J. Math. Soc. Japan
Vol. 16, No. 3, 1964

On 1-cohomology groups of infinite dimensional representations of semisimple Lie algebras

By Akira HATTORI

(Received Feb. 22, 1964)

Let $\mathfrak{g}$ be a semisimple Lie algebra over an algebraically closed field K of characteristic 0. Then all finite dimensional representations of $\mathfrak{g}$ are completely reducible. In homological language, this well-known theorem of Weyl is expressed by saying that $H^1(\mathfrak{g}, V)=0$ for every finite dimensional $\mathfrak{g}$-module V. Harish-Chandra [2] showed that the usual Cartan-Weyl theory is extendible in a large extent to some wide class of infinite dimensional representations. But the complete reducibility fails to hold for them. Indeed, simple examples show that $\mathrm{Ext}^1(K, V)=H^1(\mathfrak{g}, V)\neq 0$ for certain irreducible spaces V(see below). The purpose of the following lines is to determine the structure of $H^1(\mathfrak{g}, V)$ for irreducible spaces of that type.

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. We shall make use of the following simple exact sequence (established in more general setting in Hirata [4] and Hattori [3]):

$$0\rightarrow H^1(\mathfrak{g}, \mathfrak{h}, V)\rightarrow H^1(\mathfrak{g}, V)\rightarrow H^1(\mathfrak{h}, V). \tag{1}$$

In the general setting, the relative cohomology group $H^1(\mathfrak{g}, \mathfrak{h}, V)$ is the one defined by Hochschild [5]. In the present case, $\mathfrak{h}$ is a reductive subalgebra of $\mathfrak{g}$, and the relative cohomology groups coincide with those defined by Chevalley and Eilenberg [1] as is shown in [5].

The structure of $H^1(\mathfrak{h}, V)$ is quite simple. In general, we have

LEMMA 1. *Let $\mathfrak{h}$ be an abelian Lie algebra, and V^μ be an $\mathfrak{h}$-module such that $hv=\mu(h)v$ for every $h\in\mathfrak{h}$, $v\in V^\mu$, where μ is a linear form on $\mathfrak{h}$. Then we have, for $n=0, 1, 2, \cdots$,*

$$H^n(\mathfrak{h}, V^\mu)=\begin{cases} \mathrm{Hom}_K(E^n(\mathfrak{h}), V^\mu) & (\mu=0), \\ 0 & (\mu\neq 0), \end{cases}$$

where $E^n(\mathfrak{h})$ is the homogeneous component of degree n of the exterior algebra of $\mathfrak{h}$.

PROOF. An n-cochain $f\in C^n(\mathfrak{h}, V^\mu)$ is an n-cocycle if and only if

$$\mu(h_0)f(h_1, \cdots, h_n)-\mu(h_1)f(h_0, h_2, \cdots, h_n)+\cdots\pm\mu(h_n)f(h_0, \cdots, h_{n-1})=0$$

for every $h_0, h_1, \cdots, h_n \in \mathfrak{h}$. When $\mu \neq 0$, there is an $h_* \in \mathfrak{h}$ such that $\mu(h_*) \neq 0$. Put $g(h_1, \cdots, h_{n-1}) = \mu(h_*)^{-1} f(h_*, h_1, \cdots, h_{n-1})$, then we see immediately $\delta g = f$. When $\mu = 0$, our assertion is evident.

COROLLARY. *Assume that a* $\mathfrak{g}$*-module* V *admits a (finite or infinite) direct sum decomposition into weight spaces* V^μ *with respect to* $\mathfrak{h}$. *If* 0 *is not a weight of* V, *then* $H^n(\mathfrak{h}, V) = 0$, $n = 0, 1, 2, \cdots$.

We now fix a lexicographic ordering in the rational vector space $\mathfrak{h}_0^*$, and let $\alpha_1, \cdots, \alpha_l$ be the system of simple roots relative to this ordering. We denote the positive roots generally by α, a non-zero vector belonging to the root α (resp. $-\alpha$) by x_α (resp. y_α), and the half sum of positive roots by δ: $\delta = \frac{1}{2} \sum \alpha$ [1].

