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On irreducibility of an analytic set

By Kenkiti KASAHARA

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§1. Let D be a domain in the n-dimensional complex Euclidean space $\boldsymbol{C}^n$, and M be a k-dimensinal analytic set[1] in D $(1 \leqq k \leqq n-1)$. It is well-known that the set of all irreducible points of M is not always an open subset of M. For example, the analytic set $\{z_1^2 - z_2^2 z_3 = 0\}$ in $\boldsymbol{C}^3$ is irreducible at the origin, but there exist reducible points of the analytic set converging to the origin (Osgood [2]). We shall say that a point p is *a singular irreducible point of M*, if M is irreducible at p and there exist reducible points of M converging to p. *Let S be the set of all singular irreducible points of M.* Recently S. Hitotumatu [1] has shown that S must be empty if M is an analytic set of 1-dimension in $\boldsymbol{C}^2$. In this note, we show the following:

THEOREM. *The closure $\bar{S}$ of S in D is an analytic set in D. For each point $p \in \bar{S}$, a relation* $\dim_p \bar{S} \leqq \dim_p M - 2$ *holds.*

REMARK. For the set S itself, Theorem is not true. For example, the analytic set $\{z_4(z_1^2 - z_2^2 z_3) = 0\}$ in $\boldsymbol{C}^4$ has the set $\{z_1 = z_2 = z_3 = 0,\ z_4 \neq 0\}$ as S. For another example, the analytic set $\{z_4^4 - 2z_3^2 z_4^2 + z_3^4(1 - z_1^2 z_2) = 0\}$ in $\boldsymbol{C}^4$ is irreducible in $\boldsymbol{C}^4$. Outside the set $\{z_2 = 0\} \cup \{z_3 = 0\} \cup \{1 - z_1^2 z_2 = 0\}$, the analytic set is decomposed into the following four sets:

$$\{z_4 = z_3\sqrt{1 + z_1\sqrt{z_2}}\}, \qquad \{z_4 = -z_3\sqrt{1 + z_1\sqrt{z_2}}\},$$

$$\{z_4 = z_3\sqrt{1 - z_1\sqrt{z_2}}\} \quad \text{and} \quad \{z_4 = -z_3\sqrt{1 - z_1\sqrt{z_2}}\}.$$

We have easily

$$S = \{z_1 = z_2 = 0, z_3 = z_4\} \cup \{z_1 = z_2 = 0, z_3 = -z_4\} - \{(0, 0, 0, 0)\}.$$

But we can generally show that the set S itself has an analytic property, that is, S is locally the finite union of locally analytic sets. (cf. §4.)

First applying the Remmert-Stein's 'Einbettungssatz' ([3]) and the method of Osgood [2, Chap. II, §15], we shall define the number of components of M at a point $p \in M$. (cf. §2). In §3, we shall derive a property of roots of a polynomial. In §4, we shall consider Theorem for the case that M is

1) About the definition and related notions of an analytic set, see Remmert-Stein [4].

purely dimensional, and in §5 we shall conclude the proof of Theorem.

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§**2.** If the set S is empty, Theorem is trivial. We assume that S is not empty. In this section and §4, we assume that the analytic set M is purely k-dimensional in D. An ordinary point of M can not belong to $\bar{S}$. Let p be a singular point of M and U be an arbitrary neighborhood of p contained in D. By the Remmert-Stein's 'Einbettungssatz', after suitable non-singular analytic transformations of coordinates, the analytic set M has the following type of local representations in a neighborhood of p. We may assume the point p to be the origin of $\boldsymbol{C}^n$. We denote by $w_1, \cdots, w_{n-k}, z_1, \cdots, z_k$ the coordinates in a neighborhood of p. There exist neighborhoods W and Z_ν of the origin in the spaces $\boldsymbol{C}^{n-k}(w_1, \cdots, w_{n-k})$ and $\boldsymbol{C}^1(z_\nu)$ respectively $(\nu=1, 2, \cdots, k)$ satisfying the following:

$W \times Z_1 \times \cdots \times Z_k$ is contained in U. There exist distinguished polynomials[2)] $P_\alpha(w_\alpha; z_1, \cdots, z_k)$ in w_α of degree q_α $(\alpha=1, 2, \cdots, n-k)$ with coefficients holomorphic in $Z_1 \times \cdots \times Z_k$. For each α, P_α has no multiple factors and every system $(w_1, \cdots, w_{n-k})$ of the solutions of $P_\alpha(w_\alpha; z_1, \cdots, z_k)=0$ for any point $(z_1, \cdots, z_k) \in Z_1 \times \cdots \times Z_k$ is surely a point in W.

