Pseudo-uniform reducibility

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1. Introduction

In [1] we showed:

THEOREM A. If every recursive set is representable in a theory (T) then (T) is undecidable.

THEOREM B. If every recursive set is definable in (T) and if (T) is consistent, then the set $T_0$ of Gödel numbers of the provable sentences of (T) is recursively inseparable from the set $R_0$ of Gödel numbers of the refutable sentences of (T).

The above propositions combine notions of recursive function theory with those of mathematical logic—i.e. with the concept of a “first order theory”. In this note we obtain generalizations of these propositions which are purely recursive function theoretic in nature. We also show that the conclusions of Theorems A and B hold under still weaker hypotheses.

2. Pseudo-uniform reducibility

The word “number” shall mean natural number. We use “A”, “B”, “α”, “β” for sets of natural numbers. A set $A$ is (many-one) reducible to $\alpha$ if there is a recursive function $g(\varphi)$ (called a (many-one) reduction of $A$ to $\alpha$) such that $A = g^{-1}(\alpha)$—i.e. for each number $i$, $i \in A \iff g(i) \in \alpha$. Consider now a collection $\Sigma$ of recursively enumerable sets. The collection $\Sigma$ is uniformly reducible to $\alpha$ (as defined in [2]) if there is a recursive function $g(x,y)$ (called a uniform reduction of $\Sigma$ to $\alpha$) such that for every $i$ for which $\omega_i \in \Sigma$, the function $g(i,y)$ (as a function of the one variable $y$) is a reduction of $\omega_i$ to $\Sigma$.

Thus, if $\Sigma$ is uniformly reducible to $\alpha$, then not only is every element of $\Sigma$ reducible to $\alpha$, but given any such element $\omega_i$ (in the sense of given its index $i$) we can effectively find a reduction of it to $\alpha$.

It is trivial to verify that if some non-recursive set is reducible to $\alpha$,

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2) By $\omega_i$, we mean the set of all numbers $x$ satisfying the condition $(\exists y) T_1(i, x, y)$ where $T_1(x, x, y)$ is the predicate of Kleene's enumeration theorem [3]
then $\alpha$ is non-recursive. Hence, it follows that if every recursively enumerable set is reducible to $\alpha$, then $\alpha$ is non-recursive (since there exists a recursively enumerable set which is not recursive). This fact is well known. Suppose that every recursive set is reducible to $\alpha$; does it follow that $\alpha$ is non-recursive? Clearly not, for if $\alpha$ is any non-empty set whose complement is also non-empty, then every recursive set $A$ is reducible to $\alpha$ (just take an element $a_1$ of $\alpha$ and an element $a_2$ of $\bar{\alpha}$ and define $g(x) = a_1$ if $x \in A$; $g(x) = a_2$ if $x \notin A$). Since $A$ is recursive, $g(x)$ is a recursive function, and clearly a reduction of $A$ to $\alpha$. Suppose that the collection of all recursive sets is uniformly reducible to $\alpha$; does it follow that $\alpha$ is non-recursive? In [2], we showed that this hypothesis implies not only that $\alpha$ is non-recursive, but that the complement of $\alpha$ is productive. Thus, to establish the non-recursivity of a set $\alpha$, the hypothesis that all recursive sets be reducible to $\alpha$ is too weak, and the hypothesis of uniform reducibility is stronger than necessary. We now consider a notion which is of intermediate strength.

We shall say that $\Sigma$ is pseudo-uniformly reducible to $\alpha$ if there is a recursive function $g(x, y)$ (called a pseudo-uniform reduction of $\Sigma$ to $\alpha$) such that for every set $A \in \Sigma$, there is a number $a$ such that $g(a, y)$ (as a function of the one variable $y$) is a reduction of $A$ to $\alpha$. We note that this definition (unlike that of uniform reducibility) does not require that such a number $a$ be an index of the set $A$, nor that there be a recursive function $\varphi(x)$ which assigns to any index of $A$ such a number $a$. If there were such a recursive function $\varphi(x)$, then $\Sigma$ would indeed be uniformly reducible to $\alpha$ under the function $g(\varphi(x), y)$. We shall soon see that a sufficient condition for $\alpha$ to be non-recursive is that the collection of all recursive sets be pseudo-uniformly reducible to $\alpha$. And in light of our next proposition, we feel that this fact constitutes the mathematical essence of [Theorem] A.