For any $\lambda \in \mathfrak{h}_0^*$, there exists one and only one irreducible $\mathfrak{g}$-space having λ as the highest weight, which we denote by V_λ. V_λ admits a direct sum decomposition into weight spaces: $V_\lambda = \sum_\mu V_\lambda^\mu$, where each weight μ has the form $\mu = \lambda - \sum n_i \alpha_i$, n_i's being non-negative integers, and V_λ^λ is 1-dimensional (cf. Harish-Chandra [2], Séminaire Sophus Lie [8]). We shall prove

THEOREM. $H^1(\mathfrak{g}, V_\lambda)$ *is* 1*-dimensional over* K *for* $\lambda = -\alpha_i$, $i = 1, \cdots, l$, *and reduces to* 0 *for other* λ*'s.*

As is well-known, the Casimir operator C of V_λ is given by $C(v) = \gamma_\lambda v$ ($v \in V_\lambda$), where

$$\gamma_\lambda = \langle \lambda + \delta, \lambda + \delta \rangle - \langle \delta, \delta \rangle = \langle \lambda, \lambda \rangle + 2 \langle \lambda, \delta \rangle. \tag{2}$$

If $\gamma_\lambda \neq 0$, $H^n(\mathfrak{g}, V_\lambda)$ vanish for all $n = 0, 1, 2, \cdots$, as in the finite dimensional representations. Therefore we shall restrict our considerations to the case $\gamma_\lambda = 0$. Furthermore we may assume $\lambda \neq 0$, since $H^1(\mathfrak{g}, K) = 0$.

LEMMA 2. *Let* $\lambda = \sum m_i \alpha_i \neq 0$, $m_i \geqq 0$, $i = 1, \cdots, l$. *Then* $\gamma_\lambda > 0$.

Indeed, we see immediately that

$$\gamma_\lambda = \langle \lambda, \lambda \rangle + \sum m_i \langle \alpha_i, \alpha_i \rangle > 0.$$

By this Lemma, if $H^1(\mathfrak{g}, V_\lambda) \neq 0$, then 0 is not a weight of V_λ. Hence $H^1(\mathfrak{h}, V_\lambda) = 0$ by the Corollary of Lemma 1. It follows from the exactness of (1), that $H^1(\mathfrak{g}, V_\lambda) \cong H^1(\mathfrak{g}, \mathfrak{h}, V)$. So we shall study now $H^1(\mathfrak{g}, \mathfrak{h}, V_\lambda)$. Assume that f is a non-zero relative 1-cocycle. f satisfies

$$hf(g) = f[h, g] \qquad (h \in \mathfrak{h}, g \in \mathfrak{g}). \tag{3}$$

Putting $g = x_\alpha$ in (3), we see $f(x_\alpha) \in V_\lambda^\alpha$. Similarly $f(y_\alpha) \in V_\lambda^{-\alpha}$. Since the set $\{x_{\alpha_1}, y_{\alpha_1}, \cdots, x_{\alpha_l}, y_{\alpha_l}\}$ generates the whole $\mathfrak{g}$, one of $f(x_{\alpha_i})$'s, or $f(y_{\alpha_i})$'s is non-zero. This means that one of α_i's or $-\alpha_i$'s is a weight of V_λ. If α_{i_0} would

1) Concerning basic facts on semisimple Lie algebras, see [**6**] and [8].

be a weight, then $\lambda=\sum m_i\alpha_i$, $m_i\geqq 0$, $m_{i_0}>0$, and we should have $\gamma_\lambda>0$ by Lemma 2, contrary to the assumption. Hence $f(x_\alpha)=0$ for all positive α, while $f(y_{\alpha_i})\neq 0$ for a certain i, say i_0. As above, we have $\lambda=\sum m_i\alpha_i$, where m_i's are integers such that $m_{i_0}+1\geqq 0$, $m_i\geqq 0$ $(i\neq i_0)$. Since $\sum m_i\langle\alpha_i, \alpha_i\rangle=-\langle\lambda, \lambda\rangle<0$, m_{i_0} must be -1: $\lambda=\nu-\alpha_{i_0}$, $\nu=\sum_{i\neq i_0} m_i\alpha_i$. Let S be the reflection determined by the vector α_{i_0}, and put $\lambda'=S(\nu)$. Since $\lambda'=S(\lambda+\delta)-\delta$, we have $\gamma_{\lambda'}=\gamma_\lambda=0$. Since $S(\alpha_i)$, $i\neq i_0$, are positive roots, all m_i's must be 0. Hence $\lambda=-\alpha_{i_0}$. It is then clear that $f(y_{\alpha_i})=0$ $(i\neq i_0)$, and f is determined by the value $f(y_{\alpha_{i_0}})\in V_\lambda^\lambda$. Since V_λ^λ is 1-dimensional, $H^1(\mathfrak{g}, \mathfrak{h}, V_\lambda)$ is at most 1-dimensional. Since there is no non-zero relative 1-coboundary for $\lambda=-\alpha_{i_0}$, our proof of Theorem will be completed, if we show that there is a relative 1-cocycle f such that $f(y_{\alpha_{i_0}})\neq 0$.