The discriminant ω_α of P_α is not identically zero for each α. There exists a distinguished polynomial $\Delta_1(z_1; z_2, \cdots, z_k)$ in z_1 with coefficients holomorphic in $Z_2 \times \cdots \times Z_k$ such that

$$\{(z_1, \cdots, z_k) \in Z_1 \times \cdots \times Z_k \mid \prod_{\alpha=1}^{n-k} \omega_\alpha(z_1, \cdots, z_k)=0\}$$
$$=\{(z_1, \cdots, z_k) \in Z_1 \times \cdots \times Z_k \mid \Delta_1(z_1; z_2, \cdots, z_k)=0\}.$$

For the sake of brevity, we put often $(w_1, \cdots, w_{n-k})=w$, $z_1=v$, $(z_2, \cdots, z_k)=z$, $Z_1=V$ and $Z_2 \times \cdots \times Z_k=Z$. We may assume $\Delta_1(v; z)$ has no multiple factors and every solution of $\Delta_1(v; z)=0$ belongs to V for any $z \in Z$.

Let $\delta_1(z)$ be the discriminant of $\Delta_1(v; z)$. $\delta_1(z)$ is holomorphic in Z and not identically zero.

The set $M'=\{(w, v, z) \in W \times V \times Z \mid P_\alpha(w_\alpha; v, z)=0, (\alpha=1, 2, \cdots, n-k)\}$ is an analytic set in $W \times V \times Z$ having two properties as follows:

i) The set $M \cap (W \times V \times Z)$ is the union of some irreducible components of M' in $W \times V \times Z$. Each irreducible components of M' in $W \times V \times Z$ is the closure of a connected component of the set $M' \cap \{\prod_{\alpha=1}^{n-k} \omega_\alpha(v, z) \neq 0\}$, and conversely. Moreover each irreducible component of M' in $W \times V \times Z$ is irreducible at the origin.

2) In this note, a distinguished polynomial has generally its center at the origin.

ii) Over any point (v^0, z^0)[3] in $V\times Z$, there exists at least one point (w^0, v^0, z^0) of M. If $\prod_{\alpha=1}^{n-k}\omega_\alpha(v^0, z^0)\neq 0$, M has the same germ as M' at each point (w^0, v^0, z^0) of M over (v^0, z^0).

A point (w^0, v^0, z^0) is often called a point w^0 over (v^0, z^0). Since each point of M over (v^0, z^0) satisfying $\prod_{\alpha=1}^{n-k}\omega_\alpha(v^0, z^0)\neq 0$ is an ordinary point of M, the set $\{\prod_{\alpha=1}^{n-k}\omega_\alpha=0\}$ contains the origin and we can construct Δ_1 as above.

Take a point (w^0, v^0, z^0) in M satisfying $\Delta_1(v^0\,; z^0)=0$. Let V^0 and $\tilde{V}^0$ be two bounded simply-connected neighborhoods of v^0, and W^0, Z^0 be those of w^0, z^0. We shall say that a collection $(W^0, V^0, \tilde{V}^0, Z^0)$ is *a distinguished system of neighborhoods of* (w^0, v^0, z^0) *for* M if it satisfies the following four conditions:

(1) $W^0\subset W,\ \tilde{V}^0\Subset$[4]$V^0\subset V,\ Z^0\subset Z$,

(2) $M'\cap(W^0\times\{v^0\}\times\{z^0\})=\{(w^0, v^0, z^0)\}$,

(3) $M'\cap(\partial W^0\times V^0\times Z^0)=\phi$, ($\partial W^0$ means the boundary of W^0.)

(4) $\{v\in V^0\mid \Delta_1(v\,; z^0)=0\}=\{v^0\}$ and $\{(v, z)\in(V^0-\tilde{V}^0)\times Z^0\mid \Delta_1(v\,; z)=0\}=\phi$.

First we remark that for any given neighborhood W' of w^0 we can construct a distinguished system of neighborhoods $(W^0, V^0, \tilde{V}^0, Z^0)$ such that $W^0\subset W'$.

