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On normal contact metric manifolds

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In a series of papers [3], [4], [5], [6] S. Sasaki and his collaborators studied what they called an almost contact structure on an odd-dimensional manifold, which could be regarded as a structure corresponding to an almost complex one on an even-dimensional manifold, and was called so because of its close connection with a contact structure defined by a 1-form of maximal rank.

In the first part of this paper we treat Sasaki's theory by the method of adapted frames and in the second we investigate fundamental properties of a normal contact metric structure defined also by Sasaki and closely related to a Kaehlerian structure.

DEFINITIONS. Throughout the paper we assume manifolds and tensors to be real analytic, because we discuss a complete integrability of differential equations in complex domain. An almost contact structure, or (ϕ, ξ, η)-structure, on a $2n+1$-dimensional differentiable manifold M with local coordinates $x^1, \cdots, x^{2n+1}$ is defined by a tensor field $\phi=(\phi_i^j)$ and two vector fields $\xi=(\xi^1, \cdots, \xi^{2n+1})$, $\eta=(\eta_1, \cdots, \eta_{2n+1})$ such that

$$\begin{aligned} &\operatorname{rank}\phi=2n, \qquad \phi_j^i\xi^j=0, \qquad \phi_i^j\eta_j=0, \qquad \xi^i\eta_i=1, \\ &\phi_i^j\phi_j^k=-\delta_i^k+\xi^k\eta_i \qquad (i, j, k=1, \cdots, 2n+1). \end{aligned} \tag{1}$$

When we consider ϕ as a matrix with an element ϕ_i^j on the i-th row and the j-th column, we have

$$\phi^3+\phi=0. \tag{2}$$

We take a one dimensional space R with a coordinate t $(-\infty<t<+\infty)$ and construct an almost complex tensor $F=(f_A^B)$ on $M\times R$ as follows

$$f_i^j=\phi_i^j, \quad f_{2n+2}^i=\xi^i, \quad f_i^{2n+2}=-\eta_i, \quad f_{2n+2}^{2n+2}=0. \tag{3}$$

The manifold M is called *normal* when the Nijenhuis tensor N of the tensor F vanishes.

An almost contact metric structure, or (ϕ, ξ, η, g)-structure, is defined as an almost contact structure with a Riemannian metric $g=(g_{ij})$ such that

$$g_{ij}\xi^j=\eta_i, \quad g_{ij}\phi_h^i\phi_k^j=g_{hk}-\eta_h\eta_k. \tag{4}$$

A contact metric structure is defined as follows. On an almost contact metric manifold we construct a tensor (ϕ_{ij}) such that

$$\phi_{ij}=\phi_i^k g_{kj}. \tag{5}$$

This is an alternate tensor and a 2-form $\alpha=\phi_{ij}dx^i\wedge dx^j$ can be defined. If it is a derivative of 1-form $\beta=\eta_i dx^i$ such that $\beta\wedge(d\beta)^n\neq 0$, namely

$$d(\eta_i dx^i)=\phi_{ij}dx^i\wedge dx^j, \tag{6}$$

we call the structure *contact metric*. (6) is different from the definition of Sasaki ([**6**] p. 250) by a factor 2, but the difference is not essential.

1. Almost contact structure

1. In general we can define a Nijenhuis tensor N for any tensor field A of type (1.1) on an l-dimensional real differentiable manifold as follows. We take two arbitrary tangent vector fields X and Y and construct a vector Z such as

$$Z=-A^2[X, Y]-[AX, AY]+A[AX, Y]+A[X, AY].$$

Then a mapping $(X, Y)\rightarrow Z$ defines a Nijenhuis tensor of A. We take a coordinate neighborhood U and a complex differentiable base $X_1,\cdots,X_l$ on the tangent space of each point of U, and denote by $\omega^1,\cdots,\omega^l$ its dual base in the dual tangent space. We put

$$[X_q, X_r]=X_qX_r-X_rX_q=-c_{qr}^pX_p,\quad d\omega^p=\frac{1}{2}c_{qr}^p\omega^q\wedge\omega^r\quad (c_{qr}^p=-c_{rq}^p).$$

Here we assume that the indices run from 1 to l. We take components (a_p^q) of A with respect to the base, and for $X=u^pX_p$ we have $AX=(a_q^pu^q)X_p$. Putting $X_ra_q^p=a_{qr}^p$, namely $da_q^p=a_{qr}^p\omega^r$, we get for $N=(N_{qr}^p)$

$$N_{qr}^p=-a_q^s a_{rs}^p+a_r^s a_{qs}^p+a_s^p(a_{rq}^s-a_{qr}^s)+a_q^s a_r^t c_{st}^p-a_s^p a_q^t c_{tr}^s+a_s^p a_r^t c_{tq}^s+a_s^p a_t^s c_{qr}^t.$$

If the components a_p^q of A are all constant for our frame (which is possible when the eigenvalues are all constant), we have

$$N_{qr}^p=a_q^s a_r^t c_{st}^p-a_s^p a_q^t c_{tr}^s+a_s^p a_r^t c_{tq}^s+a_s^p a_t^s c_{qr}^t. \tag{1.1}$$

If the matrix (a_p^q) is diagonal with complex diagonal elements $a_p^p=\lambda_p$, we get (cf. [**2**] p. 142)

$$N_{qr}^p=(\lambda_p-\lambda_q)(\lambda_p-\lambda_r)c_{qr}^p\qquad(\text{not summed for } p, q, r). \tag{1.2}$$

