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Generalization of a theorem of Paley and Wiener

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1. Introduction

In his previous paper [2], the author has proved that a theorem of Planchrel and Polya [3], which contains the theorem of Paley and Wiener in one-dimensional form, can be extended to the case in which distribution is involved.

In this paper, we shall give an extension of Stein's theorem [5] in a completely general form by modifying his method.

Our aim is to show that all distributions whose Fourier transforms vanish outside a given compact symmetric and convex domain in n-space are characterized in a one-dimensional form. It is another generalization of the theorem of Paley and Wiener, through the removal of the imposed condition of boundedness on $f(x)$, which gives an extension of a theorem due to Stein [5]. It is my pleasure to thank Professor G. F. D. Duff for taking the trouble to read over this manuscript.

2. Stein's theorem

Adopting Stein's notations, we shall denote by E_n the euclidean n-space, and by $x=(x_1, \cdots, x_n)$ a generic point in it. E^n will denote the dual euclidean n-space by the inner product

$$x \cdot y=\sum_{i=1}^{n} x_i y_i .$$

The Fourier transform of $f(x) \in L^2(E_n)$ is given by

$$\mathcal{F}[f](y)=1/(2\pi)^{\frac{n}{2}} \int_{E_n} e^{-ix\cdot y} f(x)\, dx .$$

Let Ω be a compact, convex and symmetric domain in E^n, and let $\Omega^*=\{x \in E_n ; |x \cdot y| \leqq 1 \text{ for all } y \in \Omega\}$. By $c(x)$ we mean the characteristic function of Ω^*. Define

$$U_t(f)(x)=f * c_t(x)$$

where $f(x)$ is a locally integrable function and $c_t(x)=t^{-n} c(x/t)$.

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$U_t(f)(x)$ is the means of the $f(x)$ taken with respect to the domain Ω^*. Then Stein's theorem can be stated as follows:

THEOREM. *Let $f(x) \in L^2(E_n)$, $f(x)$ bounded. Then a necessary and sufficient condition that $\mathcal{F}[f](y)$ vanishes outside a compact, convex and symmetric domain Ω, is that $U_t(f)(x)$ is, for each fixed x, an analytic function of exponential type $\leqq 1$ in t.*

3. Extension of Stein's theorem

Now we state the extension of Stein's theorem which we shall prove, by the following. As an additional notation we use

$$\langle S, \varphi \rangle = \langle T, \overline{\mathcal{F}}[\varphi] \rangle ,$$

for the Fourier transform $T = \mathcal{F}[S]$ of S where $S \in (\mathcal{S}')$ and $\varphi \in (\mathcal{S})$.

THEOREM 1. *Let S be a distribution $\in (\mathcal{S}')$ and let $\mathcal{F}[S] = T$ the Fourier transform of S. Then a necessary and sufficient condition that T vanishes outside a compact, convex and symmetric domain Ω, is that $\langle e^{-ix\cdot h}S, U_t(\varphi) \rangle$ is, for each fixed $\varphi \in (\mathcal{S})$ and a fixed $h \in E^n$, an analytic function of exponential type $\leqq 1 + \rho(h)$ in t, where $U_t(\varphi) = c_t * \varphi$ and $\rho(h) = |h| \sup_{x \in \Omega^*} |x|$.*

REMARK. We note that the following two conditions are equivalent when $S = S(x) \in L^2(E_n)$ and bounded.

(1) $\langle U_t(S), \varphi \rangle$ is, for each fixed $\varphi \in (\mathcal{S})$, an analytic function of exponential type $\leqq 1$ in t.

(2) $U_t(S)(x)$ is, for each fixed $x \in E_n$, an analytic function of exponential type $\leqq 1$ in t.

The equivalence follows from the next lemma, which is known.

LEMMA A. *Let $f_\nu(z)$ be analytic functions of exponential type $\leqq \sigma$ and $|f_\nu(x)| \leqq M$ for $\nu = 1, 2, \cdots$. Then there exist an analytic function $f(z)$ of exponential type $\leqq \sigma$ and a subsequence $\{f_{\nu_k}(x)\}$ of $\{f_\nu(x)\}$ such that $f_{\nu_k}(x) \to f(x)$ uniformly on every finite interval.*

In fact, assume the condition (1) is fulfilled. Let $\alpha(x) \in (\mathcal{D})$ and $\int_{E_n} \alpha(x)dx = 1$. We form the convolutions

$$U_t(S) * \alpha_\nu(x) = \langle U_t(S), \tau_x \alpha_\nu \rangle$$

where $\alpha_\nu(x) = \nu^{-n}\alpha(\nu x)$ and τ_x is the translation operator defined by $\tau_x\alpha(x') = \alpha(x'-x)$. It is well known that $U_t(S) * \alpha_\nu \to U_t(S)$ in $L^2(E_n)$ for each fixed t, when $\nu \to \infty$.