Let $(W^0, V^0, \tilde{V}^0, Z^0)$ be a distinguished system of neighborhoods of (w^0, v^0, z^0) for M. By the condition (3), over any $(v, z)\in V^0\times Z^0$ we can find at least one point of M in W^0. Let b be a point in $V^0-\bar{\tilde{V}}^0$ and B be a simply-connected neighborhood of b contained in $V^0-\bar{\tilde{V}}^0$. Let z^1 be a point in Z^0 satisfying $\delta_1(z^1)\neq 0$ and $v_1^1, \cdots, v_t^1$ be roots of the equation $\Delta_1(v, z^1)=0$ in V^0. We take simply-connected neighborhoods V_λ^1 and Z^1 of v_λ^1 and z^1 $(\lambda=1, 2, \cdots, t)$ as follows:

(a) $Z^1\subset Z^0\cap\{\delta_1\neq 0\},\ V_\lambda^1\Subset\tilde{V}^0\ (\lambda=1, 2, \cdots, t)$,

(b) $\bar{V}_\lambda^1\cap\bar{V}_\mu^1=\phi$ for any $\lambda\neq\mu\ (\lambda, \mu=1, 2, \cdots, t)$,

(c) $\Delta_1(v\,; z)\neq 0$ for any $(v, z)\in(V^0-\bigcup_{\lambda=1}^{t}V_\lambda^1)\times Z^1$.

Let $w^1, \cdots, w^l$ be points of M in W^0 over $(v, z)\in B\times Z^0$. We denote w^μ by $w^\mu(v, z)$ or $(w_1^\mu(v, z), \cdots, w_{n-k}^\mu(v, z))$ $(\mu=1, 2, \cdots, l)$. Since for any $(v, z)\in B\times Z^0$ the equation $P_\alpha(w_\alpha\,; v, z)=0$ has distinct q_α roots which are one-valued holomorphic functions in $B\times Z^0$, the branch $w_\alpha^\mu(v, z)$ is so $(\mu=1, 2, \cdots, l;\ \alpha=1, 2, \cdots, n-k)$. Each $w_\alpha^\mu(v, z)$ can be analytically continued along any curve in $(V^0-\bigcup_{\lambda=1}^{t}\bar{V}_\lambda^1)\times Z^1$. We may assume that $w^1, \cdots, w^{l_1}$ are all of the simultaneous continuations of w^1 along some curves in $(V^0-\bigcup_{\lambda=1}^{t}\bar{V}_\lambda^1)\times Z^1$ and $w^{l_1+\cdots+l_{\nu-1}+1}, \cdots, w^{l_1+\cdots+l_{\nu-1}+l_\nu}$ are those of $w^{l_1+\cdots+l_{\nu-1}+1}$ $(\nu=1, 2, \cdots, m;\ l_1+l_2+\cdots+l_m$

3) In this note, x^y does not mean the y-th power of x unless otherwise stated.

4) $A\Subset B$ means that the closure of A is compact and is contained in B.

$=l$). We shall call m *the number of components of* M *at* (w^0, v^0, z^0). It is trivial that this number m does not depend upon a particular choice of the neighborhoods V_λ^1 and Z^1 $(\lambda=1, 2, \cdots, t)$. Under these assumptions and notations we have

LEMMA 1. *The number of components of* M *at* (w^0, v^0, z^0) *coincides with the number of irreducible components of* M *at* (w^0, v^0, z^0). *As the result of this fact, it is determined only by* M *and* (w^0, v^0, z^0), *and does not depend upon a particular choice of a point* z^1 *and a distinguished system of neighborhoods* $(W^0, V^0, \tilde{V}^0, Z^0)$ *of* (w^0, v^0, z^0) *for* M.

PROOF. It is sufficient to show that one of the systems, for example $w^1, \cdots, w^{l_1}$, makes an irreducible component of M at (w^0, v^0, z^0), and another system, for example $w^{l_1+1}, \cdots, w^{l_1+l_2}$, makes distinct one.

