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On the theory of reductive algebraic groups over a perfect field

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The purpose of this note is to prepare some basic theorems on reductive algebraic groups which will be used in our subsequent papers. The results obtained here seem more or less well-known to the mathematicians working in this field, but we found it convenient to resume them in a paper. The main idea in proving them is a systematic use of the notion of "Γ-fundamental system".

NOTATIONS AND TERMINOLOGY. In this paper, we consider exclusively affine (hence linear) algebraic groups defined over a *perfect* field k. For such a group G, G_k denotes the subgroup formed of all k-rational points in G. G° is the connected component of G containing the neutral element (except for the notation introduced in § 4). Since k is perfect, the words 'k-closed' and 'defined over k' are used quite synonymously. An isomorphism (resp. an isogeny) defined over k will be called briefly a k-isomorphism (resp. a k-isogeny). For any field k, k^* denotes the multiplicative group of all non-zero elements in k. $\mathbf{G}_m, \mathbf{G}_a$ are the multiplicative group of all non-zero elements in the universal domain and the additive group of the universal domain, respectively, considered as an algebraic group of dimension 1. For a subgroup H of an (abstract) group G, $N(H), Z(H)$ denote the normalizer and the centralizer of H in G, respectively. As usual, $\mathbf{Z}$ (resp. $\mathbf{Q}$) denotes the ring of rational integers (resp. the field of rational numbers). For any subset M of a module (resp. vector space over $\mathbf{Q}$), the symbol $M_{\mathbf{Z}}$ (resp. $M_{\mathbf{Q}}$) represents the submodule (resp. linear subspace over $\mathbf{Q}$) generated by M.

§ 1. Preliminaries.

1. Let k be a perfect field and G a connected (linear) algebraic group defined over k. It was proved by Rosenlicht [**9**] (cf. also [**5**], [**10**]) that the following four conditions on G are equivalent.

(T1) There exists a k-isomorphism from G into $T(n)$ ($=$the group of all upper triangular matrices of degree n).

(T2) G is solvable, and all characters of G (i.e. morphisms from G into $\mathbf{G}_m$)

are defined over k.

(T3) G has a composition series:

$$G=G_0\supset G_1\supset\cdots\supset G_r=\{1\}$$

such that all G_i are defined over k and that the factor groups G_{i-1}/G_i are k-isomorphic either to $\mathbf{G}_m$ or to $\mathbf{G}_a$.

(T4) If V is a transformation space of G, defined over k, which is complete and has a k-rational point, then V has a k-rational fixed point of G.

We call a connected algebraic group G satisfying one of these conditions *k-trigonalizable*. As is seen from (T2) or (T3), this property is invariant under a k-isogeny. A torus T, defined over k, is k-trigonalizable, if and only if T is k-isomorphic to $(\mathbf{G}_m)^l$, l denoting the dimension of T; such a torus is said to *split over k* or to be *k-trivial* (cf. [7]). On the other hand, a connected unipotent group (i.e. group consisting of only unipotent elements), defined over k, is always k-trigonalizable; we will call such a group simply a connected *k-unipotent group*. (Note that a connected unipotent group is necessarily nilpotent. [**4**], 6–10, Cor. 2.) It is known that all connected k-trigonalizable group G is a semi-direct product over k[1)] of a k-trivial torus A and a connected k-unipotent normal subgroup N. (This is a special case of the so-called "Chevalley decomposition", and follows from Borel [**1**], Th. 12.2 and from the existence of a maximal torus defined over k, Rosenlicht [**9**], p. 45.)

2. Let G be a connected algebraic group defined over a perfect field k. We call a maximal connected k-trigonalizable subgroup of G a *k-Borel subgroup*. The following properties of k-Borel subgroups are due to Borel (cf. [**5**], [**10**]).

(B1) All k-Borel subgroups of G are conjugate to each other with respect to inner automorphisms defined by elements in G_k, i.e. if H, H' are two k-Borel subgroups of G, we have $H'=gHg^{-1}$ with $g\in G_k$. Moreover, if $H=AN$, $H'=A'N'$ are the Chevalley decompositions of H, H', respectively, we have $A'=gAg^{-1}$, $N'=gNg^{-1}$ with $g\in G_k$.

(B2) If H is a k-Borel subgroup of G, there exists a transformation space V of G, defined over k, which is projective, and an injective k-morphism from the homogeneous space G/H into V (with respect to the structure of transformation space of G) such that $V_k=G_k\cdot\varphi(H)$, V_k denoting the set of all k-rational points in V. (In particular, the canonical map $G_k\to(G/H)_k$ is surjective.)

1) An algebraic group G, defined over k, is said to be a *semi-direct product over k* of closed subgroups G_1, G_2, both defined over k, if G is a semi-direct product of G_1, G_2 in the abstract sense and if the natural correspondence $G_1\times G_2\to G$ is a birational (hence biregular) map defined over k.

It is clear from the definition that, if G is the direct product of two algebraic groups G_1, G_2, both defined over k, then a k-Borel subgroup H of G is the direct product of k-Borel subgroups H_i of G_i $(i=1,2)$, and that, if $f: G \to G'$ is a k-isogeny and if H, H' are k-closed connected subgroups of G, G', respectively, corresponding under f, then H is a k-Borel subgroup of G if and only if H' is a k-Borel subgroup of G'.

Now let R^u denote the unipotent part of the radical (= the maximal connected solvable normal subgroup) of G, which is clearly k-closed. Then it is clear that a closed subgroup H of G is a k-Borel subgroup of G, if and only if H contains R^u and H/R^u is a k-Borel subgroup of G/R^u. This reduces the study of k-Borel subgroups to the case where G is reductive (i. e. to the case where $R^u=\{1\}$). We recall that, if G is reductive, the connected component C of the center of G is a torus defined over k, the derived group S of G is a semi-simple k-closed normal subgroup and we have $G=S\cdot C, S\cap C=$finite; or, in other words, G is k-isogeneous to the direct product of S and C. Notation: $G\sim S\times C$. We call S and C the 'semi-simple' and the 'torus part' of G, respectively.

A connected algebraic group G, defined over k, is called *of compact type over k* (or briefly *k-compact*, by abuse of language), if k-Borel subgroups of G reduce to the neutral element. From what we have mentioned above, the k-compactness is invariant under a k-isogeny, and the direct product $G=G_1\times G_2$ is k-compact if and only if G_1, G_2 are both k-compact. A connected k-compact algebraic group is necessarily reductive.

3. (In this paragraph, k may be an arbitrary field.) Let T be a torus of dimension l, defined over k, and let X be its character module. Then there exists a finite Galois extension K of k such that T splits over K ([**7**], Prop. 1.2.1); T is then K-isomorphic to $(\mathbf{G}_m)^l$ and X is isomorphic (as module) to $\mathbf{Z}^l$. Call Γ the Galois group of K/k. All $\chi\in X$ being defined over K, Γ operates in a natural manner on X. We consider Γ as operating on X (and also on any objects defined over K) from the right, so that we have $(\chi^\sigma)^\tau=\chi^{\sigma\tau}$ for all $\chi\in X, \sigma,\tau\in\Gamma$. X has thus a structure of (right) Γ-module.

PROPOSITION 1. *Let T be a torus defined over k. Then a maximal k-trivial subtorus A and a maximal k-compact subtorus T_0 are determined uniquely; and we have*

$$T=T_0\cdot A, \qquad T_0\cap A=\textit{finite};$$

in other words, T is k-isogeneous to the direct product of T_0 and A.

We call T_0, A the 'k-compact' and 'k-trivial part' of T, respectively.

PROOF. Put

$$X^\Gamma=\{\chi\in X \mid \chi^\sigma=\chi \quad \text{for all} \quad \sigma\in\Gamma\}, \tag{1}$$

(2) $$X_0 = \{\chi \in X \mid \sum_{\sigma\in\Gamma} \chi^\sigma = 0\}.$$

It is clear that X^Γ, X_0 are Γ-invariant submodules of X and that the factor modules $X/X^\Gamma, X/X_0$ have no torsion. Hence, denoting by T_0, A the annihilators of X^Γ, X_0 in T, respectively, we see that T_0, A are subtoruses of T defined over k ([7]). Now, by definition, T is k-trivial, if and only if $X^\Gamma = X$. Hence a subtorus T' of T, defined over, k, is k-trivial, if and only if the character module X' of T' satisfies the condition $X'^\Gamma = X'$. Call X_1 the annihilator of T' in X; then, X' being identified (as Γ-module) with X/X_1, this condition is equivalent to saying that $\chi^\sigma - \chi \in X_1$ for all $\chi \in X, \sigma \in \Gamma$. Since X/X_1 has no torsion, this latter is equivalent to $X_0 \subset X_1$, or $A \supset T'$. Thus A is the biggest k-trivial subtorus of T. It follows that T is k-compact, if and only if $X_0 = X$. Hence, the notations being as above, a subtorus T' is k-compact, if and only if $X_0' = X'$, or, what amounts to the same, $\sum_{\sigma\in\Gamma} \chi^\sigma \in X_1$ for all $\chi \in X$; since X/X_1 has no torsion, this is equivalent to $X^\Gamma \subset X_1$ or $T_0 \supset T'$. This proves that T_0 is the biggest k-compact subtorus of T. Finally, if we denote by $X_{\mathbf{Q}}, (X^\Gamma)_{\mathbf{Q}}, (X_0)_{\mathbf{Q}}$ the vector spaces over $\mathbf{Q}$ obtained from X, X^Γ, X_0, respectively, by extending the coefficients from $\mathbf{Z}$ to $\mathbf{Q}$, it is immediate that

(3) $$X_{\mathbf{Q}} = (X^\Gamma)_{\mathbf{Q}} + (X_0)_{\mathbf{Q}} \qquad \text{(direct sum)},$$

whence we get

$$[X : (X^\Gamma + X_0)] < \infty, \; X^\Gamma \cap X_0 = \{0\}.$$

This implies our last statement, q. e. d.

COROLLARY. *A torus defined over k is k-compact, if and only if it has no character defined over k.*

§ 2. Maximal connected (k, T)-unipotent subgroups.

4. In the following, G denotes a connected reductive algebraic group defined over a perfect field k. Let A be a maximal k-trivial torus in G. It is then easy to see that, in G, there exists a maximal torus T defined over k and containing A. (In fact, for a maximal torus T, the condition $T \supset A$ is equivalent to $T \subset Z(A)$. Hence it is enough to take a maximal torus, defined over k, in $Z(A)$ ([**9**], p. 45). Note also that $Z(A)$ is connected. [4], 6-14, Th. 6, 2).) Then, A is the k-trivial part of T. Such a T (and A) will be fixed in the following once for all. Denote by X, Y the character modules of T, A, respectively, and, as in N°**3**, by X_0 the annihilator of A in X; Y is then identified with X/X_0. For $\chi \in X$ the canonical image of χ in Y, i.e. the restriction of χ on A, will be denoted by $\pi(\chi)$.

We will often make use of a linear order (compatible with addition) in

X with the following property:

(4) $\qquad$ *If* $\chi \notin X_0, \chi > 0, \chi \equiv \chi' \pmod{X_0}$, *we have* $\chi' > 0$.

Suppose first that a linear order in X satisfying (4) is given. Call X_+ the set of all positive elements in X and put

$$X_{0+} = X_+ \cap X_0, \quad Y_+ = \pi(X_+ - X_{0+}).$$

Then it is clear that one can define linear orders in X_0 and in Y, by taking X_{0+} and Y_+ as the sets of positive elements in X_0 and in Y, respectively. Moreover, the linear order in X is determined uniquely by those 'induced' (in this sense) in X_0 and in Y, because we have $X_+ = X_{0+} \cup \pi^{-1}(Y_+)$. This consideration shows also that conversely, given linear orders in X_0 and in Y, there exists (uniquely) a linear order in X satisfying (4) and inducing the given linear orders in X_0 and in Y. If, as in N^o**3**, K denotes a finite Galois extension of k with Galois group Γ such that T splits over K, a linear order in X satisfying (4) is characterized by the property:

(4′) $\qquad$ *If* $\chi \notin X_0, \chi > 0$, *we have* $\chi^\sigma > 0$ *for all* $\sigma \in \Gamma$.