The notion of pseudo-uniform reducibility arises naturally in connection with metamathematics in the following way. Suppose we have a theory $(T)$ with standard formalizations (cf. [4]). Let $F_1, F_2, \ldots, F_n, \ldots$ be an effective enumeration of all the formulas with exactly one free variable; let $A_i$ be the numeral designating the natural number $i$; let $g$ be an effective Gödel numbering of all closed sentences; let $T$ be the set of all provable (closed) sentences and $R$ the set of all (closed) sentences whose negation is provable; let $T_0, R_0$ respectively be the set of Gödel numbers of the provable, refutable sentences of $(T)$; let $\varphi(i, j)$ be the Gödel number of $F_i(A_j)$. Under the usual requirements of "effectiveness" of the Gödel numbering and of the sequence $A_0, A_1, A_2, \ldots, A_i$, the function $\varphi(x, y)$ is (general) recursive.

A formula $F(x)$ is said to represent the set of all numbers $n$ for which $F(A_n) \in T$. We pointed out in [2] that if a set $A$ is representable in $(T)$, then
A is (many-one) reducible to $T_\alpha$. We now note the following stronger fact:

PROPOSITION 1. If each element of a collection $\Sigma$ is representable in $(T)$, then the collection $\Sigma$ is pseudo-uniformly reducible to $T_\alpha$.

PROOF. For each element $A$ of $\Sigma$ there is, by hypothesis, a formula $F_i(x)$ which represents $A$ in $(T)$. Then for every number $i,j \in A \leftrightarrow F_i(A) \in T \leftrightarrow \varphi(i,j) \in T_\alpha$. Thus $\varphi(x,y)$ is a pseudo-uniform reduction of $\Sigma$ to $T_\alpha$.

We now show

THEOREM 1. If the collection of all recursive sets is pseudo-uniformly reducible to $\alpha$, then $\alpha$ is not recursive.

We actually show Theorem 1 in the following stronger form.

THEOREM 1'. Each of the following conditions implies the next.

(a) The collection of recursive sets is pseudo-uniformly reducible to $\alpha$.

(b) There is a recursive function $g(x)$ such that for every recursive set $A$, there is a number $i$ such that $i \in A \rightarrow g(i) \in \alpha$.

(c) $\alpha$ is not recursive.

PROOF. Suppose (a); let $f(x,y)$ be such a uniform reduction. Define $g(x) = f(x,x)$. Then $g(x)$ is recursive. Let $A$ be any recursive set. By hypothesis there is a number $i$ such that for every number $y, i \in A \rightarrow f(i,y) \in \alpha$. Setting $y = i, i \in A \rightarrow f(i,i) \in \alpha \cdot i \in A \rightarrow g(i) \in \alpha$. Thus (a)$\Rightarrow$(b).

Suppose (b). We must show that $\alpha$ is not recursive. Suppose it were. Then $\bar{\alpha}$ would be recursive. Then $g^{-1}(\bar{\alpha})$ is recursive $[g^{-1}(\bar{\alpha}) = \{\varphi \rightarrow \varphi(i) \in \alpha\}$ the set of all $i$ such that $g(i) \in \bar{\alpha}$]. Then there is a number $i$ such that $i \in g^{-1}(\bar{\alpha}) \rightarrow g(i) \in \alpha$. But $i \in g^{-1}(\bar{\alpha}) \rightarrow g(i) \in \bar{\alpha}$. Hence $g(i) \in \bar{\alpha} \rightarrow g(i) \in \alpha$, which is impossible.

In view of Proposition 1, Theorem 1 is indeed a generalization of Theorem A.