On the other hand, for each fixed $x \in E_n$ we can find by the above lemma an analytic function $f_x(t)$ of exponential type $\leqq 1$ in t and a subsequence $\{U_t(S) * \alpha_{\nu'}\}$ of $\{U_t(S) * \alpha_\nu\}$ such that $f_x^{\nu'}(t) = U_t(S) * \alpha_{\nu'}(x) \to f_x(t)$ uniformly on every finite interval of t, since the $f_x^\nu(t)$ are analytic, of exponential type $\leqq 1$ in t for each fixed $x \in E_n$ and $\sup_t |f_x^\nu(t)| \leqq \omega^* \|S\|_\infty$ where ω^* is the volume of

Ω^*. Hence we have $f_x(t)=U_t(S)(x)$ for a.e. $x\in E_n$ which means that (1) implies (2).

Conversely, let the condition (2) be satisfied. Then from the above lemma it easily follows that for each fixed $\varphi\in(\mathcal{D})$,

$$\langle U_t(S),\varphi\rangle=\int_{F_n}U_t(S)(x)\varphi(x)\,dx$$

is analytic, of exponential type $\leqq 1$ in t, using the fact that $U_t(S)(x)$ is bounded for $x\in E_n$ and real t. Furthermore, letting $\varphi_\nu(x)\in(\mathcal{D})$ approach $\varphi(x)\in(\mathcal{S})$ in $L^2(E_n)$, we conclude that (2) implies (1).

First we shall prove the following lemmas which are used for proving the necessity of the condition in our theorem.

LEMMA 1. *Let* $C_t(y)=\bar{\mathcal{F}}[c_t](y)$ *and* $D_y=\dfrac{\partial^{|\alpha|}}{\partial y_1^{\alpha_1}\cdots\partial y_n^{\alpha_n}}$ *and let* $\varepsilon>0$, *be given. Then there exists a neighborhood* Ω_ε *of* Ω *such that if* $y\in\Omega_\varepsilon$,

$$|D_y\tau_hC_t(y)|=K|t|^{|\alpha|}e^{(1+\rho(h)+\varepsilon)|t|}$$

where K *is a constant independent of* y.

PROOF. Let $\Omega_\varepsilon=\{y\in E^n;|x\cdot y|\leqq 1+\varepsilon$ for all $x\in\Omega^*\}$. We shall show Ω_ε has the required property. Now we fix a $y\in\Omega_\varepsilon$. The Fourier transform of c_t, $\bar{\mathcal{F}}[c_t]$ is

$$C_t(y)=(2\pi)^{-\frac{n}{2}}\langle e^{ix\cdot y},c_t(x)\rangle=(2\pi)^{-\frac{n}{2}}\int_{\Omega^*}e^{itx\cdot y}dx\,.$$

Since the right hand side of the above equality is the integration over a bounded domain Ω^*, $C_t(y)$ is analytic in t for each fixed y, and so is $\tau_hC_t(y)$. Furthermore, by differentiating under the integral sign, we have:

$$D_yC_t(y-h)=(2\pi)^{-\frac{n}{2}}(it)^{|\alpha|}\int_{\Omega^*}x^\alpha e^{itx\cdot(y-h)}dx$$

which shows that $D_y\tau_hC_t$ is also analytic in t, for each fixed y. Let the expansion of $D_y\tau_hC_t(y)$ in t be as follows:

$$D_y\tau_hC_t(y)=\sum_{m=0}^{\infty}a_m(y,h)\frac{t^m}{m!}\,.$$

To estimate the coefficients $\dfrac{a_m(y,h)}{m!}$ of the expansion, we calculate

$$b_m(y,h)=\left[\frac{\partial^m}{\partial t^m}t^{|\alpha|}\int_{\Omega^*}e^{itx\cdot(y-h)}x^\alpha dx\right]_{t=0}.$$

It is evident that the $b_m(y,h)$ vanish for all $m<|a|$. Since

$$\frac{\partial^m}{\partial t^m}t^{|\alpha|}\int_{\Omega^*}x^\alpha e^{itx\cdot(y-h)}dx=\sum_{p=0}^{|\alpha|}\binom{m}{p}|a|\cdots(|a|-p+1)t^{|\alpha|-p}$$
$$\times\int_{\Omega^*}[ix\cdot(y-h)]^{m-p}e^{itx\cdot(y-h)}dx$$

if $m\geqq|\alpha|$, and since $|a_m(y,h)|=(2\pi)^{-\frac{n}{2}}|b_m(y,h)|$, we have, for all $y\in\Omega_\varepsilon$,

$$|b_m(y,h)|=K\frac{m!}{(m-|\alpha|)!}(1+\rho(h)+\varepsilon)^{m-|\alpha|}$$

noting that $|x\cdot(y-h)|\leqq 1+\rho(h)+\varepsilon$.