We call such a linear order in X a *Γ-linear order.*

Now let $\mathfrak{r} = \{\alpha\} \subset X$ be the "root system" of G relative to T. By definition, a *root* α relative to T is a character of T such that there exists a (uniquely determined) connected unipotent subgroup P_α of dimension 1 in G and an isomorphism x_α from $\mathbf{G}_a$ onto P_α such that we have

(5) $$t x_\alpha(\xi) t^{-1} = x_\alpha(\alpha(t)\xi) \qquad \text{for all} \quad t \in T, \xi \in \mathbf{G}_a.$$

Here x_α is uniquely determined up to a non-zero scalar multiplication in $\mathbf{G}_a$ and may be taken to be defined over K (if K is sufficiently large), so that P_α is also defined over K (see Remark on p. 229). It is known ([4], Exp. 11~13, 16) that the set $\mathfrak{r}$ of all roots satisfies the usual conditions of root system (in $X_{\mathbf{Q}}$). It is clear from the definition that $\mathfrak{r}$ is invariant under Γ and we have

(6) $$(P_\alpha)^\sigma = P_{\alpha^\sigma}$$

for all $\sigma \in \Gamma$.

Put

(7) $$\mathfrak{r}_0 = \mathfrak{r} \cap X_0, \quad \bar{\mathfrak{r}} = \pi(\mathfrak{r} - \mathfrak{r}_0).$$

Then $\mathfrak{r}_0$ becomes clearly a 'closed' subsystem of $\mathfrak{r}$, i.e. $(\mathfrak{r}_0)_{\mathbf{Z}} \cap \mathfrak{r} = \mathfrak{r}_0$. An element of $\bar{\mathfrak{r}}$ is called a *restricted root* relative to A. (In fact, as is easily seen, $\bar{\mathfrak{r}}$ does not depend on the choice of T, but only on A.) On the other hand, given a Γ-linear order in X, we use sometimes the following notation:

$$\mathfrak{r}_+ = \mathfrak{r} \cap X_+, \quad \mathfrak{r}_{0+} = \mathfrak{r}_0 \cap X_{0+},$$

$$\bar{\mathfrak{r}}_+ = \bar{\mathfrak{r}} \cap Y_+ = \pi(\mathfrak{r}_+ - \mathfrak{r}_{0+}).$$

5. For convenience, we call a connected subgroup N of G *(k, T)-unipotent,* if it is unipotent, defined over k and normalized by T. In the case of characteristic 0, every maximal connected k-unipotent subgroup of G normalized by A can be proved to be (k, T)-unipotent. (See Remark 2 on p. 218) We shall now prove the following

PROPOSITION 2. *For any maximal connected (k, T)-unipotent subgroup N of G, there exists a Γ-linear order in X such that*

$$N = \prod_{\alpha \in \mathfrak{r}_+ - \mathfrak{r}_{0+}} P_\alpha. \tag{8}$$

(This means that every element in N is expressed uniquely as a product of the elements in P_α with $\alpha \in \mathfrak{r}_+ - \mathfrak{r}_{0+}$, $\mathfrak{r}_+ - \mathfrak{r}_{0+}$ being ordered in an arbitrary way.)

PROOF. Since TN is a connected solvable subgroup of G, there exists an "absolute Borel subgroup" (i. e. Borel subgroup w. r. t. the universal domain) B containing TN; N is then contained in the unipotent part B^u of B. Therefore ([4], 13-05, Th. 1), if one denotes by $\mathfrak{r}_*$ the set of all the roots α such that $P_\alpha \subset N$, we have $N = \prod_{\alpha \in \mathfrak{r}_*} P_\alpha$. Moreover, there exists a linear order in X such that $B^u = \prod_{\alpha > 0} P_\alpha$. (In these $\prod$, the order of factors may be taken arbitrarily.) Then, from (6) and from the fact that N is invariant under Γ, $\mathfrak{r}_*$ is also invariant under Γ; therefore $\alpha \in \mathfrak{r}_*$ implies $\alpha^\sigma > 0$ for all $\sigma \in \Gamma$. It follows that $\mathfrak{r}_*$ has the following property:

(*) Any finite sum $\sum \alpha_i$ of elements in $\mathfrak{r}_*$ is not contained in X_0.

From this, and by using Zorn's lemma, we infer that there exists a subset Y_+ of Y containing $\pi(\mathfrak{r}_*)$ which is maximal with respect to the following properties:

(i) $\eta, \eta' \in Y_+$ implies $\eta + \eta' \in Y_+$,

(ii) $0 \notin Y_+$.

Then it is easy to see that for any $\eta \in Y, \eta \neq 0$, we have either $\eta \in Y_+$ or $-\eta \in Y_+$. Therefore one can define a linear order in Y, by taking Y_+ as the set of positive elements. Then, denoting by X_{0+} the set of positive elements in X_0 with respect to any linear order and taking $X_+ = X_{0+} \cup \pi^{-1}(Y_+)$ as the set of positive elements in X, one can define a Γ-linear order in X, such that one has $\mathfrak{r}_* \subset X_+$. Therefore we may assume at the beginning that the linear order corresponding to B is a Γ-linear order. Then, the subgroup N' generated by all the P_α's with $\alpha \in \mathfrak{r}_+ - \mathfrak{r}_{0+}$ is clearly connected, unipotent, normalized by T and invariant under all $\sigma \in \Gamma$, hence defined over k. From the maximality of N, we conclude that $N = N'$, i. e. $\mathfrak{r}_* = \mathfrak{r}_+ - \mathfrak{r}_{0+}$, q. e. d.

COROLLARY 1. *For a connected reductive group G, defined over k, the following two conditions are equivalent:*

1) *A maximal k-trivial torus A is contained in the center of G.*

2) *G has no connected (k, T)-unipotent subgroup of dimension >0.*

(In particular, in order that a semi-simple group G, defined over a field k of characteristic 0, be k-compact, either one of the conditions $A=\{1\}, N=\{1\}$ is necessary and sufficient.)

PROOF. It follows from the Proposition that the condition 2) is equivalent to $\mathfrak{r}\subset X_0$. On the other hand, since G is generated by T and the P_α's and since P_α centralizes A if and only if $\alpha\in X_0$, the condition $\mathfrak{r}\subset X_0$ is also equivalent to 1), q. e. d.

COROLLARY 2. *$Z(A)$ is a connected reductive subgroup of G without connected (k, T)-unipotent subgroup of dimension >0.*

PROOF. Since $Z(A)$ is a connected subgroup containing T, of a reductive group G, it is generated by T and by the P_α's which are contained in it ([**4**], 12-07, Prop. 3, 13-05, Th. 1, d)), or equivalently, by T and by the P_α's with $\alpha\in\mathfrak{r}_0$. Since $\mathfrak{r}_0$ is a 'closed' subsystem of $\mathfrak{r}$, the closed subgroup $G(\mathfrak{r}_0)$ generated by the P_α's with $\alpha\in\mathfrak{r}_0$ is semi-simple ([**4**], 17-02, Th. 1), and we have $Z(A)=T_1\cdot G(\mathfrak{r}_0)$, where T_1 is the annihilator of $(\mathfrak{r}_0)_{\mathbf{Q}}\cap X$ in T. Hence $Z(A)$ is reductive and $G(\mathfrak{r}_0)$ is its semi-simple part. It follows from the above Cor. 1 that $Z(A)$ contains no connected (k, T)-unipotent subgroup of dimension >0.

PROPOSITION 3. a) *For any linear order in Y, let N be the subgroup generated by the P_α's with $\pi(\alpha)>0$ (i. e. the subgroup defined by* (8) *with respect to a Γ-linear order in X inducing a given linear order in Y). Then $Z(A)$ normalizes N and we have $\mathfrak{N}(N)=\mathfrak{N}(AN)=\mathfrak{N}(Z(A)\cdot N)=Z(A)\cdot N$.*

b) *The connected normalizer $\mathfrak{N}^\circ(A)$ of A coincides with $Z(A)$.*

PROOF. a) To prove that $Z(A)$ normalizes N, it is enough to show that the P_α's with $\alpha\in\mathfrak{r}_0$ normalize N. Let $\alpha\in\mathfrak{r}_0, \beta\in\mathfrak{r}_+-\mathfrak{r}_{0+}$; by changing the Γ-linear order (in the X_0-part) if necessary, we may assume that $\alpha>0$. Then, by considering the closed subgroup of G corresponding to the 'closed' subsystem $\{i\alpha+j\beta \mid i, j\in\mathbf{Z}, i\alpha+j\beta\in\mathfrak{r}\}$ of $\mathfrak{r}$, we have easily

$$p_\alpha p_\beta p_\alpha^{-1} p_\beta^{-1}\in\prod_{i,j>0} P_{i\alpha+j\beta} \qquad \text{for} \quad p_\alpha\in P_\alpha, p_\beta\in P_\beta,$$

where the product is taken over all (i, j) such that $i, j>0$, $i\alpha+j\beta\in\mathfrak{r}$. Now, for $j>0$, it is clear that $i\alpha+j\beta\in\mathfrak{r}_+-\mathfrak{r}_{0+}$, so that we have $p_\alpha p_\beta p_\alpha^{-1}p_\beta^{-1}\in N$, which proves that P_α normalizes N. It follows that $Z(A)\cdot N$ is a connected subgroup, defined over k, of G, containing an absolute Borel subgroup of G. Therefore we have $\mathfrak{N}(Z(A)\cdot N)=Z(A)\cdot N$ ([**4**], 12-06, Lem. 4). On the other hand, it is clear that $Z(A)\cdot N\subset\mathfrak{N}(AN)\subset\mathfrak{N}(N)$. Hence it remains to show that $\mathfrak{N}(N)\subset Z(A)\cdot N$. Since $\mathfrak{N}(N)$ contains an absolute Borel subgroup, it is connected. (This follows also from [**4**], loc. cit.; see [**5**], 206-11.) Hence

$N(N)$, being a connected subgroup containing T of a reductive group G, is generated by T and by the P_α's which are contained in it. Hence it is enough to show that $P_\alpha \subset N(N)$ implies $\alpha \in \mathfrak{r}_+ \cup \mathfrak{r}_0$. Now, if $P_\alpha \subset N(N)$, $P_\alpha \cdot N$ is a unipotent group. On the other hand, if $\alpha \notin \mathfrak{r}_+ \cup \mathfrak{r}_0$, we have $-\alpha \in \mathfrak{r}_+ - \mathfrak{r}_{0+}$, i. e. $P_{-\alpha} \subset N$, and, as is readily seen, the closed subgroup generated by P_α, $P_{-\alpha}$ is semi-simple; thus the subgroup generated by P_α and N can not be unipotent. This proves our assertion.

b) Since $N^\circ(A)$ is a connected subgroup containing T, of a reductive group G, it is generated by T and by the P_α's which are contained in it. Now, if $P_\alpha \subset N^\circ(A)$, we have $ap_\alpha a^{-1} p_\alpha^{-1} \in A \cap P_\alpha = \{1\}$ for any $a \in A, p_\alpha \in P_\alpha$ i. e. P_α centralizes A. Hence we have $N^\circ(A) = Z(A)$, q. e. d.

COROLLARY. *In the case of characteristic* 0, $Z(A) \cdot N$ *is a semi-direct product, over k, of $Z(A)$ and N.*

REMARK. It follows from Propositions 2 and 3, a) that the notion of *maximal* connected (k, T)-unipotent subgroup does not depend actually on the choice of T, but only on A.

6. LEMMA. 1. *We have* $N(A) = (N(A) \cap N(T)) \cdot Z(A)$.

PROOF. Let $s \in N(A)$. Then sTs^{-1} being contained in $Z(A)$, T, sTs^{-1} are both maximal toruses in $Z(A)$. Hence there exists $s_1 \in Z(A)$ such that $sTs^{-1} = s_1 T s_1^{-1}$. Then, putting $s_2 = s_1^{-1} s$, we have $s = s_1 s_2$ with $s_1 \in Z(A)$, $s_2 \in N(A) \cap N(T)$, as desired, q. e. d.