We also note that the statement (b)$\Rightarrow$(c) of Theorem 1' is a stronger statement than Theorem 1, and implies the following stronger form of Theorem A (by setting $g(i) = \varphi(i,i)$).

THEOREM A'. If for every recursive set $A$, there is a number $i$ such that $i \in A \rightarrow f(A) \in T$, then $T_\alpha$ is non-recursive.

The hypothesis of Theorem A' is obviously weaker than that of Theorem A, for the latter says that for any recursive set $A$ there is a number $i$ such that for every $j$ (whether equal to $i$ or not), $j \in A \rightarrow f(A) \in T$.

3. Pseudo-uniform reducibility of ordered pairs

Let $A, B, \alpha, \beta$ be number sets. A recursive function $f(x)$ is a (many-one) reduction of the ordered pair $(A,B)$ to the ordered pair $(\alpha, \beta)$ (as defined in [2]) if $f(x)$ is simultaneously a reduction of $A$ to $\alpha$ and of $B$ to $\beta$.—i.e. for every number $i$: (1) $i \in A \rightarrow f(i) \in \alpha$; (2) $i \in B \rightarrow f(i) \in \beta$.

Consider now a collection $\Sigma$ of ordered pairs of number sets. We shall
say that \( \Sigma \) is pseudo-uniformly reducible to a pair \((\alpha, \beta)\) if there is a recursive function \( f(x, y) \) (which we will call a pseudo-uniform reduction of \( \Sigma \) to \((\alpha, \beta)\)) such that for every pair \((A, B)\) in \( \Sigma \), there is a number \( i \) such that \( f(i, y) \) (as a function of the one variable \( y \)) is a reduction of \((A, B)\) to \((\alpha, \beta)\).

The obvious analogue of [Proposition 1] is

**Proposition 2.** Let \( S \) be a collection of sets and let \( \Sigma \) be the collection of all ordered pairs \((A, \tilde{A})\) such that \( A \in S \). Then if every element of \( S \) is definable in \((T)\), and if \((T)\) is consistent, then \( \Sigma \) is pseudo-uniformly reducible to the pair \((T_0, R_0)\).

**Proof.** As in the proof of Proposition 1, let \( \varphi(i, j) \) be the Gödel number of \( F(\Delta_j) \). Let \( A \in S \). Then for some number \( i, F(x) \) defines \( A \) in \((T)\). Thus for all \( j, j \in A \Rightarrow F(\Delta_j) \in T \) and \( j \in \tilde{A} \Rightarrow F(\Delta_j) \in R \). Since \((T)\) is consistent, then \( j \in A \Rightarrow F(\Delta_j) \in T \) and \( j \in \tilde{A} \Rightarrow F(\Delta_j) \in R \). [For \( F(\Delta_j) \in T \Rightarrow F(\Delta_j) \in R \Rightarrow j \in \tilde{A} \Rightarrow j \in A \). Similarly \( F(\Delta_j) \in R \Rightarrow j \in \tilde{A} \Rightarrow j \in A \).] Thus \( f(i, y) \) is a reduction of \((A, \tilde{A})\) to \((T_0, R_0)\).

We now show

**Theorem 2.** Let \( \Sigma_R \) be the collection of all complementary pairs of recursive sets and let \( \alpha, \beta \) be disjoint. Then if \( \Sigma_R \) is pseudo-uniformly reducible to \((\alpha, \beta)\), then \((\alpha, \beta)\) is recursively inseparable.

We in fact shall show the stronger fact:

**Theorem 2'.** Each of the following conditions implies the next:

(a) \( \Sigma_R \) is pseudo-uniformly reducible to \((\alpha, \beta)\) \([\alpha, \beta \text{ are disjoint}].

(b) There is a recursive function \( g(x) \) such that for each pair \((A, \tilde{A}) \in \Sigma \), there is a number \( i \) such that \( i \in A \leftrightarrow g(i) \in \alpha \) and \( i \in \tilde{A} \leftrightarrow g(i) \in \beta \).

(c) The pair \((g^{-1}(\alpha), g^{-1}(\beta))\) is recursively inseparable.