Thus we have the inequality:

$$\begin{aligned}|D_y\tau_hC_t(y)|&\leqq K\sum_{m=|\alpha|}^{\infty}\frac{1}{(m-|\alpha|)!}(1+\rho(h)+\varepsilon)^{m-|\alpha|}|t|^m\\&=K|t|^{|\alpha|}\sum_{m=0}^{\infty}\frac{(1+\rho(h)+\varepsilon)^m|t|^m}{m!}\\&=K|t|^{|\alpha|}e^{(1+\rho(h)+\varepsilon)|t|}\end{aligned}$$

for all $y\in\Omega_\varepsilon$, which proves Lemma 1.

LEMMA 2. *Let $f(y)$ be a continuous function supported by a compact set $\subset\Omega_\varepsilon$. Then $A(t)=\int[D_y\tau_hC_t(y)]f(y)\,dy$ is analytic in t.*

PROOF. Setting

$$f_\nu(y,h)=f(y)\sum_{k=0}^{\nu}a_k(y,h)\frac{t^k}{k!},$$

it is clear that for each fixed y and t

$$|f_\nu(y,h)|\leqq K_1|t|^{|\alpha|}e^{(1+\rho(h)+\varepsilon)|t|}|f(y)|$$

and that

$$\lim_{\nu\to\infty}f_\nu(y,h)=f(y)D_y\tau_hC_t(y).$$

Therefore we observe for each fixed t

$$A(t)=\sum_{k=0}^{\infty}\frac{t^k}{k!}\int a_k(y,h)f(y)\,dy$$

which proves Lemma 2.

PROOF OF NECESSITY:

Let $\varepsilon>0$ be arbitrary, and define Ω_ε as in Lemma 1. Then T is represented by a finite sum of derivatives of continuous functions as follows:

$$T=\sum_{i=1}^{l}D_ig_i$$

where g_i are continuous functions vanishing outside Ω_ε ([4]).

For simplicity we set $T=D_yg$. Since, by the definition of Fourier transform of distribution,

$$\begin{aligned}\langle e^{ihx}S,U_t(\varphi)\rangle&=\langle\tau_hT,C_t\Phi\rangle\\&=\langle D_yg,\tau_{-h}(C_t\Phi)\rangle\\&=(-1)^{|\alpha|}\sum_{\beta\leqq\alpha}\binom{\alpha}{\beta}\langle g,D_y(\tau_{-h}C_t\tau_{-h}\Phi)\rangle.\end{aligned}$$

We have, by Lemma 1,

$$|e^{ihx}S, U_t(\varphi)| \leqq \sum_{\beta\leqq\alpha} \binom{\alpha}{\beta} K_\beta |t|^{|\beta|} \int_{\Omega_\varepsilon} |g(y) D^{\alpha-\beta}(y-h)|\, dy \times e^{|t|(1+\rho(h)+\varepsilon)}$$

$$\leqq K|t|^{|\alpha|} e^{|t|(1+\rho(h)+\varepsilon)}$$

where $\bar{\mathcal{F}}[\varphi]=\Phi$. It follows from the above equalities, by Lemma 2, that $\langle e^{ihx}S, U_t(\varphi)\rangle$ is analytic in t. Thus we have proved the necessity.

To prove the sufficiency we assume Bernstein's theorem as in [**5**]:

LEMMA B. (Bernstein's theorem) Let $F(z)$ be an analytic function of exponential type $\leqq\sigma$ in z and bounded on the real line. Then $\|F'(x)\|_\infty \leqq \sigma \|F(x)\|_\infty$.

Now consider the ease where $S\in(\mathcal{S}')$ and $\mathcal{F}[S]=T$ is a locally square integrable function $T(y)$. Then we have the following lemma:

LEMMA 3. Supp. $T\subset\Omega_\varepsilon$ *if* $\langle S, c_t * \varphi\rangle$ *is analytic of exponential type* $\leqq 1+\varepsilon$ *for each fixed* $\varphi\in(\mathcal{S})$.