THEOREM 1. a) *For any linear order in Y, the subgroup N generated by the P_α's with $\pi(\alpha) > 0$, i. e.*

$$N = \prod_{\pi(\alpha)>0} P_\alpha$$

is a maximal connected (k, T)-unipotent subgroup of G. Conversely, all maximal connected (k, T)-unipotent subgroups of G are obtained in this way.

b) *The maximal connected (k, T)-unipotent subgroups of G are conjugate to each other with respect to inner automorphisms defined by k-rational elements in $N(A)$, i. e. if N, N' are two maximal connected (k, T)-unipotent subgroups of G, we have $N' = sNs^{-1}$ with $s \in N(A)_k$. We have $sNs^{-1} = s'Ns'^{-1}$ with $s, s' \in N(A)_k$, if and only if* $s \equiv s'$ (mod $Z(A)_k$).

PROOF. a) It is clear that the subgroup N generated by the P_α's with $\pi(\alpha) > 0$ is connected and (k, T)-unipotent. To prove its maximality, suppose that there exists a connected (k, T)-unipotent subgroup N' containing N properly. Then there exists a connected (k, T)-unipotent subgroup containing N as a proper normal subgroup. In fact, it can be proved easily that there exists a connected subgroup N'' such that $N' \supset N'' \supsetneqq N$, normalized by T and normalizing N (cf. [**4**], 13-01, Lem. 1, b)). Then the subgroup generated by all the conjugates of N'' over k satisfies our requirements. Hence we may suppose that $N'\subset N(N)$. Then, since $N(N)=Z(A)\cdot N$ by Proposition 3, a), we have $N'=(N'\cap Z(A))\cdot N$ and so $N'\cap Z(A)\neq\{1\}$, which contradicts Corollary 2 to Proposition 2. This proves the direct part in a). The converse is already stated as Proposition 2.

b) Let N, N' be two maximal connected (k, T)-unipotent subgroups of G. Then, since AN, AN' are connected k-trigonalizable subgroups of G, there exist two k-Borel subgroups H, H' of G, containing AN, AN', respectively. Call $\tilde{N}, \tilde{N}'$ the unipotent parts of H, H', respectively; then it is clear that $\tilde{N}, \tilde{N}'$ are maximal connected k-unipotent subgroups of G, normalized by A and containing N, N', respectively. Hence by the property (B1) of k-Borel subgroups there exists $s\in N(A)_k$ such that $\tilde{N}'=s\tilde{N}s^{-1}$. We shall infer that $N'=sNs^{-1}$. In fact, sNs^{-1} is a connected k-unipotent subgroup of G; it is normalized by T, because by Lemma 1 we can write $s=s_1s_2$ with $s_1\in N(A)\cap N(T), s_2\in Z(A)\subset N(N)$, so that we have $sNs^{-1}=s_1Ns_1^{-1}$. It follows immediately that sNs^{-1} is a maximal connected (k, T)-unipotent subgroup of G. Therefore, by Proposition 2 there exist two Γ-linear orders in X such that, in denoting by $\mathfrak{r}'_+, \mathfrak{r}''_+$ the corresponding sets of positive roots, we have

$$N'=\prod_{\alpha\in\mathfrak{r}_+'-\mathfrak{r}_0}P_\alpha,\quad sNs^{-1}=\prod_{\alpha\in\mathfrak{r}_+''-\mathfrak{r}_0}P_\alpha.$$

Since N', sNs^{-1} are both contained in a unipotent group $\tilde{N}'$, there is no root α such that $\alpha\in\mathfrak{r}'_+-\mathfrak{r}_0, -\alpha\in\mathfrak{r}''_+-\mathfrak{r}_0$. This means that $\mathfrak{r}'_+-\mathfrak{r}_0$ is contained in the complement (in $\mathfrak{r}$) of the set $\{-\alpha\mid\alpha\in\mathfrak{r}''_+-\mathfrak{r}_0\}$, or, what amounts to the same, in $\mathfrak{r}''_+\cup\mathfrak{r}_0$. This implies clearly that $\mathfrak{r}'_+-\mathfrak{r}_0=\mathfrak{r}''_+-\mathfrak{r}_0$ and so $N'=sNs^{-1}$, as desired. Finally, if $sNs^{-1}=s'Ns'^{-1}$ with $s, s'\in N(A)$, we have $s''=s'^{-1}s\in N(N)=Z(A)\cdot N$ by Proposition 3, a), and hence $s''\in(N(A)\cap N)\cdot Z(A)$. Then, for any $a\in A$, we have $s''as''^{-1}a^{-1}\in A\cap N=\{1\}$, i.e. we have $s''\in Z(A)$, q.e.d.

COROLLARY. *Either one of the conditions* 1), 2) *in Corollary* 1 *to Proposition* 2 *is equivalent to the following condition:*

3) $N(A)_k=Z(A)_k$.

PROOF. It is clear that 1) implies 3). Conversely, suppose that 1) is not satisfied. Then there exists a root α not contained in X_0. Hence, for any linear order in Y, the subgroups $N=\prod_{\pi(\alpha)>0}P_\alpha$, $N'=\prod_{\pi(\alpha)<0}P_\alpha$ are two distinct maximal connected (k, T)-unipotent subgroups of G. Therefore by Theorem 1, b) we must have $[N(A)_k: Z(A)_k]>1$, q.e.d.

REMARK 1. The number of distinct maximal connected (k, T)-unipotent subgroups is equal to the index $[N(A)_k: Z(A)_k]$, which is finite by Proposition 3, b). It will be proved in the next section that every coset in $N(A)/Z(A)$ contains a k-rational representative, so that we have $[N(A)_k: Z(A)_k]=[N(A): Z(A)]$.

REMARK 2. It is not known to the author whether (in the case of characteristic $\neq 0$) a maximal connected k-unipotent subgroup of G normalized by A is normalized by $Z(A)$ (or equivalently, becomes a (k, T)-unipotent subgroup) or not.[2] This would be true, if (and only if) Corollary 1 to Proposition 2 remains true when the word "(k, T)-unipotent" is replaced by "k-unipotent". A sufficient condition for this is that we have $N^{\circ}(P) \neq Z^{\circ}(P)$ for any connected unipotent subgroup P of dimension 1 of G; and this condition is surely satisfied for the field of characteristic 0. (See Godement [5], 206-20/21.)

§ 3. Properties of Γ-fundamental systems.

7. We keep the notations $A, T, X, \cdots$ introduced in N^o **4** throughout this section. We call a fundamental system of $\mathfrak{r}$ corresponding to a Γ-linear order in X a *Γ-fundamental system.* From what we mentioned in N^o**4**, a fundamental system Δ of $\mathfrak{r}$ becomes a Γ-fundamental system, if and only if for $\alpha_i \in \Delta, \notin X_0$ and for all $\sigma \in \Gamma$, α_i^σ is a positive root with respect to the fundamental system Δ. For a Γ-fundamental system Δ, we put

$$\Delta_0 = \Delta \cap X_0, \quad \bar{\Delta} = \pi(\Delta - \Delta_0). \tag{9}$$

and call $\bar{\Delta}$ a *restricted fundamental system* of $\mathfrak{r}$ corresponding to a Γ-fundamental system Δ.

PROPOSITION 5. *Let Δ be a Γ-fundamental system. Then*

a) *Δ_0 is a fundamental system of the root system $\mathfrak{r}_0$.*

b) *If $\bar{\Delta} = \{\gamma_1, \cdots, \gamma_\nu\}$ (where the γ_i's are assumed to be mutually distinct), then $\gamma_1, \cdots, \gamma_\nu$ are linearly independent and every $\gamma \in \bar{\mathfrak{r}}$ can be written uniquely in the form*

$$\gamma = \pm \sum_{j=1}^{\nu} n_j \gamma_j$$

with $n_j \in \mathbf{Z}, n_j \geqq 0$.

c) *Let Δ' be another Γ-fundamental system and put $\Delta_0' = \Delta' \cap X_0, \bar{\Delta}' = \pi(\Delta' - \Delta_0')$. Then we have $\Delta = \Delta'$, if and only if we have $\Delta_0 = \Delta_0', \bar{\Delta} = \bar{\Delta}'$.*

PROOF. a) We write $\Delta = \{\alpha_1, \cdots, \alpha_l\}, \Delta_0 = \{\alpha_{l-l_0+1}, \cdots, \alpha_l\}$, where the α_i's are assumed to be mutually distinct. Take $\alpha \in \mathfrak{r}_0$ and write

$$\alpha = \pm \sum_{i=1}^{l} m_i \alpha_i \tag{10}$$

with $m_i \in \mathbf{Z}, m_i \geqq 0$. Then

$$0 = \sum_\sigma \alpha^\sigma = \pm \sum_i m_i \sum_\sigma \alpha_i^\sigma.$$

Since Δ is a Γ-fundamental system, we have $\sum_\sigma \alpha_i^\sigma > 0$ for $1 \leqq i \leqq l - l_0$, $\sum_\sigma \alpha_i^\sigma = 0$ for $l - l_0 + 1 \leqq i \leqq l$. Hence we must have $m_i = 0$ for $1 \leqq i \leqq l - l_0$. Therefore

2) Professor Tits has kindly informed me that this problem was already settled affirmatively by him.

$\Delta_0=\{\alpha_{l-l_0+1},\cdots,\alpha_l\}$ is a fundamental system of $\mathfrak{r}_0$.

b) The notations being as above, let $\alpha\in\mathfrak{r}$ and write it in the form (10). Then we have

$$\pi(\alpha)=\pm\sum_j n_j\gamma_j$$

with $n_j=\sum_{\pi(\alpha_i)=\gamma_j} m_i$; hence $n_j\in\mathbf{Z}, n_j\geqq 0$. It follows, in particular, that if $\pi(\alpha_i)=\gamma_j$, then, for any $\sigma\in\Gamma$, we have

$$\alpha_i^\sigma=\alpha_{i'}+\sum_{k=l-l_0+1}^{l} c_k\alpha_k \tag{11}$$

with $1\leqq i'\leqq l-l_0, \pi(\alpha_{i'})=\gamma_j, c_k\geqq 0$. Let us now show that $\gamma_1,\cdots,\gamma_\nu$ are linearly independent (over $\mathbf{Q}$). Put $\mathbf{E}=X_{\mathbf{Q}}$ and call $\mathbf{E}^*$ the dual space of $\mathbf{E}$ over $\mathbf{Q}$; in the notations used in the proof of Proposition 1, we may identify $Y_{\mathbf{Q}}$ with $(X^\Gamma)_{\mathbf{Q}}$, so that π may be considered as (the restriction on X of) the projection of $X_{\mathbf{Q}}$ onto $(X^\Gamma)_{\mathbf{Q}}$ with respect to the direct decomposition (3). Since $\alpha_1,\cdots,\alpha_l$ are linearly independent, there exist $\omega_1,\cdots,\omega_\nu\in\mathbf{E}^*$ such that $\langle\alpha_i,\omega_j\rangle=1$ for $\alpha_i\in\pi^{-1}(\gamma_j)$ and $=0$ for $\alpha_i\notin\pi^{-1}(\gamma_j)$. Then from (11) we have $\langle\alpha_i^\sigma,\omega_j\rangle=\langle\alpha_i,\omega_j\rangle$ for all $\sigma\in\Gamma$. Therefore, by replacing ω_j by $\frac{1}{d}\sum_{\sigma\in\Gamma}\omega_j^\sigma$ $(d=[\Gamma:1])$ if necessary, we may assume that the ω_j's are all in the annihilator of $(X_0)_{\mathbf{Q}}$. Then, we have $\langle\gamma_j,\omega_k\rangle=\langle\alpha_i,\omega_k\rangle=\delta_{jk}$ for $\gamma_j=\pi(\alpha_i)$, which proves that $\gamma_1,\cdots,\gamma_\nu$ are linearly independent.

c) Assume that $\Delta_0=\Delta_0', \bar{\Delta}=\bar{\Delta}'$. Then, for $\alpha_i\in\Delta-\Delta_0$, we have $\pi(\alpha_i)\in\bar{\Delta}=\bar{\Delta}'$. Hence α_i is positive in the Γ-linear order defining the Γ-fundamental system Δ'. It follows that the set of positive roots with respect to the fundamental system Δ is contained in that with respect to Δ'. Therefore we must have $\Delta=\Delta'$, as desired. The converse is trivial, q. e. d.