Let l_ν^α be the number of distinct branches among $w_\alpha^{l_1+\cdots+l_{\nu-1}+1}, \cdots, w_\alpha^{l_1+\cdots+l_{\nu-1}+l_\nu}$ and we make elementary symmetric functions of such l_ν^α ones $(\nu=1, 2, \cdots, m;$ $\alpha=1, 2, \cdots, n-k)$. We denote by $\Phi(v, z)$ one of them. $\Phi(v, z)$ is one-valued and holomorphic in $\{(V^0-\bigcup_{\lambda=1}^{t} \bar{V}_\lambda^1)\times Z^1\} \cup \{B\times Z^0\}$. Let c be an arbitrary closed curve passing through (b, z^1) contained in $\{V^0\times Z^0\} \cap \{\Delta_1\neq 0\}$. We continue simultaneously w^1 over (b, z^1) along c. When we come back to the point (b, z^1) again, such a continuation of w^1 must be contained in $\{w^1, \cdots, w^{l_1}\}$. We show first this fact. We may assume $\delta_1\neq 0$ on c. Let (v', z') be an arbitrary point on c and $v'_1, \cdots, v'_{t'}$ be roots of the equation $\Delta_1(v; z')=0$ in V^0. Since $\delta_1\neq 0$ on c, we have $t'=t$. Take simply-connected neighborhoods V'_λ and Z' of v'_λ and z' $(\lambda=1, 2, \cdots, t)$ satisfying the similar conditions (a), (b), (c) as V_λ^1 and Z^1. We may assume $\bigcup_{\lambda=1}^{t} \bar{V}'_\lambda \not\ni v'$. The point v' can be joined to the point b by a curve c' contained in $V^0-\bigcup_{\lambda=1}^{t} \bar{V}'_\lambda$. We continue simultaneously w^1 along c from (b, z^1) to (v', z'), along c' from (v', z') to (b, z') when z is in Z' and next along any closed curve in $(V^0-\bigcup_{\lambda=1}^{t} \bar{V}'_\lambda)\times Z'$. The set of all w^μ over (b, z') obtained by such continuations is locally invariant when (v', z') moves on c. So it is also $\{w^1, \cdots, w^{l_1}\}$. From this fact $\Phi(v, z)$ becomes holomorphic and one-valued in $\{V^0\times Z^0\} \cap \{\Delta_1\neq 0\}$. By the removable singularity theorem of Riemann, $\Phi(v, z)$ is a holomorphic and one-valued function in $V^0\times Z^0$.

Now, we have an irreducible polynomial $Q_\alpha^\nu(w_\alpha; v, z)$ in w_α of degree l_ν^α with coefficients holomorphic in $V^0\times Z^0$ such that the roots of the equation $Q_\alpha^\nu(w_\alpha; v, z)=0$ are precisely those l_ν^α distinct branches among ${w_\alpha}^{l_1+\cdots+l_{\nu-1}+1}, \cdots,$ ${w_\alpha}^{l_1+\cdots+l_{\nu-1}+l_\nu}$. By the Remmert-Stein's 'Einbettungssatz', the closure of the set which we obtain by the simultaneous continuations of w^1 along any curve contained in $\{V^0\times Z^0\} \cap \{\Delta_1\neq 0\}$ is an irreducible components of M at (w^0, v^0, z^0). Since w^{l_1+1} is not the simultaneous continuation of w^1 in

$\{V^0 \times Z^0\} \cap \{\Delta_1 \neq 0\}$, the irreducible component of M at (w^0, v^0, z^0) containing w^1 is different to that containing w^{l_1+1}. We conclude the proof.

We put $Q_\alpha(w_\alpha; v, z) = \prod_{\nu=1}^{m} Q_\alpha^\nu(w_\alpha; v, z)$ and call it *the α-th polynomial attached to M at (w^0, v^0, z^0) $(\alpha = 1, 2, \cdots, n-k)$*. It is a distinguished polynomial in w_α of degree $l_1^\alpha + \cdots + l_m^\alpha$ having its center at (w^0, v^0, z^0).

By Lemma 1, we have

LEMMA 2. *Let $(W^0, V^0, \tilde{V}^0, Z^0)$ be a distinguished system of neighborhoods of (w^0, v^0, z^0) for M and (v^1, z^1) be a point in $\tilde{V}^0 \times Z^0$. Suppose that the equation $\Delta_1(v; z^1) = 0$ has one and only one root v^1 in V^0 and over (v^1, z^1) there is one and only one point w^1 of M' in W^0. Then the number of components of M at (w^0, v^0, z^0) is equal to that at (w^1, v^1, z^1).*

PROOF. First we remark $(w^1, v^1, z^1) \in M$. We can construct a distinguished system of neighborhoods $(W^1, V^1, \tilde{V}^1, Z^1)$ of (w^1, v^1, z^1) for M such that $W^1 \subset W^0$, $V^1 = V^0$, $\tilde{V}^1 \subset \tilde{V}^0$ and $Z^1 \subset Z^0$. By Lemma 1 and the definition of the number of components, we can easily arrive at the conclusion.