PROOF. If $y^1\notin\Omega_\varepsilon$, there exists a $x^1\in\Omega^*$ such that $|x^1\cdot y^1|>1+\varepsilon$ and therefore we can find compact neighborhoods $U(x^1)$ and $V(y^1)$ and a number $\delta>1+\varepsilon$ satisfying $|x\cdot y|\geqq\delta$ for all $x\in U(x^1)\cap\Omega^*$ and $y\in V(y^1)$. On the other hand, from the analyticity of $C_t(y)$ in t and the symmetry of the domain Ω^* we have

$$C_t(y)=(2\pi)^{-\frac{n}{2}}\int_{\Omega^*} e^{itx\cdot y}dx$$

$$=\sum_{k=0}^{\infty} \frac{\gamma_{2k}(y)}{(2k)!} t^{2m}$$

where $\gamma_{2k}(y)=\left[\frac{\partial^{2k}}{\partial t^{2k}} C_t(y)\right]_{t=0}$.

Differentiation under the integral sign in the above equality shows, for $y\in V(y^1)$

$$|\gamma_{2k}(y)| \geqq (2\pi)^{-\frac{n}{2}} m(U(x^1)\cap\Omega^*)\delta^{2k}$$

where $m(\cdot)$ denotes Lebesgue measure on E_n.

Let $\Phi\in(\mathcal{D}_{V(y^1)})$ satisfy

$$\|\Phi\|_2^2=\int|\Phi(y)|^2\,dy \leqq \int_{V(y^1)} |T(y)|^2\,dy$$

and set

$$\varphi(x)=\mathcal{F}[\Phi](x).$$

Then, $F(t)=\langle U_t(S), \varphi\rangle$ is bounded on the real line. In fact,

$$F(t)=\int C_t(y)T(y)\Phi(y)\,dy$$

and

$$| C_t(y) | \leqq (2\pi)^{-\frac{n}{2}} \omega^*$$

therefore we obtain:

$$| F(t) | \leqq (2\pi)^{-\frac{n}{2}} \omega^* \| T \|^2_{V(y^1)}$$

where $\| T \|^2_{V(y^1)} = \int_{V(y^1)} | T(y) |^2 \, dy$.

Hence, Lemma B is applicable to $F(t)$ repeatedly:

$$| F^{(k)}(t) | \leqq (2\pi)^{-\frac{n}{2}} \omega^* \| T \|^2_{V(y^1)} (1+\varepsilon)^k .$$

Since by Lemma 2

$$F(t) = \sum \frac{t^{2k}}{(2k)!} \int \gamma_{2k}(y) T(y) \Phi(y) \, dy$$

we obtain

$$| F^{(2k)}(0) | = \left| \int \gamma_{2k}(y) T(y) \Phi(y) \, dy \right| \leqq (2\pi)^{-\frac{n}{2}} \omega^* \| T \|^2_{V(y^1)} (1+\varepsilon)^{2k} .$$

Now let $\Phi(y)$ approach $\overline{T(y)}$ in L^2 norm. Then from the fact that

$$| F^{(2k)}(0) | = \left| \iint_{\Omega^* \times V(y^1)} (x \cdot y)^{2k} T(y) \Phi(y) \, dx \, dy \right|$$

$$\rightarrow \iint_{\Omega^* \times V(y^1)} (x \cdot y)^{2k} | T(y) |^2 \, dx \, dy$$

it is obvious that

$$M_1 \| T \|^2_{V(y^1)} \delta^{2k} = \int_{V(y^1)} | \gamma_{2k}(y) | \cdot | T(y) |^2 \, dy$$

$$= M \| T \|^2_{V(y^1)} (1+\varepsilon)^{2k}$$

where we set $M_1 = (2\pi)^{-\frac{n}{2}} m(U(x_1) \cap \Omega^*)$, $M = (2\pi)^{-\frac{n}{2}} \omega^*$.

Therefore, noting that $\left(\frac{\delta}{1+\varepsilon}\right)^{2k} \rightarrow \infty$ with k, we have $T(y) = 0$ a. e. in $V(y^1)$ and also in $\complement \Omega_\varepsilon$ which establishes supp. $T \subset \Omega_\varepsilon$. Thus we completed the proof. Now we pass to the proof of sufficiency.

PROOF OF SUFFICIENCY:

Suppose $\langle e^{ixh} S, U_t(\varphi) \rangle$ be analytic, of exponential type $\leqq 1 + \rho(h)$ in t, for each $\varphi \in (\mathcal{S})$ and each $h \in E_n$.