8. We denote by W (resp. W_0) the Weyl group of the root system $\mathfrak{r}$ (resp. $\mathfrak{r}_0$) and put

$$W_\Gamma=\{w\in W\mid w(X_0)=X_0\}\,. \tag{12}$$

In a natural manner, W_0 is regarded as a subgroup of W_Γ. (W_0 is actually a normal subgroup of W_Γ, because we have $ww_\alpha w^{-1}=w_{w(\alpha)}$ with $w(\alpha)\in\mathfrak{r}_0$ for any $\alpha\in\mathfrak{r}_0, w\in W_\Gamma$, where w_α denotes the reflection defined by α.)

For $w\in W_\Gamma$, we denote by $\pi(w)$ (by abuse of notation) the automorphism of $Y=X/X_0$ induced from w; in other words, we define $\pi(w)$ by the formula

$$\pi(w(\chi))=\pi(w)\pi(\chi)\qquad\text{for all}\quad \chi\in X. \tag{13}$$

It is clear that $w\in W_0$ implies $\pi(w)=1$. The converse of this is also true, as is seen from the following Lemma 2. Thus, putting $\pi(W_\Gamma)=\overline{W}$, we obtain a canonical isomorphism

$$W_\Gamma/W_0\cong\overline{W}. \tag{14}$$

LEMMA 2. *Let Δ be a Γ-fundamental system of* $\mathfrak{r}$ *and let* $\bar{\Delta}$ *be the corresponding restricted fundamental system. Then, for any* $w \in W_{\Gamma}$, $w(\Delta)$ *is again a* Γ*-fundamental system of* $\mathfrak{r}$ *and* $\pi(w)(\bar{\Delta})$ *is the corresponding restricted fundamental system. We have* $\pi(w)(\bar{\Delta}) = \bar{\Delta}$, *if and only if* $w \in W_0$.

PROOF. Let X_+ be the set of positive elements in X with respect to the Γ-linear order defining the Γ-fundamental system Δ. Then, for any $w \in W_{\Gamma}$, by taking $w(X_+)$ as the set of positive elements, we can define a Γ-linear order in X which defines the fundamental system $w(\Delta)$. Hence $w(\Delta)$ is a Γ-fundamental system, and so $\pi(w(\Delta)) = \pi(w)(\bar{\Delta})$ is the corresponding restricted fundamental system. Now to prove the second half, assume further that, $\pi(w)(\bar{\Delta}) = \bar{\Delta}$. Then, $w(\Delta) \cap X_0 = w(\Delta_0)$ being a fundamental system of $\mathfrak{r}_0$, by Proposition 5, a), there exists $w_0 \in W_0$ such that $w(\Delta_0) = w_0(\Delta_0)$. Since we have $\pi(w_0)(\bar{\Delta}) = \bar{\Delta}$, it follows from Proposition 5, c) that $w(\Delta) = w_0(\Delta)$ and so we have $w = w_0$. The converse ('if' part) is trivial, q. e. d.

REMARK. By the same argument as in the above proof, one can also prove the following Lemma which is apparently more general:

LEMMA 2 bis. *Let* $\bar{\Delta}, \bar{\Delta}'$ *be restricted fundamental systems corresponding to* Γ*-fundamental systems* Δ, Δ' *of* $\mathfrak{r}$, *respectively. Then we have* $\bar{\Delta} = \bar{\Delta}'$, *if and only if there exists* $w_0 \in W_0$ *such that* $\Delta' = w_0\Delta$.

It is known ([4], especially 11-07, Th. 2) that W is canonically isomorphic to $N(T)/T$. (Note that $N^{\circ}(T) = Z(T) = T$. [4], 7-01, Th. 1, 12-09, Th. 2, a).) More precisely, if $s \in N(T)$, then the inner automorphism $I_s: g \rightarrow sgs^{-1}$ defined by s induces an automorphism of T, and so in a natural manner that of X, which we denote by w_s; we have then $w_s \in W$ and the correspondence $s \rightarrow w_s$ gives the isomorphism:

$$N(T)/T \cong W. \tag{15}$$

By definition, we have $w_s = {}^t(I_s \mid T)^{-1}$ (t denoting the 'dual') or

$$w_s(\chi)(sts^{-1}) = \chi(t) \quad \text{for all} \quad \chi \in X, t \in T, \tag{16}$$

and, in particular, for $\alpha \in \mathfrak{r}$, we have

$$sP_{\alpha}s^{-1} = P_{w_s(\alpha)}. \tag{17}$$

It is clear that $w_s \in W_{\Gamma}$ if and only if $s \in N(A)$, and $w_s \in W_0$ if and only if $s \in Z(A)$ (Lemma 2). Thus we have

$$\begin{aligned} W_{\Gamma} &\cong N(T) \cap N(A)/T, \\ W_0 &\cong N(T) \cap Z(A)/T. \end{aligned} \tag{18}$$

From (14), (18) and Lemma 1 in N°**6**, we obtain

$$\begin{aligned} \overline{W} &\cong N(T) \cap N(A)/N(T) \cap Z(A) \\ &\cong N(A)/Z(A), \end{aligned} \tag{19}$$

the canonical homomorphism $N(A) \ni s \to \bar{w}_s \in \overline{W}$ being defined by $\bar{w}_s = {}^t(I_s \mid A)^{-1}$, or

$$\bar{w}_s(\eta)(sas^{-1}) = \eta(a) \qquad \text{for all} \quad \eta \in Y, a \in A. \tag{20}$$

It is clear that for $s \in N(A) \cap N(T)$ we have $\bar{w}_s = \pi(w_s)$, or, what is the same,

$$\pi(w_s(\chi)) = \bar{w}_s(\pi(\chi)) \qquad \text{for} \quad \chi \in X. \tag{21}$$

9. The Galois group Γ operates on W in a natural way, i.e. by the formula

$$w^\sigma(\chi^\sigma) = (w(\chi))^\sigma \qquad \text{for all} \quad \chi \in X, \sigma \in \Gamma. \tag{22}$$

Denoting by w_α the reflection defined by $\alpha \in \mathfrak{r}$, we have at once

$$(w_\alpha)^\sigma = w_{\alpha^\sigma} \qquad \text{for all} \quad \alpha \in \mathfrak{r}, \sigma \in \Gamma. \tag{23}$$

In view of the fact that $X_0, \mathfrak{r}_0$ are invariant under Γ, we see from definitions that W_Γ, W_0 are both invariant under Γ. On the other hand, it follows immediately from (16), (22) that, in extending $\sigma \in \Gamma$ to an automorphism of the universal domain, we have

$$(w_s)^\sigma = w_{s^\sigma} \qquad \text{for all} \quad s \in N(T), \sigma \in \Gamma. \tag{24}$$

In particular, we have $(w_s)^\sigma = w_s$ for all $\sigma \in \Gamma$ if and only if the corresponding coset sT is k-rational. (In view of (23), this implies that all cosets in $N(T)/T$ are defined over a splitting field, e.g. K, of T.)

PROPOSITION 6. *Let Δ be a Γ-fundamental system of $\mathfrak{r}$. Then for every $\sigma \in \Gamma$ there exists a uniquely determined element w_σ in W_0 such that we have*

$$\Delta^\sigma = w_\sigma \Delta. \tag{25}$$

The w_σ's ($\sigma \in \Gamma$) satisfy the relation

$$w_\sigma^\tau w_\tau = w_{\sigma\tau}. \tag{26}$$

PROOF. If X_+ is the set of positive elements in X with respect to the Γ-linear order defining the Γ-fundamental system Δ, it is clear that X_+^σ defines a Γ-linear order to which corresponds the fundamental system Δ^σ; hence Δ^σ is a Γ-fundamental system of $\mathfrak{r}$. Now, for every $\alpha_i \in \Delta - \Delta_0$, we have $\alpha_i^\sigma - \alpha_i \in X_0$ so that $\pi(\alpha_i^\sigma) = \pi(\alpha_i)$; hence we have $\bar{\Delta}^\sigma = \bar{\Delta}$. It follows from Lemma 2 bis that there exists (a uniquely determined) $w_\sigma \in W_0$ such that $\Delta^\sigma = w_\sigma \Delta$. The relation (26) follows immediately from (25).

COROLLARY. *For $w \in W_\Gamma$, we have $w^\sigma \equiv w$* (mod W_0) *for all $\sigma \in \Gamma$.*

PROOF. Take any Γ-fundamental system Δ. Then, applying Proposition 6 to Δ and $w\Delta$, one can find $w_\sigma, w'_\sigma \in W_0$ such that $\Delta^\sigma = w_\sigma \Delta, (w\Delta)^\sigma = w'_\sigma(w\Delta)$, whence follows the relation

$$w^\sigma w_\sigma w^{-1} = w'_\sigma,$$

which proves our assertion, q. e. d.
(This Corollary is also a direct consequence of (13), (22), Lemma 2 and the fact that Γ operates trivially on $Y=X/X_0$.)

10. We shall now prove that $\bar{\mathfrak{r}}$ is a 'root system in a wider sense' in $Y_{\mathbf{Q}}$. By this, we mean that $\bar{\mathfrak{r}}$ is a finite subset of $Y_{\mathbf{Q}}$ satisfying the following conditions with respect to a suitable (positive definite) metric $\langle\ \rangle$ in $Y_{\mathbf{Q}}$.

(i) $0 \notin \bar{\mathfrak{r}}$, and for $\gamma \in \bar{\mathfrak{r}}$, we have $-\gamma \in \bar{\mathfrak{r}}$.

(ii) For $\gamma, \gamma' \in \bar{\mathfrak{r}}$, we have $c_{\gamma\gamma'}=\dfrac{2\langle\gamma,\gamma'\rangle}{\langle\gamma,\gamma\rangle} \in \mathbf{Z}$ and $\bar{w}_\gamma\gamma'=\gamma'-c_{\gamma\gamma'}\gamma \in \bar{\mathfrak{r}}$.

(As a matter of fact, this is true if and only if the metric in $Y_{\mathbf{Q}}$ is invariant under $\overline{W}$.) It will follow that, for $\gamma \in \bar{\mathfrak{r}}$, we have $c \cdot \gamma \in \bar{\mathfrak{r}}$ ($c \in \mathbf{Q}$) only for $c=\pm\frac{1}{2}, \pm 1, \pm 2$. Call $\bar{\bar{\mathfrak{r}}}$ the set formed of all $\gamma \in \bar{\mathfrak{r}}$ such that $\frac{1}{2}\gamma \notin \bar{\mathfrak{r}}$. Then, it is immediate that $\bar{\bar{\mathfrak{r}}}$ becomes a 'root system' in the usual sense, called the root system *belonging to* $\bar{\mathfrak{r}}$. A fundamental system and the Weyl group of $\bar{\bar{\mathfrak{r}}}$ are called a fundamental system and the Weyl group of $\bar{\mathfrak{r}}$, respectively. (Note that the definition of the Weyl group of a root system is independent of the choice of the metric intervening in the definition of root system.) In these terminologies, we are going to prove the following

THEOREM 2. *For a suitable* (*positive definite*) *metric in* $Y_{\mathbf{Q}}$, *the restricted root system* $\bar{\mathfrak{r}}$ *relative to* A *becomes a root system in a wider sense in* $Y_{\mathbf{Q}}$. *A restricted fundamental system* $\bar{\Delta}$ *is a fundamental system of* $\bar{\mathfrak{r}}$ *and* vice versa, *and the group* $\overline{W}$ (*defined by* (14) *or* (19)) *is identical with the Weyl group of* $\bar{\mathfrak{r}}$.

If $\bar{\mathfrak{r}}=\phi$, we have $\mathfrak{r} \subset X_0$ and $Z(A)=G$, so that (by (14) or (19)) we have $\overline{W}=\{1\}$. Hence the Theorem holds trivially. Therefore in the following we will assume that $\bar{\mathfrak{r}} \neq \phi$.