§3. Let Δ_0 be a distinguished polynomial in z_0 of degree d with coefficients holomorphic in a neighborhood V of the origin in $\boldsymbol{C}^n$ $(d > 1, n \geqq 1)$. Suppose that Δ_0 has no multiple factors. Taking suitable coordinates $z_1, \cdots, z_n$ in a neighborhood of the origin and a sufficiently small neighborhood Z_ν of the origin in the z_ν-plane $(\nu = 1, 2, \cdots, n)$, by the Weierstrass' preparation theorem we can easily show the existence of distinguished polynomials $\Delta_\mu(z_\mu; z_{\mu+1}, \cdots, z_n)$ $(\mu = 1, 2, \cdots, r;\ 1 \leqq r \leqq n)$ satisfying the following:

1) $Z_1 \times \cdots \times Z_n \subset V$.

2) Each Δ_μ is a distinguished polynomial in z_μ whose coefficients are holomorphic functions of $z_{\mu+1}, \cdots, z_n$ in $Z_{\mu+1} \times \cdots \times Z_n$. Δ_μ has no multiple factors and every solution of the equation $\Delta_\mu(z_\mu; z_{\mu+1}, \cdots, z_n) = 0$ belongs to Z_μ for any $(z_{\mu+1}, \cdots, z_n) \in Z_{\mu+1} \times \cdots \times Z_n$ $(\mu = 1, 2, \cdots, r)$.

3) We denote by δ_μ the discriminant of Δ_μ. Then the set $\{\delta_\mu = 0\}$ contains the origin of $Z_{\mu+1} \times \cdots \times Z_n$ and is contained in the set $\{\Delta_{\mu+1} = 0\}$ $(\mu = 0, 1, \cdots, r-1)$. The analytic set $\{\Delta_r = 0\}$ in $Z_r \times \cdots \times Z_n$ is ordinary at the origin. We may assume $\{\Delta_r = 0\} = \{z_r = 0\}$ in $Z_r \times \cdots \times Z_n$.

Under these assumptions, we have

LEMMA 3. *There exists a neighborhood $'Z_\nu$ of the origin contained in Z_ν $(\nu = r+1, \cdots, n)$ such that for an arbitrary point $(z_{r+1}, \cdots, z_n) \in {}'Z_{r+1} \times \cdots \times {}'Z_n$ the simultaneous equations*

$$\Delta_\mu(z_\mu; z_{\mu+1}, \cdots, z_{r-1}, 0, z_{r+1}, \cdots, z_n) = 0 \qquad (\mu = 0, 1, \cdots, r-1)$$

have one and only one system of solutions $z_\mu(z_{r+1}, \cdots, z_n)$ $(\mu = 0, 1, \cdots, r-1)$. And each $z_\mu(z_{r+1}, \cdots, z_n)$ is a holomorphic function in $'Z_{r+1} \times \cdots \times {}'Z_n$.

PROOF. First we consider the case $r = 1$. Δ_0 is uniquely decomposed into the product $\prod_{\nu=1}^{a} \Delta_0^\nu$ of irreducible polynomials Δ_0^ν. Let d_ν be the degree of Δ_0^ν, and d_0 be the least common multiple of $d_1, \cdots, d_a$. We put $z_0 = w$, $z_1 = v$, $(z_2, \cdots, z_n) = z$ and $v = t^{d_0}$ (here t^{d_0} means the d_0-th power of t). Let $(W^0, V^0, \tilde{V}^0, Z^0)$ be a distinguished system of neighborhoods of the origin $(0, 0, 0)$ for the analytic set $\{\Delta_0 = 0\}$ such that $V^0 = \{|v| < \varepsilon\}$, $\tilde{V}^0 = \{|v| < \tilde{\varepsilon}\}$ and $V^0 \times Z^0 \subset Z_1 \times \cdots \times Z_n$. By the assumptions $d > 1$ and $r = 1$, we have $\{\delta_0 = 0\} = \{v = 0\}$. We put $T = \{|t| < \sqrt[d_0]{\varepsilon}\}$, $\tilde{T} = \{|t| < \sqrt[d_0]{\tilde{\varepsilon}}\}$ and $\Delta_0^*(w; t, z) = \Delta_0(w; t^{d_0}, z)$. Denoting by δ_0^* the discriminant of Δ_0^*, we have $\{\delta_0^* = 0\} = \{t = 0\}$ in $T \times Z^0$. We put it A. The number of components of the analytic set $\{\Delta_0^* = 0\}$ at the origin is equal to the degree d of Δ_0^*. So we have $\Delta_0^* = \prod_{\nu=1}^{d} (w - w_\nu(t, z))$ where w_ν is holomorphic and one-valued in $T \times Z^0$. Denoting by $A_{\mu\nu}$ the set $\{(t, z) \in T \times Z^0 \mid w_\mu(t, z) = w_\nu(t, z)\}$ for any $\mu \neq \nu$ $(\mu, \nu = 1, 2, \cdots, d)$. Since $A_{\mu\nu}$ is not empty, it is a purely 1-codimensional analytic set in $T \times Z^0$. As A is an irreducible 1-codimensional analytic set in $T \times Z^0$ and contains $A_{\mu\nu}$, we have $A_{\mu\nu} = A$ $(\mu, \nu = 1, 2, \cdots, d;\ \mu \neq \nu)$. The roots of $\Delta_0(w; 0, z) = 0$ are those of $\Delta_0^*(w; 0, z) = 0$, and they must be $w_1(0, z)$. This concludes the proof in the case $r = 1$.