2. We assume that a $2n+1$-dimensional manifold M has an almost contact structure. By the relation (2) eigenvalues of ϕ are $i, -i$ (each n-ple), and 0 (simple). In tangent spaces at each point of any coordinate neighborhood U in M we can take real basic vectors $X_1,\cdots,X_{2n}$ on the space spanned by eigenvectors of eigenvalues i and $-i$ and an eigenvector X_{2n+1} of eigenvalue 0 in such a way that ϕ_i^j are all constant (this is possible for a suitably chosen complex base and so is true for a real base) and moreover $X_1, \cdots, X_{2n}$, X_{2n+1} are analytic on U. Then we get by virtue of (1)

$$\phi=\begin{pmatrix}\phi_0 & 0\\ 0 & 0\end{pmatrix}\ (\text{const.}),\ \phi_0^2=-E_{2n}(\text{unit}),\ \xi=(0,\cdots,0,k),\ \eta=\begin{pmatrix}0\\ \vdots\\ 0\\ l\end{pmatrix} \quad \text{with } kl=1, \tag{1.3}$$

with respect to the base $X_1, \cdots, X_{2n+1}$. If we take kX_{2n+1} instead of X_{2n+1} and denote it by X_{2n+1} anew, we have

$$k=1, \quad l=1. \tag{1.4}$$

We call the base, which we have taken in this way, an adapted frame. An almost contact structure is the structure with a tensor field ϕ under the property (2), accompanied by a fixed eigenvector of eigenvalue 0.

We take a coordinate neighborhood U of M and construct a space $U\times R$ and consider a tensor F defined by (3).

We take an adapted frame on the tangent space of U and a basic vector $\partial/\partial t$ on R and consider these together as a frame on the tangent space of $U\times R$. We denote by $\omega^1, \cdots, \omega^{2n+1}, \omega^{2n+2}=dt$ a dual base and put

$$d\omega^A=\frac{1}{2}c^A_{BC}\omega^B\wedge\omega^C \quad (c^A_{BC}=-c^A_{CB}).$$

Here we use indices in such a way that

$A, B, C, D, E=1, \cdots, 2n+2$; $i, j, k=1, \cdots, 2n+1$; $m=2n+1$; $\infty=2n+2$.

As $d\omega^\infty=0$ and $d\omega^i$ does not contain dt, we have

$$c^\infty_{BC}=0, \quad c^A_{B\infty}=0. \tag{1.5}$$

The tensor F defined on $M\times R$ by (3) has constant components

$$f^j_i=\phi^j_i, \quad f^m_\infty=1, \quad f^\infty_m=-1, \quad \text{all other } f^B_A=0 \tag{1.6}$$

with respect to our frame. Nijenhuis tensor N of F has components

$$N^A_{BC}=f^D_Bf^E_Cc^A_{DE}-f^A_Df^E_Bc^D_{EC}+f^A_Df^E_Cc^D_{EB}+f^A_Df^D_Ec^E_{BC} \tag{1.7}$$

by (1.1) and when we denote by Φ a Nijenhuis tensor of ϕ on M and by Φ^i_{jk} its components with respect to the frame $\omega^1, \cdots, \omega^{2n}, \omega^m$ we get

$$\begin{gathered} N^i_{jk}=\Phi^i_{jk} \quad \text{for} \quad i\neq m, \quad N^m_{jk}=\Phi^m_{jk}-c^m_{jk} \\ N^\infty_{jk}=\phi^h_jc^m_{hk}-\phi^h_kc^m_{hj}, \quad N^i_{j\infty}=\phi^l_jc^i_{lm}+\phi^i_lc^l_{mj}, \quad N^\infty_{\infty k}=c^m_{mk}. \end{gathered} \tag{1.8}$$

It can easily be verified that (N^i_{jk}), (N^∞_{jk}), $(N^i_{j\infty})$ are tensors and $(N^\infty_{\infty k})$ is a vector on M, which depend on its almost contact structure.

We take complex frames in such a way that $\phi=(\phi^j_i)$ reduces to a form

$$\phi=\begin{pmatrix} iE_n & 0 & 0 \\ 0 & -iE_n & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{1.9}$$

We call these frames complex adapted ones. The dual base are

$$\omega^1, \cdots, \omega^n, \omega^{n+1}=\bar{\omega}^1, \cdots, \omega^{2n}=\bar{\omega}^n, \omega^m=\theta,$$

where θ is a real 1-form. Now we use indices as

$$\alpha, \beta, \gamma=1, \cdots, 2n \qquad m=2n+1$$
$$a, b, c=1, \cdots, n \qquad a', b', c'=n+1, \cdots, 2n$$

$\Phi=(\Phi^i_{jk})$ is the Nijenhuis tensor of ϕ and for our frame we have $\lambda_a=i$, $\lambda_{a'}=-i, \lambda_m=0$ in (1.2). Hence we get

$$\begin{aligned} &\Phi^a_{b'c'}=-4c^a_{b'c'}, \quad \Phi^{a'}_{bc}=-4c^{a'}_{bc}, \quad \text{all other } \Phi^\alpha_{\beta\gamma}=0, \\ &\Phi^a_{bm}=0, \quad \Phi^a_{b'm}=-2c^a_{b'm}, \quad \Phi^{a'}_{b'm}=0, \quad \Phi^{a'}_{bm}=-2c^{a'}_{bm}, \\ &\Phi^m_{ab}=-c^m_{ab}, \quad \Phi^m_{ab'}=c^m_{ab'}, \quad \Phi^m_{a'b'}=-c^m_{a'b'}, \quad \Phi^m_{ma}=0, \quad \Phi^m_{ma'}=0. \end{aligned} \tag{1.10}$$

Hence the vanishing of the tensor (N^i_{jk}) on M, namely $N^i_{jk}=0$, is equivalent to

$$\begin{aligned} &c^a_{b'c'}=0, \quad c^{a'}_{bc}=0, \quad c^a_{b'm}=0, \quad c^{a'}_{bm}=0, \\ &c^m_{ab}=0, \quad c^m_{a'b'}=0, \quad c^m_{ma}=0, \quad c^m_{ma'}=0 \end{aligned} \tag{1.11}$$

by virtue of (1.8) and (1.10). In this case we get by (1.8)

$$N^\infty_{jk}=0, \quad N^i_{j\infty}=0, \quad N^\infty_{\infty k}=0.$$

By definition an almost contact structure is normal when its Nijenhuis tensor (N^A_{BC}) on $M\times R$ vanishes, and now it can be replaced by the vanishing of (N^i_{jk}). (1.11) is also equivalent to

$$d\omega^a \equiv 0 \ (\mathrm{mod}\ \omega^1, \cdots, \omega^n), \quad d\theta=c^m_{ab'}\,\omega^a\wedge\bar{\omega}^b. \tag{1.12}$$