For the proof, we need the notion of "Weyl chamber". Put $\mathbf{F}=Y_{\mathbf{Q}}$ and call $\mathbf{F}^*$ the dual space of $\mathbf{F}$ over $\mathbf{Q}$. For a restricted fundamental system $\bar{\Delta}=\{\gamma_1, \cdots, \gamma_\nu\}$, we define the corresponding Weyl chamber by

$$C_{\bar{\Delta}}=\{\omega \in \mathbf{F}^* \mid \langle\gamma_i, \omega\rangle>0 \quad \text{for} \quad 1 \leqq i \leqq \nu\}. \tag{27}$$

It is clear that, for any non-zero element ω_0 in $C_{\bar{\Delta}}$ and for $\gamma \in \bar{\mathfrak{r}}$, we have $\langle\gamma, \omega_0\rangle \gtrless 0$, according as γ is 'positive' or 'negative' with respect to $\bar{\Delta}$ (Proposition 5, b)). It follows first that the restricted fundamental system $\bar{\Delta}$ is uniquely determined by the corresponding Weyl chamber $C_{\bar{\Delta}}$. Next, for $\eta \in \mathbf{F}$, let us denote by H_η the annihilator of $\{\eta\}_{\mathbf{Q}}$ in $\mathbf{F}^*$ ('hyperplane' defined by η). Then we obtain the following relation:

$$\mathbf{F}^*-\bigcup_{\gamma \in \bar{\mathfrak{r}}} H_\gamma=\bigcup_{\bar{\Delta}:\, r.f.s.} C_{\bar{\Delta}}. \tag{28}$$

In fact, the inclusion $\supset$ is an immediate consequence of Proposition 5, b).

To prove the converse, take an arbitrary element ω in $\mathbf{F}^* - \bigcup_{\gamma\in\bar{\mathfrak{r}}} H_\gamma$. Then there exists a linear order in $\mathbf{F}$ such that, for $\eta\in\mathbf{F}, \langle\eta,\omega\rangle>0$ implies $\eta>0$. Call $\bar{\Delta}=\{\gamma_1,\cdots,\gamma_\nu\}$ the corresponding restricted fundamental system of $\bar{\mathfrak{r}}$. Then, since $\langle\gamma_i,\omega\rangle\neq 0$ and $\gamma_i>0$, we must have $\langle\gamma_i,\omega\rangle>0$ $(1\leqq i\leqq\nu)$, i.e. $\omega\in C_{\bar{\Delta}}$, which proves our assertion.

Next we extend $\bar{w}\in\overline{W}$ to a linear transformation of $\mathbf{F}$ in a natural manner and then to that of $\mathbf{F}^*$ by

$$\langle\bar{w}(\eta),\bar{w}(\omega)\rangle=\langle\eta,\omega\rangle \qquad \text{for} \quad \eta\in\mathbf{F},\omega\in\mathbf{F}^*. \tag{29}$$

Then, it is clear that we have $\bar{w}(C_{\bar{\Delta}})=C_{\bar{w}(\bar{\Delta})}$. We need also the following

LEMMA 3. *Let $\gamma\in\bar{\mathfrak{r}}$ and call Q_γ the connected component of the annihilator of $\{\gamma\}_{\mathbf{Z}}$ in A. Then, for $s\in N(A)$, we have $s\in Z(Q_\gamma)$, if and only if $\bar{w}_s$ leaves H_γ elementwise invariant.*

PROOF. Put

$$\hat{Y}=\{\omega\in\mathbf{F}^* \mid \langle\eta,\omega\rangle\in\mathbf{Z} \qquad \text{for all} \quad \eta\in Y\}; \tag{30}$$

then it is clear that $\mathbf{F}^*, H_\gamma$ are obtained from $\hat{Y}, H_\gamma\cap\hat{Y}$ by extending the coefficients from $\mathbf{Z}$ to $\mathbf{Q}$, respectively, and that $\bar{w}\in\overline{W}$ leaves $\hat{Y}$ invariant. Hence it is enough to show that we have $s\in Z(Q_\gamma)$ if and only if $\bar{w}_s$ leaves $H_\gamma\cap\hat{Y}$ elementwise invariant. Now, for $\omega\in\hat{Y}$, there exists a (uniquely determined) morphism f_ω from $\mathbf{G}_m$ into A such that we have

$$\eta(f_\omega(u))=u^{\langle\eta,\omega\rangle} \qquad \text{for} \quad u\in\mathbf{G}_m, \eta\in Y \tag{31}$$

([7]). It is clear that the image of f_ω is contained in Q_γ if and only if $\omega\in H_\gamma$; and moreover Q_γ is generated by those images of f_ω corresponding to $\omega\in H_\gamma\cap\hat{Y}$. Hence we have $s\in Z(Q_\gamma)$ if and only if $sf_\omega(u)s^{-1}=f_\omega(u)$ for all $\omega\in H_\gamma\cap\hat{Y}, u\in\mathbf{G}_m$. On the other hand, for any $s\in N(A)$ and $\omega\in\hat{Y}$, we get from (20), (31) and (29)

$$\eta(sf_\omega(u)s^{-1})=\bar{w}_s^{-1}(\eta)(f_\omega(u))=u^{\langle\bar{w}_s^{-1}(\eta),\omega\rangle}=u^{\langle\eta,\bar{w}_s(\omega)\rangle},$$

and from this and (31) we conclude that $sf_\omega(u)s^{-1}=f_\omega(u)$ for all $u\in\mathbf{G}_m$ if and only if $\bar{w}_s(\omega)=\omega$. This proves our assertion, q.e.d.

11. PROOF OF THEOREM 2. We devide the proof in several steps.

1°. It is clear from definitions ((7), (14)) that $\bar{w}\in\overline{W}$ leaves $\bar{\mathfrak{r}}$ invariant. Now take a restricted fundamental system $\bar{\Delta}$ of $\mathfrak{r}$. Then, for any $\bar{w}\in\overline{W}, \bar{w}(\bar{\Delta})$ is also a restricted fundamental system of $\mathfrak{r}$ and we have $\bar{w}(\bar{\Delta})=\bar{\Delta}$, if and only if $\bar{w}=1$ (Lemma 2). On the other hand, we see from Theorem 1 that the set of all restricted fundamental systems of $\mathfrak{r}$ is in one-to-one correspondence with the set of all maximal connected (k,T)-unipotent subgroups, and that the group $N(A)_k/Z(A)_k$, which may be identified by the canonical isomorphism (19) with a subgroup of $\overline{W}$, operates on the latter set simply transitively. Moreover, if a maximal connected (k, T)-unipotent subgroup N corresponds to a restricted fundamental system $\bar{\Delta}$ and if $s \in N(A)_k$ corresponds to $\bar{w}=\bar{w}_s \in \overline{W}$, then the maximal connected (k, T)-unipotent subgroup sNs^{-1} corresponds to the restricted fundamental system $\bar{w}(\bar{\Delta})$, as is readily seen from (17) and (21). Therefore, we conclude that $\overline{W} \cong N(A)_k/Z(A)_k$ and that $\overline{W}$ operates simply transitively on the set of all restricted fundamental systems of $\mathfrak{r}$. It follows also that the group $\overline{W}$, regarded as a group of linear transformations of $\mathbf{F}^*$, operates simply transitively on the set of all Weyl chambers.

2°. Let $\gamma \in \bar{\mathfrak{r}}$ and let the notations $Q_\gamma, \cdots$ be as in Lemma 3. We infer readily, just as in the proof of Corollary 2 to Proposition 2, that $Z(Q_\gamma)$ is a connected reductive subgroup, defined over k, of G; it is also immediate from the definition that $Z(Q_\gamma) \supsetneqq Z(A)$. It follows from Corollary to Theorem 1 that $N(A) \cap Z(Q_\gamma) \neq Z(A)$, i. e. there exists an element $s \in N(A) \cap Z(Q_\gamma)$ such that $\bar{w}=\bar{w}_s \neq 1$. Then, by Lemma 3, $\bar{w}$ leaves H_γ elementwise invariant. By (28) H_γ contains a "wall" of some Weyl chamber $C_{\bar{\Delta}}$. Then $\bar{w}(C_{\bar{\Delta}})$ becomes a Weyl chamber "neighbouring" to $C_{\bar{\Delta}}$, and, by the same reason, we have $\bar{w}^2(C_{\bar{\Delta}})=C_{\bar{\Delta}}$, whence, by 1°, $\bar{w}^2=1$. Thus we have proved that for each $\gamma \in \bar{\mathfrak{r}}$ there exists an element $\bar{w}$ of $\overline{W}$ such that $\bar{w} \neq 1, \bar{w}^2=1$ and leaving H_γ elementwise invariant. If we introduce in $\mathbf{F}$ any positive definite metric invariant under $\overline{W}$ and identify $\mathbf{F}^*$ with $\mathbf{F}$ by means of this metric, the above $\bar{w}$ coincides clearly with a (unique) reflection $\bar{w}_\gamma$ of $\mathbf{F}$ with respect to H_γ, i. e.

$$\bar{w}_\gamma : \eta \rightarrow \eta-\frac{2\langle\gamma_i, \eta\rangle}{\langle\gamma, \gamma\rangle}\gamma . \tag{32}$$

3°. Now we show that $\bar{\mathfrak{r}}$ is a root system in a wider sense in $Y_{\mathbf{Q}}$. It is obvious from the definition that $\bar{\mathfrak{r}}$ satisfies the condition (i), and it follows from 2° that, for $\gamma, \gamma' \in \bar{\mathfrak{r}}$, we have $\bar{w}_\gamma\gamma'=\gamma'-c_{\gamma\gamma'}\gamma \in \bar{\mathfrak{r}}$. To prove that $c_{\gamma\gamma'}=\dfrac{2\langle\gamma, \gamma'\rangle}{\langle\gamma, \gamma\rangle} \in \mathbf{Z}$, we first remark that, for $\gamma_i \in \Delta$ and $\eta \in Y$, we have $\dfrac{2\langle\gamma_i, \eta\rangle}{\langle\gamma_i, \gamma_i\rangle} \in \mathbf{Z}$. In fact, it is known that, for any $\chi \in X$ and $w \in W$, we have $w\chi-\chi \in \mathfrak{r}_{\mathbf{Z}}$ ([4], 16-09, Cor. 1). It follows that, for any $\eta \in Y$ and $\bar{w} \in \overline{W}$, we have $\bar{w}\eta-\eta \in \bar{\mathfrak{r}}_{\mathbf{Z}}$. Taking $\bar{w}$ to be $\bar{w}_{\gamma_i}$, we obtain $\bar{w}_{\gamma_i}\eta-\eta=-\dfrac{2\langle\gamma_i, \eta\rangle}{\langle\gamma_i, \gamma_i\rangle}\gamma_i \in \bar{\mathfrak{r}}_{\mathbf{Z}}$. Hence our assertion follows from Proposition 5, b).

Now let $\gamma \in \bar{\mathfrak{r}}$ and call $\mathfrak{r}_\gamma$ the set formed of all $\alpha \in \mathfrak{r}$ such that $\pi(\alpha)$ is an integral multiple of γ. Then it is clear that $\mathfrak{r}_\gamma$ is a Γ-invariant 'closed' subsystem of $\mathfrak{r}$, so that ([4], 17-02, Th. 1) there corresponds a connected semi-simple subgroup $G(\mathfrak{r}_\gamma)$, defined over k, of G. It is then clear that a restricted fundamental system relative to A of the reductive group $T \cdot G(\mathfrak{r}_\gamma)$ is given by $\{\gamma\}$. Therefore, applying the above remark to this group, we conclude that $\frac{2\langle\gamma,\eta\rangle}{\langle\gamma,\gamma\rangle}\in\mathbf{Z}$ for all $\eta\in Y$ and, in particular, that $c_{\gamma\gamma'}\in\mathbf{Z}$.