It is sufficient to show that $\langle \alpha S, c_t * \varphi \rangle$ is analytic, of exponential type $\leqq 1 + \rho(h)$ in t for an arbitrary $\alpha \in (\mathcal{S})$ with supp. $\mathcal{F}[\alpha] \subset K_{|h|}$ where $K_{|h|} = \{ y ; | y | \leqq | h | \}$.

For, then, $\mathcal{F}[\alpha S] = \mathcal{F}[\alpha] * T$ is a C^∞-function and hence by Lemma 3, we have supp. $\mathcal{F}[\alpha S] \subset \Omega_{\rho(h)}$. Taking a sequence of functions $\alpha_\nu \in (\mathcal{S})$ such that $\lim_\nu \mathcal{F}[\alpha_\nu] = \delta$ in $(\mathcal{E})$, we can conclude supp. $T \subset \Omega$. In fact, because of the convexity of Ω_ε, there holds for $\alpha \in (\mathcal{S})$ with supp. $\mathcal{F}[\alpha] \subset K_{\varepsilon_1}$

$$\overline{\text{supp.}\,\mathcal{F}[\alpha] * T} \subset \Omega_\varepsilon$$

where by $\bar{A}$ we mean the convex closure of A and $\varepsilon_1 = \varepsilon\,(\sup_{x\in\Omega^*} |x|)^{-1}$. As is well known, the theorem on supports shows

$$\overline{\text{supp.}\,\mathcal{F}[\alpha]} + \overline{\text{supp.}\,T} = \overline{\text{supp.}\,\mathcal{F}[\alpha S]}\,.$$

Therefore we have

$$\overline{\text{supp.}\,T} \subset \Omega_\varepsilon$$

for all $\varepsilon > 0$, which proves supp. $T \subset \Omega = \bigcap_{\varepsilon>0} \Omega_\varepsilon$.

Now we note that for $T \in (\mathcal{S}')$ and $\gamma \in (\mathcal{D})$

$$\gamma * T = \lim_j \sum_{\nu j} a_{\nu_j} \tau_{h_{\nu_j}} T \text{ in } (\mathcal{S}') \quad \text{(filtre convergence)}$$

where $h_\nu \in$ supp. γ ([4]).

Then there holds

$$\begin{aligned} \langle \alpha S, c_t * \varphi \rangle &= \langle \mathcal{F}[\alpha] * T, C_t \Phi \rangle \\ &= \lim_j \sum_{\nu j} a_{\nu_j} \langle T, \tau_{-h_{\nu_j}}(C_t \Phi) \rangle \\ &= \lim_j \sum_{\nu j} a_{\nu_j} \langle e^{ih_{\nu j}x} S, U_t(\varphi) \rangle\,. \end{aligned}$$

This implies that there exists a j_0 such that for all $j > j_0$

$$\Big| \sum_{\nu j} a_{\nu_j} \langle e^{ih_{\nu j}x} S, U_t(\varphi) \rangle \Big| \leqq 1 + |\langle \alpha S, U_t(\varphi) \rangle|\,.$$

Since $|\langle \alpha S, c_t * \varphi \rangle| = (2\pi)^{-\frac{n}{2}} \omega^* \int |\mathcal{F}[\alpha S] \cdot \Phi(y)|\, dy$ on the real line in t and since the integration on the right hand side exists, it is clear that $S_j(t) = \sum_{\nu j} a_{\nu_j} \langle e^{ih_{\nu j}x} S, U_t(\varphi) \rangle$ are uniformly bounded on the real line for all $j > j_0$. Since also $S_j(t)$ are analytic, of exponential type $\leqq 1+\varepsilon$ when supp. $\mathcal{F}[\alpha] \subset K_{\varepsilon_1}$, by assumption Lemma A implies that there exists a function $S(t)$ which is analytic, of exponential type $\leqq 1+\varepsilon$ in t such that

$$\lim_j S_j(t) = S(t)$$

uniformly on every finite interval of t. It is obvious that $S(t) = \langle \alpha S, U_t(\varphi) \rangle$.

Thus we have proved our theorem completely. If we take the convex domain Ω to be the sphere of radius σ with center origin, we can obtain the theorem in the following form:

COROLLARY. *Let $S \in (\mathcal{S}')$ and let $\mathcal{F}[S] = T$. Then a necessary and sufficient condition that T vanish outside the sphere of radius σ and center origin, is that*

$$t^{-n} \int_{|y| \leqq t} S * \varphi(y)\, dy$$

be, for each fixed $\varphi \in (\mathcal{S})$, an analytic function of exponential type $\leqq \sigma$ in t.

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