Thus we get

THEOREM 1. *The condition of normality of an almost contact structure is equivalent to* (1.12) *with respect to complex adapted frames.*

A. Morimoto noticed that the product $M^{2p+1}\times M^{2q+1}$ of two normal almost contact manifolds can be endowed with a complex structure. This can easily be proved by the use of (1.12). As S. Sasaki and Y. Hatakeyama noticed, a sphere S^{2n+1} of odd dimension has a normal contact structure (which we also prove in this paper later). Hence a result of Eckmann-Calabi that $S^{2p+1}\times S^{2q+1}$ has a complex structure is an example of a theorem by Morimoto.

By (1.12) two systems of differential equations

$$\omega^a=0 \quad (a=1, \cdots, n) \tag{1.13}$$

and

$$\omega^a = 0, \quad \theta = 0 \quad (a = 1, \cdots, n) \tag{1.14}$$

are completely integrable.

Conversely, if these two systems are completely integrable, we have $d\omega^a \equiv 0 \pmod{\omega^1, \cdots, \omega^n}$ and

$$d\theta = c^m_{ab'}\omega^a \wedge \bar{\omega}^b + c^m_{ma}\theta \wedge \omega^a + \bar{c}^m_{ma}\theta \wedge \bar{\omega}^a$$

because θ is real. If we assume moreover $N^\infty_{\infty k} = 0$, we have $c^m_{ma} = 0$ by (1.8) and hence $d\theta = c^m_{ab'}\omega^a \wedge \bar{\omega}^b$. Thus the condition of normality is equivalent to complete integrability of (1.13) and (1.14) accompanied by the vanishing of the vector $(N^\infty_{\infty k})$. This theorem is due to S. Sasaki and C. J. Hsu [5], where a proof is rather different from ours.

Next we discuss (1.12) more precisely. We have by the first equation of (1.12)

$$\omega^a = p^a_b(z, \bar{z}, u)dz^b,$$

where $z^1, \cdots, z^n, u$ are suitably chosen complex coordinates (u real). Then the second equation of (1.12) reduces to

$$d\theta = \Gamma_{ab}(z, \bar{z}, u)dz^a \wedge d\bar{z}^b. \tag{1.15}$$

As this form is closed we have $\partial\Gamma_{ab}/\partial u = 0$ and so $\Gamma_{ab} = \Gamma_{ab}(z, \bar{z})$. Hence if we take a form λ which can be obtained from θ by restricting u to a constant, we get $d\theta = d\lambda$. As λ is real we can put $\lambda = L_a dz^a + \bar{L}_a d\bar{z}^a$ with $L_a = L_a(z, \bar{z})$. We get by (1.15) $d'(L_a dz^a) = 0$ and we have a function $M = M(z, \bar{z})$ such that $d'M = L_a dz^a$. Hence $d\theta = d\lambda = d(d'M + d''\bar{M}) = d'd''(-M + \bar{M})$. Putting $K = i(M - \bar{M})$ (real) we get $d\theta = d'd''(iK) = -\frac{i}{2}d(d'K - d''K)$.

Hence

$$\theta = dw - \frac{i}{2}(d' - d'')K$$

with a real variable w. When we take a suitable base $\pi^a = t^a_b\omega^b$ instead of ω^a, we get

$$\pi^a = dz^a, \quad \theta = dw - \frac{i}{2}(d' - d'')K. \tag{1.16}$$

Thus we get the following theorem:

THEOREM 2. *The condition of normality of an almost contact structure is equivalent to the local existence of such a dual complex adapted frame* $\pi^1, \cdots, \pi^n, \bar{\pi}^1, \cdots, \bar{\pi}^n, \theta$ *that* (1.16) *holds good, where* $z^1, \cdots, z^n$ *are complex coordinates,* w *is a real one and* $K = K(z, \bar{z})$ *is a real function.*

2. Normal contact metric structure

1. An almost contact metric structure is defined by the existence of a tensor field $g=(g_{ij})$ satisfying (4). For adapted frames this reduces to

$$g=(g_{ij})=\begin{pmatrix} g_0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{where} \quad \phi_0 g_0 {}^t\phi_0 = g_0 .$$

For complex adapted frames we have (1.9) and so

$$g_0=\begin{pmatrix} 0 & g_1 \\ {}^tg_1 & 0 \end{pmatrix}, \quad g_1=(g_{ab}) .$$

Hence the Riemannian metric is

$$ds^2=2g_{ab}\omega^a\bar{\omega}^b+\theta^2 ,$$

where $g_{ab}=\bar{g}_{ba}$, because ds^2 is real. We assume throughout this paper that the metric tensor (g_{ij}) is positive definite. Then for a suitably chosen complex adapted frame we have

$$ds^2=\omega^a\bar{\omega}^a+\theta^2 , \tag{2.1}$$

where we mean by $\omega^a\bar{\omega}^a$ a summation with respect to $a=1, \cdots, n$.