4°. We now prove the second half of the Theorem. It is clear from Proposition 5, b) that a restricted fundamental system $\bar{\Delta}$ is a fundamental system of $\bar{\bar{\mathfrak{r}}}$ (and hence, by definition, of $\bar{\mathfrak{r}}$). The converse of this follows immediately from the fact that, by (28), the Weyl chambers of $\bar{\bar{\mathfrak{r}}}$ (in the usual sense) are all given by the $C_{\bar{\Delta}}$. Finally, the group $\overline{W}$ contains all reflections $\bar{w}_{\gamma_i}$ with $\gamma_i\in\bar{\Delta}$, by 2°, on the one hand, and $\overline{W}$ operates simply transitively on the set of all Weyl chambers, by 1°, on the other. Therefore we conclude, by a standard argument in such situation, that $\overline{W}$ is generated by the w_{γ_i}'s, i.e. $\overline{W}$ coincides with the Weyl group of $\bar{\bar{\mathfrak{r}}}$ (and hence, by definition, of $\bar{\mathfrak{r}}$), q.e.d.

COROLLARY 1. *Every coset in $N(A)/Z(A)$ contains a k-rational point.*

COROLLARY 2. *$\overline{W}$ operates simply transitively on the set of all restricted fundamental systems.*

These were established in the 1-st step of the above proof.

COROLLARY 3. *W_Γ operates simply transitively on the set of all Γ-fundamental systems.*

PROOF. We know already that if Δ is a Γ-fundamental system and if $w\in W_\Gamma$, then $w(\Delta)$ is again a Γ-fundamental system (Lemma 2). Now let Δ,Δ' be two Γ-fundamental systems of $\mathfrak{r}$ and call $\bar{\Delta},\bar{\Delta}'$ the corresponding restricted fundamental systems of $\mathfrak{r}$. Then, by Corollary 2 above, there exists a $\bar{w}\in\overline{W}$ such that $\bar{\Delta}'=\bar{w}(\bar{\Delta})$. Then, for any $w\in\pi^{-1}(\bar{w})$, the restricted fundamental system corresponding to $w(\Delta)$ is equal to $\bar{w}(\bar{\Delta})=\bar{\Delta}'$. Hence, by Lemma 2 bis, there exists a $w_0\in W_0$ such that $w_0w(\Delta)=\Delta'$, q.e.d.

COROLLARY 4. *For any $\gamma\in\bar{\mathfrak{r}}$ and $\eta\in Y$, we have* $\frac{2\langle\gamma,\eta\rangle}{\langle\gamma,\gamma\rangle}\in\mathbf{Z}$.

This was established in the 4-th step of the above proof.

REMARK. As is seen from the above proof, $\bar{\mathfrak{r}}$ satisfies the condition (ii), if and only if the metric in $Y_{\mathbf{Q}}$ is invariant under $\overline{W}$. Without refering to any metric, we can also formulate the main part of our results (Theorem 2 and Corollary 4) as follows: *For each $\gamma\in\bar{\mathfrak{r}}$, there exists a uniquely determined element γ^* in $\hat{Y}=$* Hom$(Y,\mathbf{Z})$ *such that $\langle\gamma,\gamma^*\rangle=2$ and that $\overline{W}$ contains the reflection*

$$\bar{w}_\gamma:\eta\to\eta-\langle\eta,\gamma^*\rangle\gamma\,. \tag{32'}$$

§ 4. Isomorphisms between reductive groups.

12. Let G be a connected reductive algebraic group, defined over a perfect field k, and let the notations $T,A,\cdots$ be as before. As we noted in the proof of Corollary 2 to Proposition 2, the semi-simple part of $Z(A)$ is given by the subgroup $G(\mathfrak{r}_0)$ corresponding to a closed subsystem $\mathfrak{r}_0=\mathfrak{r}\cap X_0$ of $\mathfrak{r}$. On the other hand, call C_0 the k-compact part of the torus part of G; and put

$$G_0=G(\mathfrak{r}_0)\cdot C_0\,. \tag{33}$$

Then, G_0 is a connected reductive k-closed subgroup of G, containing no k-trivial torus of dimension >0; in general, a reductive algebraic group with this property will be called *k-quasicompact*[3]. We call G_0 the *k-quasicompact kernel* of G relative to A. Since the maximal k-trivial toruses in G are conjugate to each other with respect to inner automorphisms defined by elements in G_k, so are also the k-quasicompact kernels of G.

It is clear that, if we put

$$T_0=T\cap G_0\,, \tag{34}$$

T_0 is a maximal torus defined over k in G_0, and the character module X^0 of T_0 may be identified (in the notations of the proof of Proposition 1) with the projection of X on the factor $(\mathfrak{r}_0)_{\mathbf{Q}}+\mathfrak{r}_{\mathbf{Q}}^{\perp}\cap(X_0)_{\mathbf{Q}}$ in the direct decomposition

$$X_{\mathbf{Q}}=((\mathfrak{r}_0)_{\mathbf{Q}}+\mathfrak{r}_{\mathbf{Q}}^{\perp}\cap(X_0)_{\mathbf{Q}})+(\mathfrak{r}_{\mathbf{Q}}\cap(\mathfrak{r}_0)_{\mathbf{Q}}^{\perp}+\mathfrak{r}_{\mathbf{Q}}^{\perp}\cap(X^{\Gamma})_{\mathbf{Q}});$$

then the root system of G_0 relative to T_0 is identified with $\mathfrak{r}_0$.

Next, let Δ be a Γ-fundamental system of $\mathfrak{r}$ and, for $\sigma\in\Gamma$, put

$$\bar{\varphi}_\sigma(\chi)=(w_\sigma\chi)^{\sigma^{-1}}=w_{\sigma^{-1}}^{-1}\chi^{\sigma^{-1}}\qquad\text{for all}\quad \chi\in X, \tag{35}$$

with $w_\sigma\in W_0$ defined in Proposition 6. Then it is clear that $\bar{\varphi}_\sigma$ is an automorphism of X leaving Δ,Δ_0 invariant, or, as we call more briefly, an automorphism of (X,Δ,Δ_0), and that the correspondence $\sigma\to\bar{\varphi}_\sigma$ is a homomorphism from Γ into $\mathrm{Aut}(X,\Delta,\Delta_0)$ ($=$the group of all automorphisms of (X,Δ,Δ_0)). We call $(X,\Delta,\Delta_0,\{\bar{\varphi}_\sigma\})$ a *system associated with* (G,T).

In particular, if G is k-quasicompact, we have $\Delta=\Delta_0$, so that we associate only $(X,\Delta,\{\bar{\varphi}_\sigma\})$. From what we mentioned above, it follows that, if we denote by $\bar{\varphi}_\sigma^0$ the automorphism of (X^0,Δ_0) induced from $\bar{\varphi}_\sigma$, then $(X^0,\Delta_0,\{\bar{\varphi}_\sigma^0\})$ is a system associated with (G_0,T_0). We call $(X^0,\Delta_0,\{\bar{\varphi}_\sigma^0\})$ the *restriction* of $(X,\Delta,\Delta_0,\{\bar{\varphi}_\sigma\})$ on G_0.

Now let G' be another connected reductive algebraic group, defined over k, and let the symbols with a prime (such as $T',A',\cdots$) denote the things for G' corresponding to those for G denoted by the same symbols without prime (such as $T,A,\cdots$); without loss of generality, we may assume that $K=K'$ and so $\Gamma=\Gamma'$. Let $G_0'=G'(\mathfrak{r}_0')\cdot C_0'$ and $(X',\Delta',\Delta_0',\{\bar{\varphi}_\sigma'\})$ be the k-quasicompact kernel of G' relative to A' and a system associated with (G',T'), respectively.

Suppose that there exists a K-isomorphism f from (G,T) onto (G',T') (i.e. K-isomorphism from G onto G' such that $f(T)=T'$) and put

3) According to the Tits's result (footnote 2)), the prefix 'quasi-' is superfluous.

(36) $$\psi = {}^t(f \mid T)^{-1},$$

t denoting the 'dual'. Then it is clear that ψ is an isomorphism from $(X, \mathfrak{r})$ onto $(X', \mathfrak{r}')$ (i.e. isomorphism from X onto X' such that $\psi(\mathfrak{r}) = \mathfrak{r}'$). Assume furthermore that f *satisfies the condition* $f(A) = A'$, or equivalently that ψ satisfies the condition $\psi(X_0) = X_0'$. Then it is easy to see that, for any Γ-fundamental system Δ of $\mathfrak{r}$, $\psi(\Delta)$ becomes a Γ-fundamental system of $\mathfrak{r}'$, so that, by Corollary 3 to Theorem 2, there exists a (uniquely determined) element w' in W'_Γ such that $\Delta' = w'\psi(\Delta)$. Then we see at once that $\Delta_0' = w'\psi(\Delta_0)$ and that

(37) $$w_\sigma' = w'^\sigma \psi^\sigma w_\sigma \psi^{-1} w'^{-1} \qquad \text{on} \quad \mathfrak{r}',$$

ψ^σ being defined by the formula

$$\psi^\sigma(\chi^\sigma) = (\psi(\chi))^\sigma \qquad \text{for all} \quad \chi \in X,$$

whence we get

(38) $$\bar{\varphi}_\sigma' = w'\psi\bar{\varphi}_\sigma\psi^{-1}w'^{-1} \qquad \text{on } \Delta'.$$

If, moreover, $f \mid C_0$ *is defined over* k, or equivalently $\psi^\sigma = \psi$ on $\mathfrak{r}_{\mathbf{Q}}^\perp \cap (X_0)_{\mathbf{Q}}$, we have $\psi^\sigma = \psi$ on $\mathfrak{r}_{\mathbf{Q}}^\perp$, since this holds trivially on $\mathfrak{r}_{\mathbf{Q}}^\perp \cap (X^\Gamma)_{\mathbf{Q}}$. Hence (38) holds also on $X' \cap \mathfrak{r}'^\perp_{\mathbf{Q}}$. Therefore, putting $\bar{\psi} = w'\psi$, we obtain the relations

(39) $$\Delta' = \bar{\psi}(\Delta), \quad \Delta_0' = \bar{\psi}(\Delta_0), \quad \bar{\varphi}_\sigma' = \bar{\psi}\bar{\varphi}_\sigma\bar{\psi}^{-1}.$$

In general, when (39) holds with an isomorphism $\bar{\psi}$ from X onto X', we say that two systems $(X, \Delta, \Delta_0, \{\bar{\varphi}_\sigma\})$, $(X', \Delta', \Delta_0', \{\bar{\varphi}_\sigma'\})$ are *congruent*. The above $\bar{\psi} = w'\psi$ is called a *congruence associated with* f. (Here and in the following, $\psi, \bar{\psi}, \cdots$ will be considered, whenever necessary, as to be extended in a natural manner to isomorphisms (of vector space over $\mathbf{Q}$) from $X_{\mathbf{Q}}$ onto $X'_{\mathbf{Q}}$.)

Now this consideration is first applied to the case $G = G'$. Namely, in this case, we have $T' = gTg^{-1}, A' = gAg^{-1}$ with $g \in G_K$. (In fact, A, A' being both maximal k-trivial toruses in G, there exists $g_1 \in G_k$ such that $A' = g_1 A g_1^{-1}$. Then, $g_1 T g_1^{-1}, T'$ being both maximal K-trivial toruses in $Z(A')$, there exists $g_2 \in Z(A')_K$ such that $T' = g_2 g_1 T g_1^{-1} g_2^{-1}$. Putting $g = g_1 g_2$, we get the assertion.) The inner automorphism $f = I_g$ defined by g satisfying all the above conditions, our arguments applied to it show that the system $(X, \Delta, \Delta_0, \{\bar{\varphi}_\sigma\})$ does not depend essentially on the choice of T, but is uniquely determined only by G, up to a congruence.