In the general case, the proof is inductive. If $n = 1$, the lemma is trivial. Let us assume the lemma true for $n - 1$.

We put $\Delta_\mu^*(z_\mu; z_{\mu+1}, \cdots, z_{r-1}, z_{r+1}, \cdots, z_n) = \Delta_\mu(z_\mu; z_{\mu+1}, \cdots, z_{r-1}, 0, z_{r+1}, \cdots, z_n)$ $(\mu = 0, 1, \cdots, r-1)$. Δ_μ^* is a distinguished polynomial in z_μ and not identically zero. Since $\{z_r = 0\} \not\subset \{\Delta_{\mu+1} = 0\}$, Δ_μ^* has no multiple factors $(\mu = 0, 1, \cdots, r-2)$. Denoting by δ_μ^* the discriminant of Δ_μ^*, we have $\{\delta_\mu^* = 0\} \subset \{\Delta_{\mu+1}^* = 0\}$. By the lemma of the case $r = 1$, $\Delta_{r-1}(z_{r-1}; 0, z_{r+1}, \cdots, z_n) = 0$ is equivalent to $z_{r-1} = \zeta(z_{r+1}, \cdots, z_n)$ in a neighborhood $''Z_{r+1} \times \cdots \times ''Z_n \subset Z_{r+1} \times \cdots \times Z_n$ where ζ is a holomorphic function in $''Z_{r+1} \times \cdots \times ''Z_n$. By the transformations of coordinates $'z_\nu = z_\nu$ $(\nu = 1, 2, \cdots, n;\ \nu \neq r-1)$ and $'z_{r-1} = z_{-1} - \zeta(z_{r+1}, \cdots, z_n)$, $Z_1 \times \cdots \times Z_r \times ''Z_{r+1} \times \cdots \times ''Z_n$ can be regarded as a neighborhood in $('z_1, \cdots, 'z_n)$-space. From the hypothesis of the induction, there exists a neighborhood $'Z_{r+1} \times \cdots \times 'Z_n$ such that for any $('z_{r+1}, \cdots, 'z_n) \in 'Z_{r+1} \times \cdots \times 'Z_n$ the simultaneous equations $\Delta_\mu^*('z_\mu; 'z_{\mu+1}, \cdots, 'z_{r-2}, 0, 'z_{r+1}, \cdots, 'z_n) = 0$ $(\mu = 0, 1, \cdots, r-2)$ have one and only one system of solutions $'z_\mu = 'z_\mu('z_{r+1}, \cdots, 'z_n)$. $'Z_{r+1} \times \cdots \times 'Z_n$ can be regarded as a neighborhood of the origin in $(z_{r+1}, \cdots, z_n)$-space. This yields the lemma.

§ 4. In this section we use the same assumptions and notations as in § 2. Taking suitable coordinates $w_1, \cdots, w_{n-k}, z_1, \cdots, z_k$ in a neighborhood of the origin and making neighborhoods $W, Z_1, \cdots, Z_k$ small, we may assume that all of the hypothesis in § 2 hold and furthermore the following:

If $\delta_1(0) \neq 0$, we have $\{\Delta_1(v; z) = 0\} = \{v = 0\}$ in $V \times Z$; we put then $r = 1$.