For simplicity, let $i_0=1$. Denote also x_{α_i} (resp. y_{α_i}) as x_i (resp. y_i). x_i's (resp. y_i's) generate the nilpotent subalgebra $\mathfrak{n}^+$ (resp. $\mathfrak{n}^-$) spanned by the vectors belonging to positive (resp. negative) roots: $\mathfrak{g}=\mathfrak{n}^-+\mathfrak{h}+\mathfrak{n}^+$. Let U^+ (resp. U^-) be the universal enveloping algebra of $\mathfrak{n}^+$ (resp. $\mathfrak{n}^-$), and I^+ (resp. I^-) the kernel of the canonical epimorphism $U^+\to K$ (resp. $U^-\to K$). We have $I^-=U^-y_1+\cdots+U^-y_l$. Put $J=U^-y_2+\cdots+U^-y_l$. Then $I^-=U^-y_1+J$, and I^-/J is generated by y_1+J as a U^--module. Now, let Kw be the 1-dimensional zero-representation of $\mathfrak{h}$, and construct the *induced* $\mathfrak{g}$-module $U^-U^+\otimes w$ [2]. Put $V=(U^-U^+\otimes w)/(U^-I^+\otimes w)$, and $w_1=(1\otimes w)+(U^-I^+\otimes w)$. Then V is a $\mathfrak{g}$-module generated by w_1 as U^--module, and is isomorphic to U^- as U^--module by the correspondence $uw_1\leftrightarrow u$ $(u\in U^-)$. I^-w_1 and Jw_1 are $\mathfrak{g}$-submodules of V, and I^-w_1/Jw_1 is generated by $v=y_1w_1+Jw_1$. v is annihilated by I^+ and belongs to the weight $-\alpha_1$. Hence I^-w_1/Jw_1 has a (unique) irreducible factor $V_{-\alpha_1}$ (see [8, exposé **17**]). Thus, we have an $\mathfrak{h}$-trivial extension M of $V/I^-w_1=K$ with the kernel $V_{-\alpha_1}$, and the characteristic cocycle of this extension satisfies $f(y_1)=v$, as desired.

REMARKS 1. In [3] we introduced the subspace $H^1(\mathfrak{h}, V)^{\mathfrak{g}}$ of $H^1(\mathfrak{h}, V)$ consisting of the *stable* cohomology classes. A simple calculation shows that

$$H^1(\mathfrak{h}, V_\lambda)^{\mathfrak{g}}=\begin{cases}\mathfrak{h}^* & (\lambda=0)\,,\\ 0 & (\lambda\neq 0)\,.\end{cases}$$

Therefore the isomorphism $H^1(\mathfrak{g}, V_\lambda)\cong H^1(\mathfrak{g}, \mathfrak{h}, V_\lambda)$ can also be deduced from the exactness of the following sequence [3]:

---

2) Let U and U^0 be the universal enveloping algebras of $\mathfrak{g}$ and $\mathfrak{h}$ respectively. Then $U=U^-U^+U^0$. Any $\mathfrak{h}$-module ($=U^0$-module)W induces a $\mathfrak{g}$-module $W^{\mathfrak{g}}=U\otimes_{U^0}W=U^-U^+\otimes_K W$. We see that $hy_1\otimes w=-\alpha_1(h)y_1\otimes w$ for $h\in\mathfrak{h}$, $w\in W$, and $x_iy_1\otimes w=y_1x_i\otimes w$, $i=1,\cdots,l$.

$$0 \to H^1(\mathfrak{g}, \mathfrak{h}, V) \to H^1(\mathfrak{g}, V) \to H^1(\mathfrak{h}, V)^{\mathfrak{g}}.$$

2. The study of general $\mathrm{Ext}^1_{\mathfrak{g}}(U, V)$ is closely related to the study of tensor products and of the multiplicity of weights. A forthcoming paper of H. Kimura [7] deals with these subject-matters.

Tokyo University of Education

References

[1] C. Chevalley and S. Eilenberg, Cohomology theory of Lie groups and Lie algebras, Trans. Amer. Math. Soc., **63** (1948), 85-124.

[2] Harish-Chandra, Some applications of the universal enveloping algebra of a semi-simple Lie algebra, Trans. Amer. Math. Soc., **70** (1951), 28-99.

[3] A. Hattori, On fundamental exact sequences, J. Math. Soc. Japan, **12** (1960), 65-80.

[4] K. Hirata, On module extensions, Sci. Rep. Tokyo Kyôiku Daigaku Sect. A., **6** (1959), 265-269.

[5] G. Hochschild, Relative homological algebra, Trans. Amer. Math. Soc., **82** (1956), 246-269.

[6] N. Jacobson, Lie algebras, Interscience, 1962.

[7] H. Kimura, On some infinite dimensional representations of semi-simple Lie algebras, to appear.

[8] Séminaire "Sophus Lie", **1** (1954/55), Ecole Norm. Sup.