(d) The pair \((\alpha, \beta)\) is recursively inseparable—in fact, the subset \( gg^{-1}\alpha \) of \( \alpha \) is recursively inseparable from the subset \( gg^{-1}\beta \) of \( \beta \).

**Proof.** (1) (a) \( \Rightarrow \) (b). Let \( f(x, y) \) be a pseudo-uniform reduction of \( \Sigma_R \) to \((\alpha, \beta)\). As in the proof of Theorem 1', let \( g(x) \) be the recursive function \( f(x, x) \). Let \((A, \tilde{A}) \in \Sigma \) and let \( i \) be such that \( f(i, y) \) is a reduction of \((A, \tilde{A})\) to \((\alpha, \beta)\). Since \( f(i, y) \) is a reduction of \( A \) to \( \alpha \), then (by the argument in the proof of Theorem 1') \( i \in A \Leftrightarrow g(i) \in \alpha \). Similarly, since \( f(i, y) \) is a reduction of \( \tilde{A} \) to \( \beta \), then \( i \in \tilde{A} \Leftrightarrow g(i) \in \beta \).

(2) (b) \( \Rightarrow \) (c). Suppose \( g(x) \) is as in (b). Suppose \((g^{-1}(\alpha), g^{-1}(\beta))\) were

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3) Again, this notion is midway in strength between the notions: (1) every element of \( \Sigma \) is reducible to \((\alpha, \beta)\); (2) \( \Sigma \) is uniformly reducible to \((\alpha, \beta)\), as defined in [2]. The latter says that given indices \( i, j \) of \( A, B \) where \((A, B) \in \Sigma \), we can effectively find a number \( i \) such that \( f(i, y) \) is a reduction of \((A, B)\) to \((\alpha, \beta)\).

4) A pair is called recursively inseparable if there is no recursive superset of one disjoint from the other.
Pseudo-uniform reducibility

133

to recursively separable. Then there is a recursive superset $A$ of $g^{-i}(\beta)$ disjoint from $g^{-i}(\alpha)$. Hence, $g^{-i}(\beta) \subseteq A$; $g^{-i}(\alpha) \subseteq \tilde{A}$. By the hypothesis of (b), there is an $i$ such that $i \in A \rightarrow g(i) \in \alpha$ and $i \in \tilde{A} \rightarrow g(i) \in \beta$. Hence, $i \in A \Rightarrow g(i) \in \alpha \Leftrightarrow i \in g^{-i}(\alpha) \Rightarrow i \in \tilde{A}$, and $i \in \tilde{A} \Rightarrow g(i) \in \beta \Rightarrow i \in g^{-i}(\beta) \Rightarrow i \in A$.

Thus $i \in A \lhd i \in \tilde{A}$, which is impossible. Hence $g^{-i}(\alpha), g^{-i}(\beta)$ are recursively inseparable.

(3) (c) $\Rightarrow$ (d). We have shown in [2] (p. 62, Proposition 4, Ch. II) that if $(A_1, A_2)$ is recursively inseparable and if $(A_1, A_2)$ is reducible to $(B_1, B_2)$ (or even if there is a recursive function which maps $A_1$ into $B_1$ and $A_2$ into $B_2$) then $(B_1, B_2)$ is in turn recursively inseparable. But clearly $g$ maps $g^{-i}(\alpha)$ into $g g^{-i} \alpha$ and $g^{-i}(\beta)$ into $g g^{-i} \beta$.

Theorem 2 and Proposition 2 clearly imply Theorem B. But again, the statement (b) $\Rightarrow$ (d) of Theorem 2' is stronger than Theorem 2, and implies the following stronger form of Theorem B.

Theorem B'. A sufficient condition for the nuclei $(T_0, R_0)$ of a consistent theory $(T)$ to be recursively inseparable is that for every recursive set $A$ there exists a number $i$ such that $i \in A \rightarrow F_i(A) \in T$ and $i \in \tilde{A} \rightarrow F_i(A) \in R$.

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References