($v=z_1, z=(z_2,\cdots,z_k)$, $V=Z_1$ and $Z=Z_2\times\cdots\times Z_k$). If $\delta_1(0)=0$, there exist distinguished polynomials $\Delta_\mu(z_\mu; z_{\mu+1},\cdots,z_k)$ ($\mu=2,\cdots,r$) satisfying the similar conditions 2), 3) as in § 3.

Let L^λ_ν be connected components of the set

$$L^\lambda=M\cap(W\times V\times Z)\cap\{\Delta_1=0\}\cap\cdots\cap\{\Delta_\lambda=0\}\cap\{\Delta_{\lambda+1}\neq 0\}$$

such that $L^\lambda=\bigcup_{\nu=1}^{t_\lambda}L^\lambda_\nu$ ($\lambda=1,2,\cdots,r$; we put $\Delta_{r+1}\equiv 1$).

LEMMA 4. *If M is irreducible (reducible) at a point in L^λ_ν, then M is also irreducible (reducible) at any point in L^λ_ν.*

PROOF. Let (w^0, v^0, z^0) be a point in L^λ_ν. We can take a distinguished system of neighborhoods $(W^0, V^0, \tilde{V}^0, Z^0)$ of (w^0, v^0, z^0) for M such that $\{z_\mu\in Z^0_\mu\,|\,\Delta_\mu(z_\mu; z^0_{\mu+1},\cdots,z^0_k)=0\}=\{z^0_\mu\}$ and $(\partial Z^0_\mu\times Z^0_{\mu+1}\times\cdots\times Z^0_k)\cap\{\Delta_\mu=0\}=\phi$ for $\mu=1,2,\cdots,\lambda$, where $v^0=z^0_1$, $z^0=(z^0_2,\cdots,z^0_k)$ and $Z^0=Z^0_2\times\cdots\times Z^0_k$. We may assume $\Delta_{\lambda+1}\neq 0$ in $Z^0_{\lambda+1}\times\cdots\times Z^0_k$ and furthermore in $Z^0_\mu\times\cdots\times Z^0_k$ Δ_μ is equivalent to a distinguished polynomial $'\Delta_\mu$ in z_μ having its center at $(z^0_\mu,\cdots,z^0_k)$ ($\mu=1,2,\cdots,\lambda$). Let $Q_\alpha(w_\alpha; v, z)$ be the α-th polynomial attached to M at (w^0, v^0, z^0) ($\alpha=1,2,\cdots,n-k$). Q_α and $'\Delta_\mu$ ($\mu=1,2,\cdots,\lambda$) satisfy all assumptions of Lemma 3. Making the neighborhood $Z^0_{\lambda+1}\times\cdots\times Z^0_k$ small as in Lemma 3, by Lemma 1 and Lemma 2 our assertion is proved.

LEMMA 5. *If $\bar{L}^\lambda_\nu\cap L^{\lambda'}_{\nu'}\neq\phi$, then we have $\lambda\leqq\lambda'$ and $\bar{L}^\lambda_\nu\supset L^{\lambda'}_{\nu'}$.*

PROOF. By the definition of L^λ_ν, the fact $\lambda\leqq\lambda'$ is trivial. If $\lambda=\lambda'$, we have $\nu=\nu'$. Suppose $\lambda<\lambda'$. Let N_σ be irreducible components of an analytic set $N=M\cap\{\Delta_1=0\}\cap\cdots\cap\{\Delta_\lambda=0\}\cap\{\Delta_{\lambda+1}=0\}$ in $W\times V\times Z$ such that $N=\bigcup_{\sigma=1}^{t'}N_\sigma$. By the Remmert-Stein's continuation theorem ([3]), $\bar{L}^\lambda_\nu$ is a purely $k-\lambda$ dimensional analytic set in $W\times V\times Z$ and we have either $\bar{L}^\lambda_\nu\supset N_\sigma$ or $\bar{L}^\lambda_\nu\cap N_\sigma=\phi$ for each σ ($\sigma=1,2,\cdots,t'$). Since $N\supset L^{\lambda'}_{\nu'}$, the relation $L^{\lambda'}_{\nu'}=\bigcup_{\sigma=1}^{t'}(N_\sigma\cap L^{\lambda'}_{\nu'})$ holds. Suppose that $N_\sigma\not\supset L^{\lambda'}_{\nu'}$ for each σ. Then for each σ $\dim L^{\lambda'}_{\nu'}>\dim(N_\sigma\cap L^{\lambda'}_{\nu'})$ at each point of $L^{\lambda'}_{\nu'}$, because $L^{\lambda'}_{\nu'}$ is a connected locally analytic set without singularities in $W\times V\times Z$ by Lemma 3. This is a contradiction. Hence $N_\sigma\supset L^{\lambda'}_{\nu'}$ for some σ. Since $\bar{L}^\lambda_\nu\cap L^{\lambda'}_{\nu'}\neq\phi$, we have $N_\sigma\cap\bar{L}^\lambda_\nu\neq\phi$ and $\bar{L}^\lambda_\nu\supset N_\sigma$ for this σ. This concludes the proof.