Secondly, the same arguments can also be applied to any k-isomorphism f from (G, T) onto (G', T'). In this case, f induces a k-isomorphism f_0 from (G_0, T_0) onto (G_0', T_0'), and so a congruence $\bar{\psi}^0$ from $(X^0, \Delta_0, \{\bar{\varphi}_\sigma^0\})$ to $(X'^0, \Delta_0', \{\bar{\varphi}_\sigma'^0\})$ associated with f_0. Then, naturally, $\bar{\psi}^0$ coincides with the restriction on X^0 of the congruence $\bar{\psi}$ associated with f. Now the main purpose of this section is to establish the converse of this statement. Namely, we are going to prove the following

THEOREM 3. *Let G, G' be connected reductive algebraic groups, defined over a perfect field k, and T, T' maximal toruses in G, G', also defined over k, containing maximal k-trivial toruses A, A' in G, G', respectively. Call G_0, G_0' the k-quasicompact kernels of G, G' relative to A, A', respectively, and put $T_0 = T \cap G_0$, $T_0' = T' \cap G_0'$; and let $(X, \Delta, \Delta_0, \{\bar{\varphi}_\sigma\})$, $(X', \Delta', \Delta_0', \{\bar{\varphi}_\sigma'\})$ be systems associated with $(G, T), (G', T')$, respectively, and let $(X^0, \Delta_0, \{\bar{\varphi}_\sigma^0\}), (X'^0, \Delta_0', \{\bar{\varphi}_\sigma'^0\})$ be their restrictions on G_0, G_0', respectively. Suppose that the following conditions are satisfied:*

(i) *There exists a congruence $\bar{\psi}$ from $(X, \Delta, \Delta_0, \{\bar{\varphi}_\sigma\})$ to $(X', \Delta', \Delta_0', \{\bar{\varphi}_\sigma'\})$;*

(ii) *There exists a k-isomorphism f_0 from (G_0, T_0) onto (G_0', T_0');*

(iii) *The congruence $\bar{\psi}^0$ from $(X^0, \Delta_0, \{\bar{\varphi}_\sigma^0\})$ to $(X'^0, \Delta_0', \{\bar{\varphi}_\sigma'^0\})$ associated with f_0 coincides with the restriction of $\bar{\psi}$ on X^0.*

Then there exists a k-isomorphism f from (G, T) onto (G', T') extending f_0 such that $\bar{\psi}$ is a congruence associated with f.

13. For the proof of Theorem 3, we need a more detailed investigation on isomorphisms between reductive algebraic groups. To begin with, let G be a connected reductive algebraic group, defined over k, and let the notations be as before. Then, applying $\sigma \in \Gamma$ on the both sides of (5), we get from the 'uniqueness' of x_α the relation

$$x_\alpha^\sigma(\xi) = x_{\alpha^\sigma}(\xi_{\sigma,\alpha}\xi) \tag{40}$$

with $\xi_{\sigma,\alpha} \in K^*$. The system $\{\xi_{\sigma,\alpha}\}$ satisfies clearly the relation

$$\xi_{\sigma,\alpha}^\tau \xi_{\tau,\alpha^\sigma} = \xi_{\sigma\tau,\alpha}\,. \tag{41}$$

REMARK. Incidentally we notice that x_α may be taken to be defined over any perfect field over which α is defined (hence a fortiori over any splitting field for T). In fact, if we call Γ_α the subgroup of Γ formed of all $\sigma \in \Gamma$ leaving α invariant, the system $\{\xi_{\sigma,\alpha}\ (\sigma \in \Gamma_\alpha)\}$ becomes, by (41), a 1-cocycle of Γ_α in K^*. Therefore, by the Theorem 90 of Hilbert, we have $\xi_{\sigma,\alpha} = \eta_\alpha^{\sigma-1}$ with $\eta_\alpha \in K^*$. Then, putting $\bar{x}_\alpha(\xi) = x_\alpha(\eta_\alpha^{-1}\xi)$, we see that $\bar{x}_\alpha$ is invariant under all $\sigma \in \Gamma_\alpha$, i.e. defined over the smallest field containing k over which α is defined.

Now let G' be another connected reductive algebraic group, defined over k, and let the notations be as mentioned in N° **12**; in particular, we define the system $\{\xi'_{\sigma,\alpha'}\}$ by means of $x'_{\alpha'}$ through a formula similar to (40). Suppose that a K-isomorphism f from (G, T) onto (G', T') is given. Then, applying f on the both sides of (5), we get again by the uniqueness of $x'_{\alpha'}$

$$f(x_\alpha(\xi)) = x'_{\psi(\alpha)}(\eta_\alpha\xi) \tag{42}$$

with ψ defined by (36) and $\eta_\alpha \in K^*$. Fixing x_α's and x'_α's once for all, we attach $\{\psi, \eta_\alpha\}$ to f. Notation: $f \longleftrightarrow \{\psi, \eta_\alpha\}$. Clearly f is uniquely determined by $\{\psi, \eta_\alpha\}$.

LEMMA 4. a) *The notations being as above, we have*

$$f^\sigma \longleftrightarrow \left\{\psi^\sigma,\ \eta^\sigma_{\alpha^{\sigma-1}} \frac{\xi'_{\sigma,\psi(\alpha^{\sigma-1})}}{\xi_{\sigma,\alpha^{\sigma-1}}}\right\}$$

for all $\sigma \in \Gamma$. *In particular, f is defined over k, if and only if the following conditions are satisfied:*

$$\psi^\sigma = \psi\,, \tag{43}$$

$$\xi'_{\sigma,\psi(\alpha)} = \xi_{\sigma,\alpha} \frac{\eta_{\alpha^\sigma}}{\eta^\sigma_\alpha} \tag{44}$$

for all $\sigma \in \Gamma, \alpha \in \mathfrak{r}$.

b) *The notations being as above, let G'' be a third connected reductive algebraic group, defined over k, and T'' a maximal torus in G'', also defined over k and split over K, and let f' be a K-isomorphism from (G', T') onto (G'', T''). If $f' \longleftrightarrow \{\psi', \eta'_{\alpha'}\}$ in the above sense, we have*

$$f' \circ f \longleftrightarrow \{\psi' \circ \psi,\ \eta'_{\psi(\alpha)}\eta_\alpha\}\,.$$

The proof of this Lemma is straightforward.

LEMMA 5. *Let Δ be any fundamental system of $\mathfrak{r}$. If $f \longleftrightarrow \{\psi, \eta_\alpha\}$, f is uniquely determined by ψ and by the η_α's for $\alpha \in \Delta$.*

By Lemma 4, b), the proof is reduced to the case, where f is a K-automorphism of (G, T), the case which was already established in [4], 17-08/09. For convenience, we call a system $\{\eta_\alpha\ (\alpha \in \mathfrak{r})\}$ *coherent with respect to* ψ, if there exists an isomorphism $f \longleftrightarrow \{\psi, \eta_\alpha\}$. Then, Lemma 5 may be stated as follows: *A coherent system $\{\eta_\alpha\ (\alpha \in \mathfrak{r})\}$ with respect to an isomorphism ψ: $(X, \mathfrak{r}) \to (X', \mathfrak{r}')$ is uniquely determined by the η_α's corresponding to $\alpha \in \Delta$.*

REMARK 1. In the case of characteristic 0, we have more explicitly the following result. Namely, we have $x_\alpha(\xi) = \exp(\xi E_\alpha)$ with an element E_α in the Lie algebra of G; and, if $\alpha, \beta, \alpha+\beta \in \mathfrak{r}$, we have

$$[E_\alpha, E_\beta] = N_{\alpha,\beta} E_{\alpha+\beta}$$

with $N_{\alpha,\beta} \in K^*$ (which we may take in $\mathbf{Z}$). Then, for a suitable $\{E_\alpha\}$ (so-called 'Weyl basis'), we obtain the following relations characterizing a coherent system:

$$\begin{cases} \eta_{\alpha+\beta} = \eta_\alpha \eta_\beta \dfrac{N_{\psi(\alpha),\psi(\beta)}}{N_{\alpha,\beta}}\,, \\ \eta_{-\alpha} = \eta_\alpha^{-1}\,. \end{cases} \tag{45}$$

Even in the case of characteristic $\neq 0$, similar formulas may be obtained, by using a method of Chevalley [3]. But no such explicit formulas will be needed in the following.

REMARK 2. Applying Lemma 5 to $f^{-1}\circ f^{\sigma}$, we see that in the statement of Lemma 4, a) the word 'for all $\alpha\in\mathfrak{r}$' may be replaced by 'for all $\alpha\in\Delta$'.

LEMMA 6. *If I_t denotes the inner automorphism of (G, T) determined by $t\in T$, we have*

$$I_t\leftrightarrow\{1, \alpha(t)\}.$$

Conversely, if $\Psi\in\mathrm{Aut}(G, T)$ is such that $\Psi\leftrightarrow\{1, *\}$, we have $\Psi=I_t$ with some $t\in T$. I_t is defined over K, if and only if $\alpha(t)\in K^*$ for all $\alpha\in\mathfrak{r}$, or, what amounts to the same, t (mod. Z) is K-rational (as an element in T/Z), Z denoting the center of G.

The first and the third assertions are obvious, and the second follows from the first and Lemma 5.

LEMMA 5 bis. *For any isomorphism ψ from $(X, \mathfrak{r})$ onto $(X', \mathfrak{r}')$ and for any $\eta_\alpha\in K^*$ $(\alpha\in\Delta)$, there exists (uniquely) a system $\{\eta_\alpha\,(\alpha\in\mathfrak{r})\}$ in K^* coherent with respect to ψ, (provided K is sufficiently large).*

By the main result of Chevalley [4] (Exp. 18~24), there exists, for any ψ, at least one K-isomorphism f from (G, T) onto (G', T') such that $f\leftrightarrow\{\psi, *\}$, if K is taken sufficiently large (depending on ψ, for the moment). The Lemma then follows from Lemmas 4, b) and 6. It will be ascertained afterwards (by Corollary 2 to Theorem 3) that it is not necessary to extend K.

14. PROOF OF THEOREM 3. It is enough to show that there exists a K-isomorphism f from (G, T) onto (G', T'), (K being taken sufficiently large), such that $f\leftrightarrow\{\psi, \eta_\alpha\}$, in the notations of N^o **13**, with $\{\psi, \eta_\alpha\}$ satisfying the following conditions:

a) (43), (44) hold;

b) $f_0\leftrightarrow\{\psi^0, \eta_\alpha\,(\alpha\in\mathfrak{r}_0)\}$ (with respect to the x_α's $(\alpha\in\mathfrak{r}_0)$ and the $x'_{\alpha'}$'s $(\alpha'\in\mathfrak{r}'_0)$) with $\psi^0=\psi\mid X^0$;

c) $\bar{\psi}=w'\psi$ with $w'\in W'_\Gamma$.

To start with, let $f_0\leftrightarrow\{\psi^0, \eta_\alpha\,(\alpha\in\mathfrak{r}_0)\}$. By definition, we have $\bar{\psi}^0=w'\psi^0$ with $w'\in W'_0$, and $\bar{\psi}^0(\Delta_0)=\Delta'_0$. Put

$$\psi=w'^{-1}\circ\bar{\psi}.$$

Then, it is clear that ψ is an isomorphism from $(X, \mathfrak{r})$ onto $(X', \mathfrak{r}')$ such that $\psi(X^0)=X'^0$ and $\psi\mid X^0=\psi^0$, by the condition (iii). We now assert that ψ satisfies the relation (43), i.e. $\psi^\sigma=\psi$ for all $\sigma\in\Gamma$. In fact, since $X_{\mathbf{Q}}=(X^0)_{\mathbf{Q}}+(X^0)^{\perp}_{\mathbf{Q}}$, it is enough to show that this relation holds both on $(X^0)_{\mathbf{Q}}$ and on $(X^0)^{\perp}_{\mathbf{Q}}$. First, since f_0 is a k-isomorphism, we have $\psi^{0\sigma}=\psi^0$ for all $\sigma\in\Gamma$, i.e. (43) holds on $(X^0)_{\mathbf{Q}}$. On the other hand, since $w_\sigma\in W_0$, $w'_\sigma, w'\in W'_0$ and since $\mathfrak{r}_0\subset X^0$, $\mathfrak{r}'_0\subset X'^0$, we have

$$\bar{\varphi}_\sigma(\chi)=\chi^{\sigma^{-1}},\quad \bar{\psi}(\chi)=\psi(\chi)\quad \text{for all}\quad \chi\in(X^0)^{\perp}_{\mathbf{Q}},$$
$$\bar{\varphi}'_\sigma(\chi')=\chi'^{\sigma^{-1}}\qquad\qquad \text{for all}\quad \chi'\in(X'^0)^{\perp}_{\mathbf{Q}},$$

so that (the third equality in) (39) implies that (43) holds on $(X^0)^{\perp}_{\mathbf{Q}}$. This proves our assertion.