Next a contact form $\eta=\eta_i dx^i$ is in our case θ and as to $\phi_{ij}=\phi_i^k g_{kj}$

$$(\phi_{ij})=\phi g=\begin{pmatrix} 0 & \frac{i}{2}E_n & 0 \\ -\frac{i}{2}E_n & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{because} \quad g_0=\begin{pmatrix} 0 & \frac{1}{2}E_n \\ \frac{1}{2}E_n & 0 \end{pmatrix}.$$

Here a 2-form $\phi_{ij}dx^i \wedge dx^j$ is $i\omega^a \wedge \bar{\omega}^a$. A condition for a contact structure is (6) and in this case it is $d\theta=i\omega^a \wedge \bar{\omega}^a$. As the condition of normality is equivalent to (1.12) we get the following theorem:

THEOREM 3. *The metric structure given by* (2.1) *is normal contact when and only when*

$$d\omega^a \equiv 0 \pmod{\omega^1, \cdots, \omega^n} \tag{2.2}$$

$$d\theta=i\omega^a \wedge \bar{\omega}^a . \tag{2.3}$$

On a normal contact metric manifold we have by virtue of (2.2)

$$\omega^a=p_b^a(z, \bar{z}, u)dz^b$$

if we take suitable complex coordinates $z^1, \cdots, z^n, u$ (u being real). Putting

$$\omega^a\bar{\omega}^a=g_{ab}(z, \bar{z}, u)dz^a d\bar{z}^b$$

we get

$$\omega^a \wedge \bar{\omega}^a=g_{ab}(z, \bar{z}, u)dz^a \wedge d\bar{z}^b . \tag{2.4}$$

By virtue of (2.3) we have

$$d(\omega^a \wedge \bar{\omega}^a)=0 \tag{2.5}$$

and by (2.4) we get $\partial g_{ab}/\partial u=0$ and so $g_{ab}=g_{ab}(z,\bar{z})$. Thus we have

$$d\sigma^2=\omega^a\bar{\omega}^a=g_{ab}(z,\bar{z})dz^a d\bar{z}^b\,.$$

This is in itself a Kaehlerian metric on account of (2.5). Hence there exists a real function K such that

$$d\sigma^2=\frac{\partial^2 K}{\partial z^a\partial\bar{z}^b}dz^a d\bar{z}^b\,. \tag{2.6}$$

When we put

$$\varphi=-\frac{i}{2}(d'-d'')K \tag{2.7}$$

which is real, we get $d\varphi=i\dfrac{\partial^2 K}{\partial z^a\partial\bar{z}^b}dz^a\wedge d\bar{z}^b=i\omega^a\wedge\bar{\omega}^a$ and by (2.3) $d\theta=d\varphi$. Hence there exists locally a real function w such that $\theta=dw+\varphi$. Thus we get

THEOREM 4. *The Riemannian metric of a normal contact metric structure is locally given by*

$$ds^2=d\sigma^2+(dw+\varphi)^2\,, \tag{2.8}$$

where $d\sigma^2$ is a Kaehlerian metric and φ is a 1*-form* (2.7) *derived from the metric $d\sigma^2$.*

Hereafter we investigate a normal contact metric structure on the base of Theorem 4 or (2.2) (2.3).

2. Sasaki and Hatakeyama [**6**] showed that a sphere of odd dimension gives an example of a normal contact metric manifold. Here we consider it in detail and prepare for our later discussion.

Here indices run as $a,b,c=1,\cdots,n$; $p=0,1,\cdots,n$. We take a complex vector space C_{n+1} with a flat metric

$$d\Sigma^2=dz^p d\bar{z}^p \tag{2.9}$$

in complex coordinates $z^0,z^1,\cdots,z^n$. When we put

$$r=(z^p\bar{z}^p)^{\frac{1}{2}},\quad z^p=ru^p$$

we have

$$u^p\bar{u}^p=1,$$

and

$$d\Sigma^2=dr^2+r^2du^p d\bar{u}^p\,. \tag{2.10}$$

We denote by ds^2 an induced Riemannian metric on a unit hypersphere $r=1$ and we get

$$ds^2=du^p d\bar{u}^p\,. \tag{2.11}$$

We take a unitary base $e_0,e_1,\cdots,e_n$ with e_0 on a unit vector $u=(u^0,u^1,\cdots,u^n)$ and put

$$\omega^a=(de_0,e_a),\quad \theta=-i(de_0,e_0) \tag{2.12}$$

where brackets mean hermitian inner product. Then (2.11) is represented as

$$ds^2 = d\sigma^2 + \theta^2, \quad \text{with} \quad d\sigma^2 = \omega^a \bar{\omega}^a,$$

and

$$d\theta = i\omega^a \wedge \bar{\omega}^a .$$

$d\sigma^2$ is a Kaehlerian metric of constant holomorphic curvature 1 which is the usual elliptic metric on a complex projective space. These we will explain next in a more general situation. Although the results are well known, the author thinks it proper to describe briefly, because they are rarely treated by the method of moving frames except in [**1**] and some others.

3. In a complex vector space C_{n+1} of dimension $n+1$ with coordinates $z^0, z^1, \cdots, z^n$ we define an inner product as

$$\langle z, w \rangle = \frac{1}{K} z^0 \bar{w}^0 + z^a \bar{w}^a \qquad (K \text{ real const.}).$$

Then all the linear transformations preserving the inner product form a group, which we call G. We take a quadric Q: $\langle z, z \rangle = \frac{1}{K}$ in C_{n+1}. Then G operates transitively on it, from which we can take a set of vectors $A = e_0, e_1, \cdots, e_n$, such that

$$\langle A, A \rangle = \frac{1}{K}, \quad \langle A, e_a \rangle = 0, \quad \langle e_a, e_b \rangle = \delta_{ab} .$$

A set of these vectors forms a frame on Q with a point A on Q and $e_1, \cdots, e_n$ on the tangent hyperplane at A. For a differentiable set of such frames we put

$$dA = \omega^0 A + \omega^a e_a, \quad de_a = \omega_a^0 A + \omega_a^b e_b .$$

Then we have

$$\omega^0 = -\bar{\omega}^0, \quad \omega_a^0 = -K\bar{\omega}^a, \quad \omega_a^b = -\bar{\omega}_b^a .$$

