) semi-norm $p(x)$ and $x\in\mathfrak{X}$, the estimate

$$\begin{aligned} p(\lambda R(\lambda)x) &\leqq\int_0^\infty|\lambda|e^{-\mathrm{Re}(\lambda e^{i\omega})\tau}d\tau\cdot\sup_{\arg t=\omega}p(T_t x) \\ &=\sec(\arg\lambda+\omega)q_\omega(x)\,, \end{aligned} \tag{9.1}$$

where $q_\omega(x)$ is a (τ,σ) semi-norm depending on ω. This proves (II).

2° (II) is equivalent to (III). Proposition 8.12 and the relation

$$R(\lambda)^{n+1}=\frac{(-1)^n}{n\,!}\frac{d^n}{d\lambda^n}R(\lambda) \tag{9.2}$$

proves that (II) implies (III). Conversely if (III) holds, we can prove (II) by the same way as in the proof of Proposition 6.1.

3° (II) implies (IV). First we prove that when (II) holds, we have the representation

$$T_t x=\frac{1}{2\pi i}\int_\Gamma e^{t\lambda}R(\lambda)xd\lambda \tag{9.3}$$

for any $x\in\mathfrak{X}$ and $t>0$, where the integral is in the sense of an improper τ-Riemann integral and the path of integration Γ runs from $e^{-i\omega_1}\infty$ to $e^{i\omega_2}\infty$ with $\pi/2<\omega_i<\theta+\pi/2$.

It follows from Proposition 8.1 and (9.2) that for any $x\in\mathfrak{X}$ and $n\geqq 1$ we have

$$\begin{aligned} \left(I-\frac{tA}{n}\right)^{-n}x=\left(\frac{n}{t}R\left(\frac{n}{t}\right)\right)^n x &=\frac{-1}{2\pi i}\oint_{|\lambda-n/t|=\varepsilon}\left(1-\frac{t\lambda}{n}\right)^{-n}R(\lambda)xd\lambda \\ &=\frac{1}{2\pi i}\int_\Gamma\left(1-\frac{t\lambda}{n}\right)^{-n}R(\lambda)xd\lambda \end{aligned} \tag{9.4}$$

in the sense of a τ-Riemann integral. Let n tends to infinity. We see easily that the right hand side of (9.4) tends to that of (9.3) relative to τ. On the other hand

$$\left(\frac{n}{t}R\left(\frac{n}{t}\right)\right)^n=\frac{1}{(n-1)!}\int_0^\infty\left(\frac{n}{t}\right)^n s^{n-1}e^{-ns/t}T_sxds$$

shows that the left hand side of (9.4) converges to T_tx relative to σ. This proves (9.3).

Next we shall show that if $t>0$, then $T_tx\in\mathfrak{D}(A)$ for every x and we have

$$AT_tx=\frac{1}{2\pi i}\int_\Gamma \lambda e^{\lambda t}R(\lambda)xd\lambda\,. \tag{9.5}$$

Let μ be a positive number. We see easily that the integral

$$S_{\mu,t}x=\frac{1}{2\pi i}\int_\Gamma(\mu-\lambda)e^{\lambda t}R(\lambda)xd\lambda$$

converges in the sense of a τ-Riemann integral. We have

$$\begin{aligned}R(\mu)S_{\mu,t}x&=\frac{1}{2\pi i}\int_\Gamma(\mu-\lambda)e^{\lambda t}R(\mu)R(\lambda)xd\lambda\\&=\frac{1}{2\pi i}\int_\Gamma e^{\lambda t}R(\lambda)xd\lambda-\frac{1}{2\pi i}\int_\Gamma e^{\lambda t}R(\mu)xd\lambda\\&=T_tx\,.\end{aligned}$$

Hence we obtain $T_tx\in\mathfrak{D}(A)$ and by the equation $AT_tx=(\mu R(\mu)-I)S_{\mu,t}x$ (9.5) follows. Repeating the same arguments, we can prove that if $t>0$, $T_tx\in\mathfrak{D}(A^n)$ for every x and every n, and that

$$A^nT_tx=\frac{1}{2\pi i}\int_\Gamma\lambda^n e^{\lambda t}R(\lambda)xd\lambda$$

holds. It follows by Proposition 3.13 that $A^nT_{nt}x=(AT_t)^nx$.

Now let ω be the solution of $\cos\omega=-c$. We can take for Γ a curve arbitrarily close to the rays $|\arg\lambda|=\omega$. Therefore for any τ-continuous semi-norm $p(x)$ we have

$$\begin{aligned}p((ctAT_{t/n})^n/n!)&\leqq\frac{1}{\pi}\int_0^\infty\frac{(ctr)^{n-1}}{n!}e^{-ctr}ctdr\sup_{|\arg\lambda|=\omega}p(\lambda R(\lambda)x)\\&=\frac{1}{n\pi}q_\omega(x)\,.\end{aligned}$$

with a continuous semi-norm $q_\omega(x)$ depending on ω.

4° (IV) implies (I). If it is shown that T_tx is infinitely differentiable relative to τ and that the derivatives are given by

$$\left(\frac{d}{dt}\right)^nT_tx=(AT_{t/n})^nx\,, \tag{9.6}$$

then (I) is an easy consequence of Proposition 8.7.

$AT_tx=T_{t-t_0}AT_{t_0}x\in\mathfrak{D}(A)$ proves $T_tx\in\mathfrak{D}(A_0)$. Thus it follows from Proposition 4.4 that $T_t x$ is differentiable relative to τ and $d/dt\ T_t x = AT_t x$. Similarly $A^2 T_t x = AT_{t-t_0}\ AT_{t_0} x = T_{t-t_0} A^2 T_{t_0} x \in \mathfrak{D}(A)$ proves the differentiability of $d/dt\ T_t x$ and (9.6) for $n=2$. The argument is the same for $n>2$.

PROPOSITION 9.3. *Let A be a closed linear operator with a dense domain relative to τ. Then, in order that A be the generator of a (τ, τ) analytic semi-group of class $\mathfrak{H}(\theta)$ it is necessary and sufficient that A has the resolvent $R(\lambda) = R(\lambda, A)$ which satisfies one of the conditions* (II) *or* (III) *of Proposition* 9.2.

PROOF. We know that (II) and (III) are equivalent and that they are necessary. To prove the sufficiency we could use the representation (9.3) as in Hille-Phillips [4], Yosida [**10**] and Kato [**6**]. But we shall adopt here a method related to the proof of Proposition 6.1.

We define

$$T_t^{(n)} x = \left(I - \frac{tA}{n}\right)^{-n} x$$

for t with $|\arg t| < \theta$ and $x \in \mathfrak{X}$. From (III) it follows that $T_t^{(n)}$ is analytic and equicontinuous in every subsector $|\arg t| \leqq \theta - \varepsilon$, $\varepsilon > 0$. By the same way as in § 6, we can show that

$$T_t x = \lim_{n\to\infty} T_t^{(n)} x$$

exists for every x and t with $|\arg t| < \theta$ and that the convergence is uniform on every compact set in the sector. Therefore $T_t x$ is analytic in t in the sector $|\arg t| < \theta$, and $\{T_t\}$ is, as the set of limits of operators from an equicontinuous set, equicontinuous in every subsector $|\arg t| \leqq \theta - \varepsilon$, $\varepsilon > 0$. Clearly this T_t gives an analytic continuation of T_t constructed in § 6.

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