It remains to show that $\{\eta_\alpha(\alpha\in\mathfrak{r}_0)\}$ can be extended to a system $\{\eta_\alpha(\alpha\in\mathfrak{r})\}$ coherent with respect to ψ and satisfying (44). For each $\alpha_i\in\Delta-\Delta_0$, denote by $\mathfrak{r}^{(i)}$ the subsystem of $\mathfrak{r}$ formed of all roots which are linear combinations of α_i and of the roots in $\mathfrak{r}_0$, and denote by Γ_{α_i} the subgroup of Γ formed of all $\sigma\in\Gamma$ such that $\bar{\varphi}_\sigma(\alpha_i)=\alpha_i$. Then the closed subgroup $G(\mathfrak{r}^{(i)})$ of G corresponding to the closed subsystem $\mathfrak{r}^{(i)}$ of $\mathfrak{r}$ is clearly invariant under Γ_{α_i} so that defined over the intermediate field of K/k corresponding to Γ_{α_i}. We shall first show that $\{\eta_\alpha(\alpha\in\mathfrak{r}_0)\}$ can be extended to a coherent system $\{\eta_\alpha(\alpha\in\mathfrak{r}^{(i)})\}$ (with respect to the restriction of ψ on $\mathfrak{r}^{(i)}_{\mathbf{Q}}$) satisfying (44) for all $\sigma\in\Gamma_{\alpha_i}$. In fact, taking $\eta_{\alpha_i}\in K^*$ arbitrarily, $\{\eta_\alpha(\alpha\in\mathfrak{r}_0),\eta_{\alpha_i}\}$ can be extended (uniquely) to a coherent system $\{\eta_\alpha(\alpha\in\mathfrak{r}^{(i)})\}$ (Lemma 5 bis). Let $f^{(i)}\leftrightarrow\{\psi\mid\mathfrak{r}^{(i)}_{\mathbf{Q}},\ \eta_\alpha(\alpha\in\mathfrak{r}^{(i)})\}$. It follows from Lemma 4 and the relation (43) that we have

$$f^{(i)-1}\circ f^{(i)\sigma}\leftrightarrow\{1,\zeta_{\sigma,\alpha^{\sigma^{-1}}}\}$$

with

$$\zeta_{\sigma,\alpha}=\frac{\xi'_{\sigma,\psi(\alpha)}\eta_\alpha^\sigma}{\xi_{\sigma,\alpha}\eta_{\alpha^\sigma}}\qquad \text{for}\quad \alpha\in\mathfrak{r}^{(i)},\ \sigma\in\Gamma_{\alpha_i}.$$

Therefore, in view of Lemma 6, we get

$$\zeta^\tau_{\sigma,\alpha}\zeta_{\tau,\alpha^\sigma}=\zeta_{\sigma\tau,\alpha},$$
$$\zeta_{\sigma,\alpha}\zeta_{\sigma,\beta}=\zeta_{\sigma,\alpha+\beta},\quad \zeta_{\sigma,-\alpha}=\zeta^{-1}_{\sigma,\alpha};$$

moreover, since $f^{(i)}\mid G(\mathfrak{r}_0)$ is defined over k, $\zeta_{\sigma,\alpha}$ depends only on the class of α modulo $\mathfrak{r}_{0\mathbf{Q}}$. It follows that for $\sigma\in\Gamma_{\alpha_i}$ we have $\zeta_{\tau,\alpha_i^\sigma}=\zeta_{\tau,\bar{\varphi}_\sigma^{-1}w^{-1}_{\sigma^{-1}}(\alpha_i)}=\zeta_{\tau,\alpha_i}$, so that $\{\zeta_{\sigma,\alpha_i}(\sigma\in\Gamma_{\alpha_i})\}$ forms a 1-cocycle of Γ_{α_i} in K^*. Hence by the Theorem 90 of Hilbert, we can find $\omega\in K^*$ such that $\zeta_{\sigma,\alpha_i}=\omega^{\sigma-1}$ for all $\sigma\in\Gamma_{\alpha_i}$. Then, replacing η_{α_i} by $\omega^{-1}\eta_{\alpha_i}$, we obtain $\zeta_{\sigma,\alpha_i}=1$, hence $\zeta_{\sigma,\alpha}=1$ for all $\alpha\in\mathfrak{r}^{(i)}$, $\sigma\in\Gamma_{\alpha_i}$, which proves our assertion. For convenience, we say that η_{α_i} is 'properly' chosen if we have $\zeta_{\sigma,\alpha_i}=1$ for all $\sigma\in\Gamma_{\alpha_i}$. Now we divide $\Delta-\Delta_0$ into orbits of $\{\bar{\varphi}_\sigma(\sigma\in\Gamma)\}$ and call Δ_i $(1\leqq i\leqq r)$ the totality of the orbits. For each i, take a representative α_{k_i} in Δ_i and choose $\eta_{\alpha_{k_i}}$ 'properly'; and then extend $\{\eta_\alpha(\alpha\in\mathfrak{r}_0),\eta_{\alpha_{k_i}}\}$ to a coherent system $\{\eta_\alpha(\alpha\in\mathfrak{r}^{(k_i)})\}$. For every $\alpha_j\in\Delta_i$, there exists $\sigma\in\Gamma$ such that $\alpha_j=\bar{\varphi}_\sigma(\alpha_{k_i})=(w_\sigma(\alpha_{k_i}))^{\sigma^{-1}}$. We define η_{α_j} by

$$\eta_{\alpha_j}=\eta^{\sigma^{-1}}_{w_\sigma(\alpha_{k_i})}\frac{\xi'_{\sigma^{-1},\psi(w_\sigma(\alpha_{k_i}))}}{\xi_{\sigma^{-1},w_\sigma(\alpha_{k_i})}}.$$

Then, from our choice of the $\eta_{\alpha_{k_i}}$ and from the relation (41) for $\{\xi_{\sigma,\alpha}\}$, $\{\xi'_{\sigma,\alpha'}\}$, we see immediately that this definition of η_{α_j} does not depend on the choice of $\sigma\in\Gamma$ such that $\alpha_j=\bar{\varphi}_\sigma(\alpha_{k_i})$. This means that, with η_{α_j} thus defined, the relation (44) is satisfied for $\alpha=\alpha_j$ and for all $\sigma\in\Gamma$ such that $\alpha_j=\bar{\varphi}_\sigma(\alpha_{k_i})$. Now, starting from the given $\eta_{\alpha_j}(\alpha_j\in\Delta_0)$ and the $\eta_{\alpha_j}(\alpha_j\in\Delta-\Delta_0)$ thus defined, we extend the definition of η_α to a system $\{\eta_\alpha\ (\alpha\in\mathfrak{r})\}$ coherent with respect to ψ. Then, it is easy to see that (44) is satisfied for all $\alpha\in\Delta$ and $\sigma\in\Gamma$; hence also for all $\alpha\in\mathfrak{r}$, as we remarked in Remark 2 after Lemma 5, q. e. d.

15. We add here several consequences of Theorem 3. First, in the special case where G_0, G_0' reduce to the neutral element, the condition (ii), (iii) becoming void, we obtain the following

COROLLARY 1. *Let G, G' be connected reductive algebraic groups defined over a perfect field k, such that G_0, G_0' reduce to the neutral element. In the notation of Theorem 3, suppose that there exists a congruence $\bar{\psi}$ from $(X, \Delta, \phi, \{\bar{\varphi}_\sigma\})$ to $(X', \Delta', \phi, \{\bar{\varphi}_\sigma'\})$. Then there exists a k-isomorphism f from (G, T) onto (G', T') such that $\bar{\psi}$ is associated with f.*

It is well-known that there exists no quasicompact semi-simple algebraic group, of dimension >0, over a finite field. Therefore Corollary 1 can be applied for any connected semi-simple algebraic groups G, G', defined over a finite field k. On the other hand, another special case where Corollary 1 applies, is the case of so-called groups of Chevalley type. Namely, we call connected reductive algebraic group G, defined over k, *of Chevalley type* over k, if it has a k-trivial maximal torus T, or, in our notation, if $T=A$; if this is the case, we have surely $G_0=\{1\}$. Hence we obtain from Corollary 1 the following 'uniqueness' of groups of Chevalley type.

COROLLARY 2. *Let G, G' be connected reductive algebraic groups of Chevalley type over a perfect field k. In the notation of N°**12**, if there exists an isomorphism ψ from $(X, \mathfrak{r})$ onto $(X', \mathfrak{r}')$, then there exists a k-isomorphism f from (G, T) onto (G', T') such that $\psi={}^t(f\,|\,T)^{-1}$.*

Finally we make some remarks on the classification theory of semi-simple algebraic groups over a perfect field k. First of all, the results of Chevalley ([**3**], supplemented by Ono [**6**], and [**4**], supplemented by Corollary 2 above) give a complete classification of groups of Chevalley type over k. More precisely, the isomorphism classes of connected semi-simple algebraic groups G of Chevalley type over k are in one-to-one correspondence with the isomorphism classes of the pairs $(X, \mathfrak{r})$ formed of a free submodule X of the maximal rank in a metric vector space $\mathbf{E}$ over $\mathbf{Q}$ and of a root system $\mathfrak{r}$ in $\mathbf{E}$ (in the usual sense) satisfying the conditions $\mathfrak{r}\subset X$ and that $\dfrac{2\langle\alpha,\chi\rangle}{\langle\alpha,\alpha\rangle}\in\mathbf{Z}$ for all $\alpha\in\mathfrak{r}, \chi\in X$. Therefore, in order to classify all connected semi-simple algebraic groups over k, it is enough to fix a finite Galois extension K/k and a connected semi-simple group $\tilde{G}$ of Chevalley type over K, and to classify all "k-forms" G of $\tilde{G}$ over K (i.e. algebraic groups, defined over k, which are K-isomorphic to $\tilde{G}$) such that G contains a K-trivial maximal torus T, defined over k, containing a maximal k-trivial torus A. When $\tilde{G}$ is corresponding to $(X, \mathfrak{r})$, we say that such a pair (G, T) belongs to $K/k, X, \mathfrak{r}$. Now, by Theorem 3, our problem is reduced to the following two problems:

PROBLEM 1. *For given $K/k, X^0, \mathfrak{r}_0$, classify all k-quasicompact pairs (G_0, T_0) belonging to them up to k-isomorphisms.*

PROBLEM 2. *For given $K/k, X, \mathfrak{r}, \mathfrak{r}_0 = \mathfrak{r} \cap X_0$ and a k-isomorphism class of k-quasicompact pair (G_0, T_0) belonging to $K/k, X^0, \mathfrak{r}_0$, (X^0 denoting the orthogonal projection of X on $(\mathfrak{r}_0)_{\mathbf{Q}}$), classify all 'admissible' systems $(X, \Delta, \Delta_0, \{\bar{\varphi}_\sigma\})$ (i.e. systems associated actually with a pair (G, T) belonging to $K/k, X, \mathfrak{r}$ and such that $G_0 = G(\mathfrak{r}_0)$), up to congruences (compatible with k-isomorphisms of G_0).*

Moreover, it is easy to see that Problem 2 can be reduced further to the 'absolutely irreducible' case of 'rank 1' (i.e. the case where Δ is connected and $\{\bar{\varphi}_\sigma\}$ operates transitively on $\Delta - \Delta_0$). In this way, we can actually achieve a complete classification over a finite field and over the real number field[4)5)].

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4) Cf. R. Steinberg, Variations on a theme of Chevalley, Pacific J. Math., **9** (1959), 875-891, D. Hertzig, Forms of algebraic groups, Proc. of Amer. Math. Soc., **12** (1961), 657-660, and S. Araki, On root systems and an infinitesimal classification of irreducible symmetric domains, J. of Math. Osaka City Univ., **13** (1963), 1-34. On the other hand, over a field of characteristic 0 and for classical groups (except D_4), Problem 2 can be solved by a result of A. Weil, (J. of Indian Math. Soc., **24** (1960), 589-623), and for exceptional groups of type G_2, F_4 by a result of H. Hijikata (J. Math. Soc. Japan, **15** (1963), 159-164).

5) The author was informed, after completion of this paper, that similar results as in §4 had been announced, without proof, by Tits [13], [14].

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