Now we identify points A, A' on Q such that $A' = e^{i\alpha} A$ (α real) and we get a manifold F of real dimension $2n$. We can prove that a metric

$$d\sigma^2 = \omega^a \bar{\omega}^a \tag{2.13}$$

is a metric on F. By virtue of structure equations in C_{n+1} we get

$$d\omega^a = \omega^b \wedge \pi_b^a, \quad \text{where } \pi_b^a = -\bar{\pi}_a^b = \omega_b^a - \delta_b^a \omega^0$$

and our metric $d\sigma^2$ is Kaehlerian. The curvature forms of the metric are

$$\Pi_b^a = d\pi_b^a - \pi_b^c \wedge \pi_c^a = K(\omega^a \wedge \bar{\omega}^b + \delta_b^a \omega^c \wedge \bar{\omega}^c) \tag{2.14}$$

and also

$$d\omega^0 = -K\omega^a \wedge \bar{\omega}^a . \tag{2.15}$$

These we obtain from structure equations in C_{n+1}.

The manifold F with the metric (2.13) is of constant holomorphic curvature K and the metric is called Fubini metric. We denote this metric by $\mathfrak{F}_n(K)$ hereafter. It is proved that curvature forms (2.14) characterize our Kaehlerian metric, in other words, under this condition the space is locally isometric with F.

The Riemannian metric constructed on Q by

$$ds^2=\omega^a\bar{\omega}^a+\theta^2, \quad \text{where} \quad \theta=-\frac{i}{K}\omega^0$$

is normal contact as we see by (2.15). The case $K=1$ is the one at the end of the preceding paragraph. On the other hand the metric

$$\langle dA, dA\rangle=\frac{1}{K}\omega^0\bar{\omega}^0+\omega^a\bar{\omega}^a=\omega^a\bar{\omega}^a+K\theta^2$$

is invariant on Q for the transformation group G.

3. Imbedding theorem

1. As we have said, a sphere in an even-dimensional real euclidean space is an example of a normal contact metric manifold. We will show that in general a normal contact metric manifold of dimension $2n+1$ can locally be imbedded into a Kaehlerian manifold of complex dimension $n+1$.

We take a normal contact metric manifold M with a metric

$$ds^2=d\sigma^2+\theta^2\,, \tag{3.1}$$

where

$$d\sigma^2=\omega^a\bar{\omega}^a \text{ (Kaehlerian)}, \quad d\theta=i\omega^a\wedge\bar{\omega}^a\,. \tag{3.2}$$

As $d\sigma^2$ is Kaehlerian we have connection forms (ω^a_b) such that

$$d\omega^a=\omega^b\wedge\omega^a_b\,, \quad \omega^a_b=-\bar{\omega}^b_a\,. \tag{3.3}$$

We construct a Riemannian metric of dimension $2n+2$

$$d\Sigma^2=dr^2+s^2d\sigma^2+t^2\theta^2\,, \tag{3.4}$$

where s and t are real functions of a real variable r. We put

$$\pi^a=s\omega^a, \quad \pi^{n+1}=dr+it\theta \tag{3.5}$$

and we get

$$d\Sigma^2=\pi^j\bar{\pi}^j\,. \tag{3.6}$$

By (3.2) (3.3) (3.5) we get $d\pi^j\equiv 0 \pmod{\pi^1, \cdots, \pi^{n+1}}$ $(j=1, \cdots, n+1)$. Hence there exist complex coordinates $(v^1, \cdots, v^{n+1})$ such that $\pi^j\equiv 0 \pmod{dv^1, \cdots, dv^{n+1}}$ and the metric (3.6) is hermitian.

Next we have

$$\pi^a\wedge\bar{\pi}^a+\pi^{n+1}\wedge\bar{\pi}^{n+1}=-i(s^2d\theta+2tdr\wedge\theta)$$

by virtue of (3.2) and (3.5). This is closed when and only when

$$t=ss' \quad \left(s'=\frac{ds}{dr}\right). \tag{3.7}$$

This we assume and our metric is Kaehlerian.

We can take functions $s=s(r), t=t(r)$ in such a way that $s(r_0)=1, s'(r_0)=1$ for $r=r_0\ (\neq 0)$. Then a hypersurface defined by $r=r_0$ in a Kaehlerian manifold with a metric (3.4) has a normal contact metric (3.1) as an induced one. Thus we get

THEOREM 5. *A normal contact metric manifold can locally be imbedded into a Kaehlerian manifold as a hypersurface in the way stated above.*

The induced metric on any hypersurface $r=r_1$ in (3.4) is homothetic to a normal contact metric $(c\omega^a)(c\bar{\omega}^a)+(c^2\theta)^2$, where $c=s(r_1)^{-1}t(r_1)$. Y. Tashiro [8] treated the case $s=t=r$ of Theorem 5.

Next we seek for connection forms of our Kaehlerian metric (3.6). We have by (3.5)

$$d\pi^a=\pi^b\wedge\omega_b^a+s^{-1}s'dr\wedge\pi^a$$

$$d\pi^{n+1}=it\,d\theta+it'\,dr\wedge\theta=-s^{-1}s'\pi^a\wedge\bar{\pi}^a+it'\,dr\wedge\theta \tag{3.7}$$

$$dr=\frac{1}{2}(\pi^{n+1}+\bar{\pi}^{n+1}),\quad it\,\theta=\frac{1}{2}(\pi^{n+1}-\bar{\pi}^{n+1}),\quad it\,dr\wedge\theta=-\frac{1}{2}\pi^{n+1}\wedge\bar{\pi}^{n+1}. \tag{3.8}$$

Hence when we put

$$\pi_b^a=\omega_b^a+i(s')^2\theta\delta_b^a,\quad \pi_{n+1}^a=-\bar{\pi}_a^{n+1}=s^{-1}s'\pi^a,\quad \pi_{n+1}^{n+1}=it'\theta, \tag{3.9}$$

we get

$$d\pi^a=\pi^b\wedge\pi_b^a+\pi^{n+1}\wedge\pi_{n+1}^a,\quad d\pi^{n+1}=\pi^a\wedge\pi_a^{n+1}+\pi^{n+1}\wedge\pi_{n+1}^{n+1},$$

namely

$$d\pi^i=\pi^j\wedge\pi_j^i \quad\text{with}\quad \pi_j^i=-\bar{\pi}_i^j,\quad (i,j=1,\cdots,n+1)$$

and so π_j^i are connection forms of the Kaehlerian metric (3.6).