Now, we can prove our Theorem when M is purely dimensional. Let p' be an arbitrary point of S in $W\times V\times Z$. Since p' is not an ordinary point of M, there exists one and only one $L^{\lambda'}_{\nu'}$, such that $p'\in L^{\lambda'}_{\nu'}$. By Lemma 4, M is irreducible at each point of $L^{\lambda'}_{\nu'}$. By the definition of the set S, there must exist L^λ_ν such that $\bar{L}^\lambda_\nu\ni p'$ and every point of L^λ_ν is a reducible point of M. By Lemma 5, we have $\bar{L}^\lambda_\nu\supset L^{\lambda'}_{\nu'}$ and $\lambda<\lambda'$. From this fact we have $\lambda'\geqq 2$ and $S\supset L^{\lambda'}_{\nu'}$. Thus we obtain the relation $\bar{S}\cap(W\times V\times Z)=\bigcup_{L^\lambda_\nu\subset S}\bar{L}^\lambda_\nu$. Since $\bar{L}^\lambda_\nu$

is a purely $k-\lambda$ dimensional analytic set in $W\times V\times Z$, our assertion is proved.

We remark that the set S must be empty when M is 1-dimensional.

§5. Suppose M is not purely dimensional. M is decomposed uniquely into the union of purely dimensional analytic set in D. We denote it by $M=M_0\cup M_1\cup\cdots\cup M_k$, where M_ν is either empty or purely ν-dimensional analytic set in D and no irreducible components of M_ν in D is contained in $M_{\nu'}$ for $\nu\neq\nu'$. Let S_ν be the set of all singular irreducible points of M_ν. $\bar{S}_\nu$ is an at most $(\nu-2)$-dimensional analytic set in D. We have easily $\bar{S}\subset\bar{S}_2\cup\cdots\cup\bar{S}_k$.

Take a point p in $\bar{S}_\nu\cap\bar{S}^c$. We take a small neighborhood G of p such that $G\subset D$ and $G\cap\bar{S}=\phi$. Then every point of $S_\nu\cap G$ must be a reducible point of M and must be contained in some $M_{\nu'}$ $(\nu'\neq\nu)$. Hence $\bar{S}_\nu\cap G\subset(\bigcup_{\substack{\nu'=1\\ \nu'\neq\nu}}^{k}M_\nu\cap M_{\nu'})\cap G$. From this fact, we can conclude that any irreducible component of $\bar{S}_\nu$ in D passing through a point $p\in\bar{S}_\nu\cap\bar{S}^c$ must be contained in $\bigcup_{\substack{\nu'=1\\ \nu'\neq\nu}}^{k}(M_\nu\cap M_{\nu'})$.

Let S'_ν be the union of all irreducible components of $\bar{S}_\nu$ in D not contained in $\bigcup_{\substack{\nu'=1\\ \nu'\neq\nu}}^{k}(M_\nu\cap M_{\nu'})$. S'_ν is an at most $(\nu-2)$-dimensional analytic set in D. We have easily $\bigcup_{\nu=2}^{k}S'_\nu\subset\bar{S}$. Take a point p' in $\bar{S}$. Let $\nu_1,\cdots,\nu_t$ be all indices such that $p'\in\bar{S}_{\nu_\rho}$ $(\rho=1,2,\cdots,t)$. We take a point p'' of S in a sufficiently small neighborhood of p'. p'' must belong to $\bigcup_{\rho=1}^{t}S_{\nu_\rho}$. Since M is irreducible at p'', p' must belong to $\bigcup_{\rho=1}^{t}S'_{\nu_\rho}$. We have $\bar{S}=\bigcup_{\nu=2}^{k}S'_\nu$, and this concludes the proof.

Chuo University

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