Next we put

$$\Omega_b^a=d\omega_b^a-\omega_b^c\wedge\omega_c^a \quad (a,b,c=1,\cdots,n)$$

which are curvature forms of the Kaehlerian metric $d\sigma^2$. Then by (3.9) curvature forms of $d\Sigma^2$ with respect to the complex frame $\pi^1,\cdots,\pi^{n+1}$ are

$$\begin{aligned}
\Pi_b^a&=d\pi_b^a-\pi_b^c\wedge\pi_c^a-\pi_b^{n+1}\wedge\pi_{n+1}^a\\
&=\Omega_b^a+i\delta_b^a\,d((s')^2\theta)-(s^{-1}s')^2\pi^a\wedge\bar{\pi}^b\\
\Pi_{n+1}^a&=d\pi_{n+1}^a-\pi_{n+1}^b\wedge\pi_b^a-\pi_{n+1}^{n+1}\wedge\pi_{n+1}^a\\
&=d(s^{-1}s'\pi^a)-s^{-1}s'\pi^b\wedge(\omega_b^a+i(s')^2\theta\delta_b^a)-is^{-1}s't'\theta\wedge\pi^a\\
\Pi_{n+1}^{n+1}&=d\pi_{n+1}^{n+1}-\pi_{n+1}^a\wedge\pi_a^{n+1}=d(it'\theta)+(s^{-1}s')^2\pi^a\wedge\bar{\pi}^a.
\end{aligned}$$

Taking (3.2) (3.7) (3.8) into consideration we get

$$\Pi_b^a=\Omega_b^a-((s^{-1}s')^2\pi^c\wedge\bar{\pi}^c+s^{-1}s''\pi^{n+1}\wedge\bar{\pi}^{n+1})\delta_b^a-(s^{-1}s')^2\pi^a\wedge\bar{\pi}^b$$

$$\Pi^a_{n+1} = -s^{-1}s''\pi^a \wedge \bar{\pi}^{n+1} \tag{3.10}$$

$$\Pi^{n+1}_{n+1} = -s^{-1}s''\pi^a \wedge \bar{\pi}^a - \frac{1}{2}t^{-1}t''\pi^{n+1} \wedge \bar{\pi}^{n+1}.$$

Here we consider special cases, where $d\Sigma^2$ reduces to Fubini metric $\mathfrak{F}_{n+1}(K)$.

(I) $s = \sin r$, consequently $t = \frac{1}{2}\sin 2r$

In this case we have by (3.10)

$$\Pi^a_b = \Omega^a_b + (-\cot^2 r \cdot \pi^c \wedge \bar{\pi}^c + \pi^{n+1} \wedge \bar{\pi}^{n+1})\delta^a_b - \cot^2 r \cdot \pi^a \wedge \bar{\pi}^b$$

$$\Pi^a_{n+1} = \pi^a \wedge \bar{\pi}^{n+1}, \quad \Pi^{n+1}_{n+1} = \pi^c \wedge \bar{\pi}^c + 2\pi^{n+1} \wedge \bar{\pi}^{n+1}.$$

If the metric $d\sigma^2$ is $\mathfrak{F}_n(1)$ we have by (2.14) and (3.5)

$$\Omega^a_b = \omega^a \wedge \bar{\omega}^b + \delta^a_b \omega^c \wedge \bar{\omega}^c = \operatorname{cosec}^2 r(\pi^a \wedge \bar{\pi}^b + \delta^a_b \pi^c \wedge \bar{\pi}^c)$$

and so

$$\Pi^i_j = \pi^i \wedge \bar{\pi}^j + \delta^i_j \pi^k \wedge \bar{\pi}^k.$$

Hence the metric $d\Sigma^2$ is $\mathfrak{F}_{n+1}(1)$. Thus we get

$$d\Sigma^2 = dr^2 + \sin^2 r(d\sigma^2 + \cos^2 r \cdot \theta^2). \tag{3.11}$$

(II) $s = \sinh r$, consequently $t = \frac{1}{2}\sinh 2r$

In this case we get

$$\Pi^i_j = -(\pi^i \wedge \bar{\pi}^j + \delta^i_j \pi^k \wedge \bar{\pi}^k)$$

when $d\sigma^2$ is $\mathfrak{F}_n(1)$. Hence $d\Sigma^2$ is $\mathfrak{F}_{n+1}(-1)$, and we have

$$d\Sigma^2 = dr^2 + \sinh^2 r(d\sigma^2 + \cosh^2 r \cdot \theta^2). \tag{3.12}$$

(III) $s = r$, consequently $t = r$

In this case we have

$$\Pi^i_j = 0$$

if the metric $d\sigma^2$ is $\mathfrak{F}_n(1)$, and we have for a flat metric $d\Sigma^2$

$$d\Sigma^2 = dr^2 + r^2(d\sigma^2 + \theta^2). \tag{3.13}$$

Hypersurfaces $r = r_0$ (const.) in Kaehlerian manifolds (3.11) (3.12) (3.13) have metrics which are homothetic to normal contact metrics, by the remark after Theorem 5, namely

THEOREM 6. *Fubini space with the metric $\mathfrak{F}_{n+1}(\pm 1)$ has a family of hypersurfaces whose induced metrics are homothetic to normal contact metrics.*

This has been also proved by S. Tachibana.

4. Hypersurfaces in Kaehlerian manifolds

1. A hypersurface in Kaehlerian manifold has an induced almost contact metric structure and the necessary and sufficient condition for the metric to be normal contact was given by Y. Tashiro [5]. Here we treat this by our method and in addition prove that a hypersurface with a normal contact metric structure in flat C_{n+1} is necessarily a hypersphere.

As a preparation we consider a hypersurface S in a $k+1$-dimensional Riemannian manifold in which the metric is given by

$$ds^2 = g_{pq}\pi^p\pi^q + (\pi^{k+1})^2 \quad (p, q = 1, \cdots, k)$$

with base $\pi^1, \cdots, \pi^{k+1}$ (not necessarily real) on the dual tangent space, and we assume $\pi^{k+1} = 0$ along S. We denote by (π^A_B) forms of Riemannian connection and by ω^p, ρ_p restrictions of π^p, π^{k+1}_p to S respectively. Then

$$\mathrm{II} = \omega^p \rho_p \tag{4.1}$$

gives the second fundamental form of S.

2. In this paragraph we use indices as

$$i, j, k = 1, \cdots, n+1 \qquad a, b, c = 1, \cdots, n\,.$$

We take a manifold K of complex dimension $n+1$ with a Kaehlerian metric

$$d\Sigma^2 = \pi^j \bar{\pi}^j\,. \tag{4.2}$$

Forms π^i_j of the Kaehlerian connection are such that

$$d\pi^i = \pi^j \wedge \pi^i_j, \quad \pi^i_j + \bar{\pi}^j_i = 0\,. \tag{4.3}$$

We consider a real hypersurface S in K and denote its equation by $H(z, \bar{z}) = 0$ with a real function $H(z, \bar{z})$ in local coordinates $z^1, \cdots, z^{n+1}$. We assume that base $\pi^1, \cdots, \pi^n, \pi^{n+1}$ are taken in such a way that $\pi^{n+1} = ild'H$ (l real), and put

$$\lambda = \frac{1}{2}(\pi^{n+1} + \bar{\pi}^{n+1}), \quad \rho = \frac{1}{2i}(\pi^{n+1} - \bar{\pi}^{n+1})\,.$$

Then we have

$$d\Sigma^2 = \pi^a \bar{\pi}^a + \lambda^2 + \rho^2\,. \tag{4.4}$$

We denote by P a matrix of a base transformation from $X = (\pi^1, \cdots, \pi^n, \bar{\pi}^1, \cdots, \bar{\pi}^n, \pi^{n+1}, \bar{\pi}^{n+1})$ to $Y = (\pi^1, \cdots, \pi^n, \bar{\pi}^1, \cdots, \bar{\pi}^n, \lambda, \rho)$, namely $XP = Y$. Then a matrix $\Pi = (\mu^A_B)$ of forms of the Riemannian connection with respect to a base X is transformed to $\Gamma = (\gamma^A_B)$ with respect to a base Y in such a way that

$$\Gamma = P^{-1} \Pi P\,,$$

because P is a constant matrix. Here the relation can be expressed as

$$\Pi = (\mu^A_B) = \begin{pmatrix} \Pi_0 & 0 & \tau & 0 \\ 0 & \bar{\Pi}_0 & 0 & \bar{\tau} \\ -{}^t\bar{\tau} & 0 & i\alpha & 0 \\ 0 & -{}^t\tau & 0 & -i\alpha \end{pmatrix}, \quad \Gamma = (\gamma^A_B) = \begin{pmatrix} \Pi_0 & 0 & \frac{1}{2}\tau & -\frac{i}{2}\tau \\ 0 & \bar{\Pi}_0 & \frac{1}{2}\bar{\tau} & \frac{i}{2}\bar{\tau} \\ -{}^t\bar{\tau} & -{}^t\tau & 0 & \alpha \\ -i{}^t\bar{\tau} & i{}^t\tau & -\alpha & 0 \end{pmatrix} \tag{4.5}$$

where α is a real 1-form.

Now we restrict (4.4) to S and denote the restrictions of π^a, λ by ω^a, θ respectively. The restriction of $d\Sigma^2$ to S is

$$ds^2 = \omega^a \bar{\omega}^a + \theta^2 . \tag{4.6}$$

Then on account of the relation $\rho = 0$, (4.3) and the reality of θ we get

$$\begin{aligned} d\omega^a &= \frac{1}{2} A^a_{bc} \omega^b \wedge \omega^c + B^a_{bc} \omega^b \wedge \bar{\omega}^c + C^a_b \omega^b \wedge \theta + D^a_b \bar{\omega}^b \wedge \theta \quad (A^a_{bc} = -A^a_{cb}) \\ d\theta &= L^a_b \omega^a \wedge \bar{\omega}^b + \frac{1}{2} (M_a \omega^a + \bar{M}_a \bar{\omega}^a) \wedge \theta \quad (L^a_b = -\bar{L}^b_a) . \end{aligned} \tag{4.7}$$

As $d\Sigma^2$ is Kaehlerian, we have $d(\pi^j \wedge \bar{\pi}^j) = 0$ and hence $d(\omega^a \wedge \bar{\omega}^a) = 0$. So we get from (4.7)

$$A^a_{bc} = -\bar{B}^b_{ac} + \bar{B}^c_{ab}, \quad C^a_b = -\bar{C}^b_a, \quad D^a_b = D^b_a .$$

When we put

$$\begin{aligned} \omega^a_b &= B^a_{bc} \bar{\omega}^c - \bar{B}^b_{ac} \omega^c + (C^a_b - \bar{L}^a_b)\theta \quad (\text{hence } \omega^a_b = -\bar{\omega}^b_a) \\ \theta^m_a &= \frac{1}{2} L^a_b \bar{\omega}^b + \frac{1}{2} \bar{D}^a_b \omega^b + \frac{1}{2} M_a \theta \quad (m = 2n+1) \\ \theta^a_m &= -2\bar{\theta}^m_a = -\bar{L}^a_b \omega^b - D^a_b \bar{\omega}^b - \bar{M}_a \theta , \quad \theta^m_m = 0 \end{aligned} \tag{4.8}$$

we get by (4.7)

$$d\omega^a = \omega^b \wedge \omega^a_b + \theta \wedge \theta^a_m, \quad d\theta = \omega^b \wedge \theta^m_b + \bar{\omega}^b \wedge \bar{\theta}^m_b . \tag{4.9}$$

Forms of Riemannian connection of (4.6) are obtained by eliminating $2n+2$-th row and $2n+2$-th column from Γ in (4.5) and restricting to S, and they are nothing but

$$\Omega = \begin{pmatrix} \Omega_0 & 0 & \frac{1}{2}\sigma \\ 0 & \bar{\Omega}_0 & \frac{1}{2}\bar{\sigma} \\ -{}^t\bar{\sigma} & -{}^t\sigma & 0 \end{pmatrix} \quad \text{with} \quad \begin{aligned} \Omega_0 &= (\omega^a_b) \\ -{}^t\bar{\sigma} &= (\theta^1_m, \cdots, \theta^n_m) \end{aligned} \quad \frac{1}{2}\sigma = \begin{pmatrix} \theta^m_1 \\ \vdots \\ \theta^m_n \end{pmatrix} . \tag{4.10}$$

This can be verified by (4.9) and by the property that the fundamental metric tensor of (4.6) is parallel for the connection Ω. The restriction of Γ to S is by (4.5)

$$\begin{pmatrix} \Omega_0 & 0 & \frac{1}{2}\sigma & -\frac{i}{2}\sigma \\ 0 & \bar{\Omega}_0 & \frac{1}{2}\bar{\sigma} & \frac{i}{2}\bar{\sigma} \\ -{}^t\bar{\sigma} & -{}^t\sigma & 0 & \beta \\ -i{}^t\bar{\sigma} & i{}^t\sigma & -\beta & 0 \end{pmatrix} . \tag{4.11}$$

We put $(\omega) = (\omega^1, \cdots, \omega^n)$ in matric form. Then we get by a restriction of structure equation

$$d\rho = (\omega) \wedge \left(-\frac{i}{2}\sigma\right) + (\bar{\omega}) \wedge \left(\frac{i}{2}\bar{\sigma}\right) + \theta \wedge \beta .$$

We have $\rho = 0$ on S and hence by (4.10) (4.8)

$$0=d\rho=\omega^a\wedge(-i\theta_a^m)+\bar{\omega}^a\wedge(i\bar{\theta}_a^m)+\theta\wedge\beta$$
$$=\frac{i}{2}(-M_a\omega^a+\bar{M}_a\bar{\omega}^a)\wedge\theta+\theta\wedge\beta$$

and so

$$\beta=\frac{i}{2}(-M_a\omega^a+\bar{M}_a\bar{\omega}^a)+k\theta \qquad (k \text{ real}). \tag{4.12}$$

Now the fundamental second form of S is by (4.1)

$$\mathrm{II}=(\omega)\Big(-\frac{i}{2}\sigma\Big)+(\bar{\omega})\Big(\frac{i}{2}\bar{\sigma}\Big)+\theta\beta,$$

where the products are those of matrices. Hence we have by (4.10) (4.12)

$$\mathrm{II}=-i(\omega^a\theta_a^m-\bar{\omega}^a\bar{\theta}_a^m)+\theta\beta$$
$$=-i(L_b^a\,\omega^a\bar{\omega}^b+\frac{1}{2}\bar{D}_b^a\,\omega^a\omega^b-\frac{1}{2}D_b^a\,\bar{\omega}^a\bar{\omega}^b+\frac{1}{2}M_a\omega^a\theta-\frac{1}{2}\bar{M}_a\bar{\omega}^a\theta)+k\theta^2.$$

The condition for an almost contact metric to be normal contact is by (1.12)

$$d\omega^a\equiv 0 \quad (\mathrm{mod}\ \omega^1,\cdots,\omega^n), \quad d\theta=i\omega^a\wedge\bar{\omega}^a.$$

In our case these conditions are by (4.7)

$$L_b^a=i\delta_b^a, \quad D_b^a=0, \quad M_a=0 \tag{4.14}$$

which is equivalent to

$$\mathrm{II}=\omega^a\bar{\omega}^a+k\theta^2. \tag{4.15}$$

Thus we get the following theorem due to Tashiro [8]:

THEOREM 7. *In order that an induced almost contact metric on a hypersurface in Kaehlerian manifold is normal contact it is necessary and sufficient that the second fundamental form is of the form* (4.15).

Next we restrict the curvature forms of $d\Sigma^2$ to S and denote by Θ_m^a the one corresponding to θ_m^a. Then by (4.11) and $-i^t\bar{\sigma}=(i\theta_m^1,\cdots,i\theta_m^n)$ we have on S

$$\Theta_m^a=d\theta_m^a-\theta_m^b\wedge\omega_b^a-\beta\wedge i\theta_m^a.$$

When we treat the case in which S is normal contact, we have by (4.8) (4.12) (4.14)

$$\theta_m^a=i\omega^a, \quad \beta=k\theta$$

and so

$$\Theta_m^a=i(d\omega^a-\omega^b\wedge\omega_b^a)+k\theta\wedge\omega^a=i\theta\wedge\theta_m^a+k\theta\wedge\omega^a.$$

Hence

$$\Theta_m^a=(k-1)\theta\wedge\omega^a.$$

If $d\Sigma^2$ is flat, we have $\Theta_m^a=0$ and hence $k=1$. Then by (4.15) S is umbilical. Since an umbilical hypersurface in the euclidean space is locally a hypersphere, we get the following:

THEOREM 8. *A hypersurface in the even-dimensional euclidean space whose induced metric is normal contact is necessarily a hypershere.*

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